Chapter 7

Learning Elementary Number Theory Through a Chain of Discovery: Preservice Teachers' Encounter with Pentominoes

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Elementary number theory concepts form the structural underpinnings of much of the arithmetic that is taught in the intermediate grades (Grades 4–7). Unfortunately, these underpinnings often remain an implicit part of a student's mathematical experiences, being displaced by a focus on the more obvious, and seemingly more important, algorithm for performing calculations. As such, students often graduate from their intermediate years with an impoverished view of, not only arithmetic but also mathematics; a view that promotes the memorization of facts and the mastery of algorithms over the comprehension of deeper mathematical structures (Borwein & Jörgenson, 2001). This is especially true for intermediate students' conceptions of division with remainder, the teaching of which is largely comprised of, if not completely displaced by, a focus on the algorithm for doing long division. As such, these students develop an impoverished view of division with remainder that hampers their ability to correctly answer even relatively simple story problems (Cai & Silver, 1995; Silver, 1992; Silver, Shapiro, & Deutsch, 1993). Ironically, this impoverished view is not restricted to the students. Research has shown that preservice elementary school teachers also have difficulty mastering concepts related to division with remainder (Campbell, 2002; Silver & Burkett, 1994; Zazkis, 1999, 2000; Zazkis &

Campbell, 1996a, 1996b). As such, any attempts to change the experiences of the students must start with changing the experiences of the teachers. In particular, teachers need to experience division with remainder concepts in a context other than in simple computation. They need to experience these concepts in a context that will allow for deeper exploration of the concepts themselves, and allow them to make explicit some of the understandings that are inherent in the algorithms they teach. Finally, teachers need to experience division with remainder concepts in an environment that is not only cognitively meaningful but also affectively rich, an environment that will foster positive beliefs and attitudes about mathematics in general (Liljedahl, 2002, 2004a, 2005) and division with remainder in particular.

In this chapter, I present a study that examines such an experience—a group of preservice teachers' exploring division with remainder in the context of a very specific type of problem-solving environment designed around, what I call, a *chain of discovery*.

A CHAIN OF DISCOVERY

Simply put, a chain of discovery is an experience in which a person, in the process of solving a mathematical problem, makes a series of mathematical discoveries. However, there are subtleties about such an experience that are lost through such a simple description. As such, a more detailed and descriptive explanation is necessary. In what follows, the idea of a chain of discovery is more thoroughly developed through the presentation of a progressively narrowing focus on specific types of mathematical problem solving. I begin with a cursory discussion of mathematical problem solving, and then examine three specific types of problem solving, of which chain of discovery is the third.

Mathematical Problem Solving

Much has been written about mathematical problem solving and to try to summarize it in a concise fashion is difficult; difficult primarily because opinions vary as to what constitutes a mathematical problem-solving experience, what constitutes mathematical problem solving, and even, what constitutes a mathematical problem. As such, rather than embark on either a synthesizing or a differentiating analysis of these varying understandings of mathematical problem solving, I present a brief working definition that will allow me to discuss the three subsequent specific cases of problem solving more effectively. First of all, mathematical problem solving can be thought of as being divided into two distinct, but related, processes—the logical and the extralogical; the logical processes will be dealt with in this section, the extra-logical processes will be treated in the next section.

The primary logical processes of problem solving are alternatively known as *problem solving by design* (Rusbult, 2000). The process begins with a clearly defined goal or objective after which there is a great reliance on relevant past experience, referred to as *repertoire* (Bruner, 1964; Schön, 1987), to produce possible options that will lead toward a solution of the problem (Poincaré, 1952). These options are then examined through a process of conscious evaluations (Dewey, 1933) to determine their suitability for advancing the problem toward the final goal. In very simple terms, problem solving by design is the process of deducing the solution from that which is already known.

Mayer (1982), Schoenfeld (1982), and Silver (1982) stated that prior knowledge is a key element in the problem-solving process. Prior knowledge influences the problem solver's understanding of the problem as well as the choice of strategies that will be called on in trying to solve the problem. In fact, prior knowledge and prior experiences is ALL that a solver has to draw on when first attacking a problem. As a result, all problem-solving heuristics must incorporate this resource of past experiences and prior knowledge into their initial attack on a problem, and all do. Some heuristics refine these ideas, and some heuristics extend them. Of the heuristics that refine, none is more influential than the one created by George Pólya (1887–1985) in his book *How to Solve It* (1957), in which Pólya laid out a problem-solving heuristic that relies heavily on a repertoire of past experience. He summarized the four-step process of his heuristic as follows:

1. UNDERSTANDING THE PROBLEM
    - *First*. You have to understand the problem.
    - What is the unknown? What are the data? What is the condition?
    - Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
    - Draw a figure. Introduce suitable notation.
    - Separate the various parts of the condition. Can you write them down?
2. DEVISING A PLAN
    - *Second*. Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should eventually obtain a plan of the solution.
    - Have you seen it before? Or have you seen the same problem in a slightly different form?
    - Do you know a related problem? Do you know a theorem that could be useful?
    - Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

- Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. CARRYING OUT THE PLAN
    - *Third*. Carry out your plan.
    - Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?
4. LOOKING BACK
    - *Fourth*. Examine the solution obtained.
    - Can you check the result? Can you check the argument?
    - Can you derive the solution differently? Can you see it at a glance?
    - Can you use the result, or the method, for some other problem?

The emphasis on auxiliary problems, related problems, and analogous problems that are, in themselves, also familiar problems is an explicit manifestation of this strategy. The use of familiar problems also requires an ability to deduce from these related problems a recognizable and relevant attribute that will transfer to the problem at hand. The mechanism that allows for this transfer of knowledge between analogous problems is known as analogical reasoning (English, 1997, 1998; Novick, 1988, 1990, 1995; Novick & Holyoak, 1991; Vosniadou & Ortony, 1989) and has been shown to be an effective, but not always accessible, thinking strategy.

Step four in Pólya's heuristic, looking back, is also a manifestation of utilizing prior knowledge to solve problems, albeit an implicit one. Looking back makes connections "in memory to previously acquired knowledge … and further establishes knowledge in long-term memory that may be elaborated in later problem-solving encounters" (Silver, 1982, p. 20). That is, looking back is a forward-looking investment into future problem-solving encounters; it sets up connections that may later be needed.

Mathematical Discovery

Perkins (2000) made a clear distinction between problems that a person cannot solve and problems that a person has not yet solved. A problem that does not yield to a process of design is not necessarily unsolvable, but may merely be problematic. Such problems will require input from the extralogical processes in order for a solution to be found. The extralogical processes of problem solving are those processes that lie outside of the "theories of logical forms" (Dewey, 1938). Included in the cadre of the extralogical are such mysterious phenomena as intuition, imagination, insight, illumination, serendipity, and aesthetics, each of which may contribute to the solving of a problem in a fashion that may defy explanation, or more appropriately, defy logic. Also included in this collection are two phenomena that received much attention within the field of mathematics over the course of the last 100 years, creativity and discovery, both of which were born from the minds of French mathematicians.

In 1908, Henri Poincaré (1854–1912) gave a presentation to the French Psychological Society in Paris entitled "Mathematical Creation." This presentation, as well as the essay it spawned, stands to this day as one of the most insightful and thorough treatments of the topic of mathematical invention and put forth the proposition that the unconscious mind plays an invaluable role in the creative process. In particular, the anecdote of Poincaré's own discovery of Fuschian function transformations stands as the most famous contemporary account of mathematical creation.

> Just at this time, I left Caen, where I was living, to go on a geological excursion under the auspices of the School of Mines. The incident of the travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuschian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had the time, as, upon taking my seat in the omnibus, I went on with the conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake, I verified the results at my leisure. (Poincaré, 1952, p. 53)

Inspired by this presentation, Jacques Hadamard (1865–1963) began his own empirical investigation into mathematical invention. The results of this seminal work culminated in a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City in 1943. These talks were subsequently published as *The Psychology of Mathematical Invention in the Mathematical Field* (Hadamard, 1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology was an entertaining, and sometimes humorous, look at the eccentric nature of mathematicians and their ritualistic practices. His work is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process in general, turned them into a stage theory. This theory still stands as a viable and reasonable description of the process of mathematical discovery. In what follows, I present this theory, referenced not only to Hadamard and Poincaré, but also to some of the many researchers who's work has informed and verified different aspects of the theory.

The phenomenon of mathematical discovery, although marked by sudden illumination, consists of four separate stages stretched out over time, of which illumination is but one part. These stages are *initiation, incubation, illumination,* and *verification* (Hadamard, 1945). The first of these stages, the initiation phase, consists of deliberate and conscious work. This would constitute a person's voluntary, and seemingly fruitless, engagement with a problem and be characterized by an attempt to solve the problem by trolling through a repertoire of past experiences (Bruner, 1964; Schön, 1987). This is an important part of the inventive process because it creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Barnes, 2000; Davis & Hersh, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997).

Following the initiation stage, the solver, unable to come to a solution, stops working on the problem at a conscious level (Dewey, 1933) and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952). This is referred to as the incubation stage of the inventive process and it is inextricably linked to the conscious and intentional effort that precedes it.

> There is another remark to be made about the conditions of this unconscious work: it is possible, and of a certainty it is only fruitful, if it is on the one hand preceded and on the other hand followed by a period of conscious work. These sudden inspirations never happen except after some days of voluntary effort which has appeared absolutely fruitless and whence nothing good seems to have come … (Poincaré, 1952, p. 56)

After the period of incubation, a rapid coming to mind of a solution, referred to as illumination, may occur. This is accompanied by a feeling of certainty (Poincaré, 1952) and positive emotions (Barnes, 2000; Burton

1999; Rota, 1997). With regards to the phenomenon of illumination, it is clear that this phase is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind (Poincaré, 1952), a coming to (conscious) mind of an idea or solution. However, what brings the idea forward to consciousness is unclear. There are theories on aesthetic qualities of the idea (Poincaré, 1952'; Sinclair, 2002), surprise/shock of recognition (Bruner, 1964), fluency of processing (Whittlesea & Williams, 2001), or breaking functional fixedness (Ashcraft, 1989) only one of which will be expanded upon here.

Poincaré proposed that ideas that were stimulated during initiation remained stimulated during incubation. However, freed from the constraints of conscious thought and deliberate calculation, these ideas would begin to come together in rapid and random unions so that "their mutual impacts may produce new combinations" (Poincaré, 1952, p. 61). These new combinations, or ideas, would then be evaluated for viability using an aesthetic sieve (Sinclair, 2002), which allowed through to the conscious mind only the "right combinations" (Poincaré, 1952, p. 62). It is important to note, however, that good or aesthetic does not necessarily mean correct. As such, correctness is evaluated during the fourth and final stage—*verification*.

Flow and Discovered Complexity

Flow (Csikszentmihalyi, 1996) is most simply described as the pleasurable state that a person may find himself or herself in when doing an activity. It is a state where one's actions are "automatic, effortless, yet it is also a highly focused state of consciousness" (p. 110), and enjoyment and engagement are at a maximum. Csikszentmihalyi identified nine key elements in people's descriptions of such states.

1. There are clear goals every step of the way.
2. There is immediate feedback on one's actions.
3. There is a balance between challenges and skills.
4. Attention is focused on one's actions.
5. Distractions are excluded from consciousness.
6. There is no worry of failure.
7. Self-consciousness disappears.
8. The sense of time becomes distorted.
9. The activity becomes satisfying in its own right.

Williams (2001) has taken Csikszentmihalyi's idea of flow and applied it to a specific instance of problem solving that she refers to as *discovered complexity*. Discovered complexity is a state that occurs when a problem solver, or a group of problem solvers, encounter complexities that were not evident at the onset of the task and that are within their zone of proximal development (Vygotsky, 1978). This occurs when the solver(s) "spontaneously formulate a question (intellectual challenge) that is resolved as they work with unfamiliar mathematical ideas" (p. 378). Such an encounter will capture and hold the engagement of the problem solver(s) in a way that satisfies the conditions of flow. What Williams' framework describes is the deep engagement that is sometimes observed in students working on a problem-solving task during a single-problem solving session. What it does not describe, however, is a student's willingness to return to the same task, again and again, over an extended period of time (several days or weeks) until the problem is solved. Such willingness requires a different theoretical framework to explain, a framework built from a chain of discovery.

A Chain of Discovery

Willingness to engage in a problem-solving activity resides within a student's affective domain. It may reside within the student's beliefs and attitudes or it may reside within the student's emotions (McLeod, 1992). Beliefs are just that, what students believe; what they believe to be true about mathematics and what they believe about their ability to do mathematics. Beliefs about mathematics are often based on their own experiences with mathematics. For example, beliefs that mathematics is "difficult," "useless," "all about one answer," or "all about memorizing formulas" stem from experiences that have first introduced these ideas and then reinforced them.

A qualitatively different form of belief is with regards to a person's beliefs in their ability to do mathematics, often referred to as efficacy, or self-efficacy. Self-efficacy, like the aforementioned belief structures, is a product of an individual's experiences with mathematics, and is likewise slow to form and difficult to change. Self-efficacy with regard to mathematics has most often been dealt with in the context of negative belief structures (Ponte, Matos, Guimarães, Cunha Leal, & Canavarro, 1992) such as "I can't do math," "I don't have a mathematical mind," or even "girls aren't good at math."

Attitudes can be defined as "a disposition to respond favourably or unfavourably to an object, person, institution, or event" (Ajzen, 1988, p. 4) and can be thought of as the responses that students have to their belief structures. That is, attitudes are the manifestations of beliefs. For example, beliefs such as "math is difficult," "math is useless," or "I can't do math" may result in an attitude such as "math sucks." A belief that "math is all about formulas" may manifest itself as an attitude of disregard for explanations in anticipation of the eventual presentation of a formula.

Emotions, on the other hand, are relatively unstable (Eynde, De Corte, & Verschaffel, 2001). They are rooted more in the immediacy of a situation or a task and as a result, are often fleeting. Students with generally negative beliefs and attitudes can experience moments of positive emotions about a task at hand or, conversely, students with generally positive outlooks can experience negative emotions.

So, the willingness of a student to engage with a problem during a single problem-solving session may, in fact, be due to a temporarily heightened positive emotional response to the situation. This is certainly in keeping with the foundation of "enjoyment" existing within the frameworks of both Csikszentmihalyi (1996) and Williams (2001). To extend this engagement across several sessions, however, beliefs and attitudes about the task must also be in a positive state. Within some student this may already be the case in that the student may have positive beliefs and attitudes about either problem solving in general, or the topic in which the problem is set in particular. Within other students, however, this may be more problematic.

For a student who has already developed negative beliefs and attitudes about mathematics and/or problem solving, engagement in a problem-solving task across many sessions will be difficult. Research has shown that, although negative beliefs and attitudes are slow to form in a learner, they are equally slow to change once formed (Eynde, De Corte, & Verschaffel, 2001). Ironically, change is most often achieved through the emotional dimension in that repeated positive experiences will eventually produce positive beliefs and attitudes. However, change is slow and will generally require a large number of successive successes before any long-term change is observed. As such, the "enjoyment" that may be experienced in the first session of an extended series of problem solving will likely not effect enough change to encourage engagement in subsequent sessions.

However, there is one mechanism by which change in attitudes and beliefs can be quickly realized. Research has shown that the experience of discovery in the context of mathematical problem solving has an immediate and powerfully transformative effect on learners beliefs about mathematics and their ability to do mathematics and the attitudes that govern their behavior in the context of doing mathematics (Liljedahl, 2002, 2004a, 2005). These changes can be as wide sweeping as "I now like mathematics," but are more likely to fall into the domain of "I can do this." The exact mechanism within discovery that facilities this change is not clear but it may be linked to feelings of certainty (Burton, 1999; Fischbein, 1987), bliss (Rota, 1997), and the release of the tension from unresolved effort that occurs at the flash of illumination; "that moment when the connection is made, in that synaptic spasm of completion when the thought drives through the red fuse, is our keenest pleasure" (Harris, 2000, p. 132). Extrapolating this positive effect across a series of discoveries will not only magnify change in the affective domain, but will also maintain the engagement of the student in the problem-solving task across several distinct sessions.

A chain of discoveries will facilitate such an extrapolation. It occurs when a problem solver, or a group of problem solvers, encounter successive discoveries in the course of solving a problem over an extended period of time requiring an extended number of problem-solving sessions. Each new discovery provides the solver(s) with new information and new tools to aid in the advancement toward an eventual solution, as well as provides the necessary changes in the affective domain to sustain engagement across these many sessions.

OCCASIONING A CHAIN OF DISCOVERY

The idea that discovery in general, and a chain of discovery in particular, can be orchestrated has to be qualified with the fact that such experiences are largely dependent on chance; intrinsic and extrinsic chance. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new understanding. This was discussed by Hadamard (1945) as well as by a host of others under the name of "the chance hypothesis." Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article, a chance encounter with an individual or some piece of mathematical knowledge, any of which contributes to the eventual resolution of the problem on which one is working. However, the idea that mathematical discovery often relies on the fleeting and unpredictable occurrence of chance encounters is starkly contradictory to the image projected by mathematics as a field reliant on logic and deductive reasoning. Ironically, this contributory role of chance emerged from a study done with prominent research mathematicians in which a portion of Hadamard's (1945) famous survey was resurrected in order to solicit responses pertaining to illumination, creativity, and insight (Liljedahl, 2004a, 2004b). The upshot of this strong dependence on chance means that the orchestration of discovery can best be described as the *occasioning* (Kieren, Simmt, & Mgembelo, 1997) of discovery. That is, the environment for such an experience can be orchestrated, but the experience itself cannot.

In general, an environment conducive to discovery is one that provides interaction, time, and a rich task. Research has shown that much more is achieved in the context of problem solving from interacting with others, whether that interaction is collaborative, or merely conversational (Liljedahl, 2004a). People learn from each other and they learn in conjunction with others. In fact, research mathematicians claim that they are much more likely to garner something that will help them solve a particular problem through a conversation with someone than from reading something (Liljedahl, 2004a).

Time is also an important element in the discovery process. A discovery is much more likely to occur after much time has been dedicated to the solving of a problem, and is much more likely to occur at the moment when a problem is revisited (Liljedahl, 2004a). This means that, to begin with, a lot of time has to be provided in the form of allocated time as well as extended deadlines. It also means that within this allocation of time, there needs to be many opportunities to revisit the problem.

Finally, a discovery is much more likely to occur in the context of working on a good problem. Schoenfeld (1982) has come up with a list of characteristics of a good problem, summarized as follows:

- The problem needs to be accessible. That is, it is easily understood, and does not require specific knowledge to get into.
- The problem can be approached from a number of different ways.
- The problem should serve as an introduction to important mathematical ideas.
- The problem should serve as a starting point for rich mathematical exploration and lead to more good problems.

However, a good problem for the purposes of occasioning a discovery is slightly different than what is more generally considered a good problem. Perkins (2000) felt that what is important is that a problem be "problematic." That is, the problem must cause the solver to get stuck. For the purposes of occasioning discovery and engaging the participants in the process of problem solving, the first two of Schoenfeld's points as well as Perkins' requirement have the greatest relevance. Research has also shown that problems that produce early success and problems that stimulate conversation are the most likely to foster a discovery (Liljedahl, 2004a). For the purposes of occasioning a chain of discovery in particular, the last of Schoenfeld's conditions is most relevant.

CHAIN OF DISCOVERY, ELEMENTARY NUMBER THEORY, AND PRESERVICE TEACHERS: A RESEARCH STUDY

In this section, I present a study that exemplifies a chain of discovery, as well as demonstrates the depth to which mathematical content can be accessed through such a chain. This study details the results of a group of preservice elementary school teachers working to solve a problem steeped in one particular elementary number theory concept, the concept of division with remainder. Cai and Silver (1995), Silver et al. (1993), and Silver (1992) examined this concept as it related to "real life" situations. Campbell (2002) looked at the concept as it related to partitive and quotitive division and the division algorithm. Both showed that participants' understandings of remainder were firmly rooted in the algorithm for performing long division and any understanding of the remainder in terms of the nature of the dividend was rare and limited. My research contributes in exemplifying how a concrete understanding of the remainder as it relates to the nature of the dividend can be achieved or extended through the chain of discovery experienced in attempting to solve the pentominoe problem.

The Participants

Participants in this study are preservice elementary school teachers enrolled in six different offerings of a Designs for Learning Elementary Mathematics course for which I was the instructor, spanning six semesters and 4 years and involving a total of more than 180 students. The course, which runs for 13 weeks, is specifically designed to provide preservice teachers with the foundational knowledge and skills requisite to teach mathematics at the elementary (k–7) level. As such, the focus of the course is more on teaching than on the specifics of mathematics content, although such content is often used as the context in which teaching and learning are discussed. Because of the nature of the courses, and the timing with which they are offered, the majority of the students who take these courses were enrolled in the teacher preparation program offered at the university.

Each offering of the course enrolls between 28 and 35 students with about 90% being female. In addition, the vast majority of the students in each offering are extremely fearful of having to take mathematics and even more so of having to teach mathematics. This fear resides, most often, within their negative beliefs and attitudes about their ability to learn and do mathematics. At the same time, however, as apprehensive and fearful of mathematics as these students are, they are extremely open to, and appreciative of, any ideas that may help them to become better mathematics teachers. They are willing to engage in discussions regarding pedagogy and philosophy and are willing to reflect on how to apply these in their teaching.

The Problem

The problem,[1] as given to the participants, can be seen in Fig. 7.1.

Clearly this is a problem that places much focus on the elementary number theory concepts of divisibility and division with remainder.

The Context

Although the participants are eventually provided with a hard copy of the problem (as presented earlier) their actual exposure to the problem is done in pieces. To begin with, they are simply given the task of determining how

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[1]This problem was presented to me by Ralph Mason. For many other problems related to pentominoes, see Solomon Golomb's (1996) *Polyominoes*.

A pentominoe is a shape that is created by the joining of five squares such that every square touches at least one other square along a full face. Now consider a 100's chart.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

If a pentominoe is placed somewhere on a 100's chart will the sum of the numbers it covers be divisible by 5? If not, what will the remainder be? Explain how you can know **quickly**!

FIG. 7.1. The pentominoe assignment.

many pentominoes there are. This is done as an in-class, group problem-solving activity in which they use small square tiles as manipulatives. Once the class as a whole settles on the fact that there are 12 such shapes, they are presented with the problem as shown in Fig. 7.2.

I allow them to discuss the problem for a few minutes within their groups and then I answer any clarifying questions they may have. Invariably, one of the questions that is asked has to do with what is meant by "quickly." Rather than answer this question, I perform a demonstration. I ask a student to come up and, using their square tile manipulatives, construct a pentominoe of their choice anywhere on a 100s chart that is set up on the overhead projector. While they do this, I close my eyes. On their command, I look at what they have constructed and immediately (within 1 or 2 secs) tell them what the remainder is. This is an important aspect of the problem for it is the participants' attempts to perform the calculation "quickly" that invariably moves them through a chain of discoveries; at each link learning something more about the problem.

The participants then work in groups, both in and out of class, over the remainder of the course to solve the problem. I also make the announce-

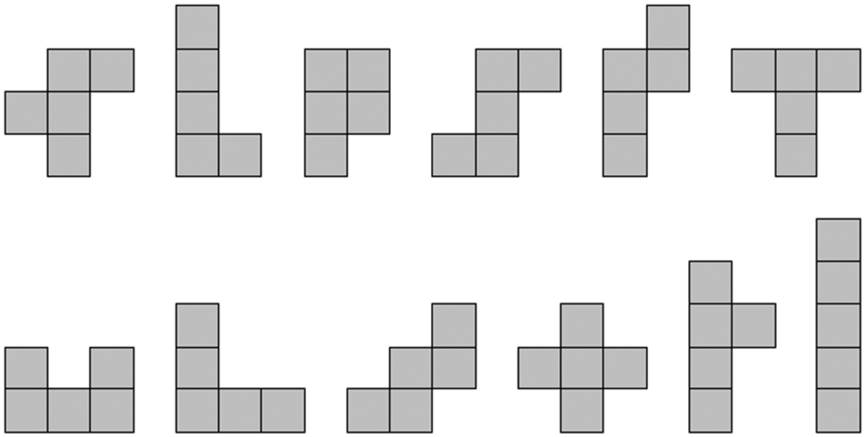


FIG. 7.2. The 12 pentominoes.

ment that they are not likely to arrive at the final solution until they have spent more than 10 hours working on it. This announcement, like the requirement of quickly helps them to move through the chain of discoveries instead of getting caught at one link with the thought that they are done. It also helps to prepare them for the work that they will be required to put into the problem. Finally, the 10-hour minimum serves as a challenge to try to solve the problem in less time.

The Source of the Data

Data for this study comes from a variety of different journals that each participant kept during their enrollment in the course. These journals ranged from personal problem-solving journals, to group problem-solving journals, to personal reflective journals. Personal problem-solving journals were used to track the participants' own trials, progress, successes, failures, and breakthroughs detailing both the mathematics that they are grappling with and the feelings they are experiencing as they work to solve the pentominoe problem. The group problem-solving journals were very similar in nature to the aforementioned personal journals except that they were to be used to track the efforts of the group as they worked on the problem. As such, entries in these journals detailed the goings-on of the group meetings and the collaborative work that occurred in these meetings. Finally, the personal reflective journals provided a place in which they respond to prompts regarding their thoughts and experiences during the course. The entries in this journal are much more contemplative in nature and provide insights into how the participants thinking has changed through their problem-solving experiences within the course.

Together these three journals provide an invaluable picture of what went on while the participants struggled to solve the pentominoe problem, both on their own and in groups. All the journals were collected and sorted according to the groups that worked together. The relevant entries from all of these journals were then coded for emergent themes and cross-checked with journals from other group members.

Results and Analysis

This analysis produced a wealth of data on the phenomenon of chain of discovery in the context of solving the pentominoe problem. In what follows I present, with the use of exemplifying excerpts, this chain of discoveries as it emerges from the data. I illustrate the relevant division with remainder concepts that the participants encounter at each link within the chain as well as discuss the nature of the discovery that predicate the advancement to each link.

Position Doesn't Matter. The first, and most obvious discovery that the students make is that the position of the pentominoe on the 100s chart does not affect the remainder. This very simple discovery seems to serve three very distinct and useful purposes. First, as already mentioned, in a problem as complex and as lengthy as this one, it is useful to have some early success to motivate the students to keep working (Liljedahl, 2004a). In this case, once the remainder for each of the 12 pentominoes has been determined for each of their four orientations, the students have, in essence, established a solution. Given the inelegance of this method, they realize that it is not *the* solution, but having at least *a* solution makes them feel somewhat safe. Rebecca states this nicely in her journal:

> Right away we figured out that the position doesn't matter. This was a huge relief to us because the problem suddenly went from something that seemed completely undoable to something that we could mange.

More important, however, the realization that the position is irrelevant gets the students focusing on the remainder. Although this is an explicit requirement of the problem, the students do not immediately focus on this one attribute alone. This is likely due to the fact that their first attempts at this problem are very mechanical; placing a pentominoe on the 100s chart, summing the numbers that are covered, and dividing by 5. This series of steps produces two results—a quotient and a remainder. Moving a given pentominoe around changes many quantities within this procedure; it changes the numbers that are covered, it changes their sum (the dividend), and it changes the quotient. It does not, however, change the remainder. Until this discovery is made, the students tend to track all of the seemingly relevant information (the specific numbers to be summed, the sum of these numbers, the quotient, and the remainder) as they move their pentominoes around the 100s chart. Once the discovery is made, however, they dispense with tracking all but the remainder. This letting go of the particulars is a positive step toward occasioning discovery (Liljedahl, 2004a, 2004b, 2005).

Finally, the student's attempts to explain why the position does not matter leads to some very useful understanding with regards to division with remainder. In this particular case, moving a pentominoe vertically by one cell on the 100s chart changes the dividend by 50, and moving it horizontally by one cell changes the dividend by 5. When the divisor is 5, neither of these changes in the dividend will produce a change in the remainder. This property is nicely demonstrated in Dave's comments:

> We realized right away that the remainder was the same no matter where we put the shape. It took a lot longer to figure out why this was so. It wasn't until we got a hint from Peter [the instructor] that we started thinking about moving the shape just slightly. If we take a shape, say the L,[2] and we put it on the table so that it covers the 25, 35, 45, 46, and 47 then the sum will be 198. If we now move the L one square to the right then the numbers that are covered will be 26, 36, 46, 47, and 48. We just increased each of the numbers by 1. So, the new total is now 203, 5 more than it was before. But this doesn't matter, because when we divide by 5 we still have a remainder of 3. Likewise, if we move the L down then the numbers that are now covered will be 35, 45, 55, 56, and 57. Basically, we increased each of the numbers by 10, for a total of 50. Adding 50 to a number doesn't change the remainder because 5 goes into 50.

This is a specific case of a more general property of division with remainder; namely that changing the dividend by multiple of the divisor does not change the remainder.[3]

Stemming from the realization that the position of the pentominoe does not affect the remainder emerges a strategy of intentionally placing the pentominoes in specific regions of the grid in order to simplify the calculations. In particular, students begin to place the pentominoes where the in-

[2]The participants usually named the pentominoe shapes according to the letter or symbol that they most closely resembled. The L, for example, is for this participant the pentominoe .

[3]Notice that Dave presents all of his arguments in arithmetic terms (with specific numbers and operations) as opposed to algebraic terms. Of the more than 180 participants who have worked on this problem, none have submitted solutions that use algebraic symbolization.

dividual numbers are the smallest; that is, in the top left hand corner of the 100s chart. In fact, adopting this strategy allows them to reduce the 100s chart to a 5 × 5 grid (Fig. 7.3a). This repositioning is often accompanied with a strategy of ignoring the 10s digit of the individual cell numbers in the chart as seen in the comments of Jennifer:

> We can just ignore the 10s part of the number because adding on all those 10s doesn't change the last number, and it is the last number that determines the remainder.

Sometimes this also results in an explicit renumbering of the chart using only the one's digits (see Fig. 7.3b).

Combining these two strategies produces a strategy of focusing on only the top left-hand corner of the renumbered grid (Fig. 7.3c) as invoked by Jonathan.

1	**2**	**3**	**4**	**5**	6	7	8	9	10
11	**12**	**13**	**14**	**15**	16	17	18	19	20
21	**22**	**23**	**24**	**25**	26	27	28	29	30
31	**32**	**33**	**34**	**35**	36	37	38	39	40
41	**42**	**43**	**44**	**45**	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

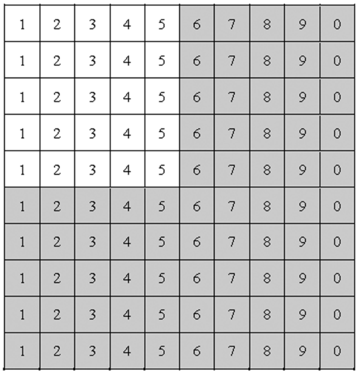
(a)

FIG. 7.3a. Reducing the grid.

1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0
1	2	3	4	5	6	7	8	9	0

(b)

FIG. 7.3b. Renumbering the grid.



(c)

FIG. 7.3c. Reducing and renumbering the grid.

> Because the position of the pentominoe doesn't matter we'll just put it in the top left-hand corner of our new chart. This way the numbers are small and every possible pentominoe will still fit in there.

These methods of streamlining the calculations of the remainders allow the students to dispense with the use of a calculator. In all of these cases, however, these strategies all involve, in one form or the other, the explicit summing of the dividend as a means to access the remainder. Although a much improved method for calculating the remainder, further improvement is needed if the students want to be able to "very quickly" perform the calculation. This improvement comes when the students begin to consider how each piece of the pentominoe will contribute to the final remainder, as opposed to the dividend, and is discussed in great detail in a subsequent section.

Orientation Does Matter[4]

Whereas the position of the pentominoe does not affect the remainder, the orientation of the pentominoe does. This discovery is often made simultaneously with the previous one, and although more difficult to explain than the irrelevance of position, does lead to some interesting patterns regarding symmetry, reflections, and rotations.

The participants very quickly discover that those shapes that are symmetrical about a vertical axis (cf.) have a remainder of zero, and those shapes that are equally weighted or balanced on both sides of a vertical axis

[4]In strictly mathematical terms, "orientation" is part of "position." However, as a theme emerging from the data, the orientation of the pentominoe is distinct from its position.

(cf.) also have a remainder of zero. Both of these discoveries are nicely demonstrated in the words of Veronica:

> When we looked at the pentominoes that had no remainder, the first thing we noticed was that from the midpoint of the shape there was always an equal number of blocks on each side. Sometimes the shapes were symmetrical like the +, the U and the T. Other times the shapes were not symmetrical like the Z. We decided that all of these shapes were somehow balanced and the blocks must cancel each other out and make a remainder of zero.

A further discovery is that for those shapes that do not have a remainder of zero, the reflection of the pentominoe about any vertical axis will produce a remainder that is the 5s compliment of the original remainder. For example, while has a remainder of 3, has a remainder of 2. Betty makes this observation using a different set of pentominoes than the one provided in the previous example.

> When we looked at the little L we noticed that its remainder and the remainder of its reflection added up to 5 (Fig. 7.4). When we looked at more shapes we discovered that mirror reflections of pentominoe remainders *always* add up to 5.

Further to this point of reflections, it is also quickly discovered that any reflections across a horizontal axis produce no change in the remainder.

Similar to the complements produced by reflection are the complements produced by half-turn rotations. That is, the half-turn rotation of a given pentominoe will also produce a 5s compliment of the original remainder. So, while has a remainder of 4, has a remainder of one. John demonstrates this realization in his problem-solving journal in the context of exploring rotations in general:

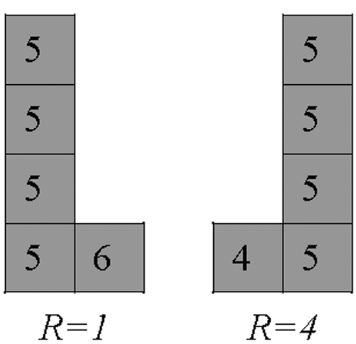


FIG. 7.4. Betty's example of reflection complements.

> We looked at each of the 18 pentominoe shapes[5] with their 4 rotations. We started to notice that if a shape had no remainder then neither did its 180° rotation. Then we noticed that if the remainder was not 0, then the 180° rotation would be the opposite of the original remainder. For example, the P has a remainder of 2 and its 180° rotation has a remainder of 3. They always add up to 5.

Although not evident in the previous passage, this "noticing" of the pattern was much more of a discovery than John indicates here. In his reflective journal, John writes about this discovery:

> We had all been working on the pentominoe problem separately. We had each taken three of the pentominoe shapes and had calculated what each of the remainders were depending on how you turned them. When we met that first time we shared our results and it was just a mess. It was pretty obvious that the symmetrical shapes had a remainder of zero. Reva also had noticed that the reflections of her shapes always added up to five and when we checked our own shapes we found this was always true. We were beginning to talk about how we would divide up the rest of the work when all of a sudden it hit me—Reva's idea about reflection also worked for rotation. It was right there on the paper and it just jumped out at me all of a sudden ...

These four discoveries (symmetry, balance, reflection complements, and rotation complements) emerge through the students' attempts to catalogue the remainders for each pentominoe and its various orientations. The identification of these four properties serves to reduce the solution space from many seemingly unrelated remainders to less than many sometimes related remainders. However, more connections and patterns are needed in order to be able to decode the remainder of a given pentominoe placed in a given position, and a given orientation somewhere on the 100s chart. Unlike the discovery concerning position, however, insights into these four discoveries are not forthcoming at this point in the students' problem-solving efforts. That is, although they are able to identify these patterns, they are not yet able to explain them. Furthermore, they are often not able to explain these patterns until a dynamic understanding of the problem is achieved near the very end of the problem-solving process (see following section).

Summing of Remainder. An altogether different strategy for calculating the remainder comes from considering how each cell would individually contribute to the remainder. That is, when division with remainder is performed on a sum, then the remainder of the sum will equal the sum of

[5]When the original 12 shapes are considered in conjunction with their reflections, then there are 18 distinct shapes.

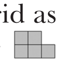
the remainders.[6] This is another property that this problem seems to make explicit for the students, as can be seen in the following reference from a group journal:

> It was our fourth meeting and so far, we hadn't gotten very far. When Jessica arrived, she showed us something that she had been trying on her own. She had renumbered the entire grid 1, 2, 3, 4, 0, 1, 2, 3, 4, 0. She explained that these numbers are the remainders that each number on the grid has. This was a much easier grid to work with.... Why it works—when we take five numbers and add them up and divide by five to get the remainder, we are really just looking at how many groups of five we can remove from the total and how much is left over. But, this means that we have to first add a bunch of groups of five first and then remove them later. Renumbering the grid, all we really do is remove the groups of five first. So, a number like 23 is really just 4 groups of 5 and a 3 left over. If our pentominoe shape covers the 23 we only have to worry about the 3.

A closer look at Jessica's reflective journal reveals how she came up with this idea.

> At our last meeting we had decided that we were going to renumber the 100s chart by ignoring the first part of the number. This was fine, I was doing this in my head anyway. I had made a new chart on my computer, and I was filling in the rows [columns] one by one. When I got to the fifth row [column] I was suddenly struck by the idea that I didn't need this row. It was going to be full of 5s anyway and 5s didn't change the remainder. This was a very strange idea. It felt both right and wrong at the same time ...

The students who chose to follow this strategy all renumbered the 100s chart such that each cell is replaced with its remainder in division by five (see Fig. 7.5a).

At first glance this may not seem to produce much of a change in the calculation of the remainder, especially if the student is invoking a strategy of placing the pentominoes in the top left-hand corner of the grid as in the previous section. Consider, for example, the pentominoe  being placed in the top left-hand corner of the renumbered grid in Fig. 7.5b and in the top left-hand corner of Fig. 7.3c. In both cases, the calculation of the remainder would involve the summing of (1, 2, 1, 2, 3) and the eventual division of the sum by 5 and the extraction of the remainder. So, at the level of arithmetic, the renumbering of the grid presented in Fig. 7.5a produces no change. However, at the level of conception, this renumbering of the grid is profound. As will be seen in subsequent sections, this renumbering is a necessary step toward the liberation of the solution from the particulars of the

[6]That is, if $x \equiv a \pmod{k}$, and $y \equiv b \pmod{k}$, then $x + y \equiv a + b \pmod{k}$.

1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0

(a)

FIG. 7.5a. Reducing the grid to remainders.

numbers. More immediately, however, it allows the students to place the pentominoes in a more strategic position on the grid (see Fig. 7.5c); a placement that, yet again, reduces the number of calculations that are necessary. This is seen in Leslie's work:

> We realized that we can really speed up our calculations if we put the pentominoe in the centre of the 100s chart. This way some of the squares that are covered would be zeros.

In general, the initial urge for the students is to place the pentominoes on the "remainder grid" so that the left most column of the pentominoe sits on the "zero line." However, they quickly come to the realization that certain columns of the pentominoe are better placed on the "zero line." Leanne nicely articulates this:

1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0
1	2	3	4	0	1	2	3	4	0

(b)

FIG. 7.5b. Placing pentominoe on new grid.

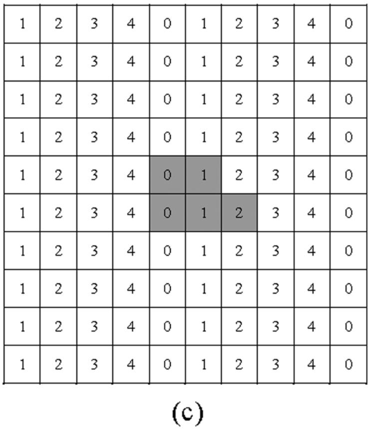


FIG. 7.5c. Replacing pentominoe on new grid.

> At first we were just putting the pentominoes on the fives column and to the right. This way we started with zero and then just counted over how many 1s we had, how many 2s we had, etc. Then Alyssa suddenly suggested that for some shapes we should put the pentominoe down so that we have the most number of squares on the fives column. For example, look at the backwards L (Fig. 7.6). The calculation is easier if we put the long part on the zeros.

Looking at Alyssa's reflective journal reveals more about the nature of this idea:

> My most powerful discovery occurred when we were working on the pentominoe problem in class. We had just figured out that we could renumber the grid 1, 2, 3, 4, 0, 1, 2, 3, 4, 0. At first I had argued against this. It didn't seem to be that different from what we were doing and it was just one more thing that we had to explain to Peter [the instructor]. Anyway, we had decided that we would try using this grid because it kept the numbers small. We were putting the different shapes on the zero line and then racing to see how quickly we could figure out what the remainder was. Someone put the L shape down and suddenly I saw that were doing it wrong. We didn't have to start on the zero line. We could start anywhere, and the best thing would be if the zero line knocked out the most number of squares …

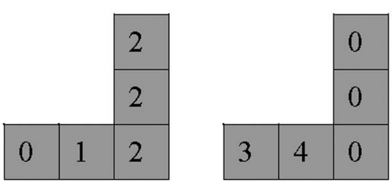


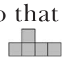
FIG. 7.6. Leanne's example of the backwards "L."

Negative Remainders. The next discovery comes in the realization that a remainder of 4 is, in fact, the same as a number being one short of a multiple of 5, or a remainder of –1. Likewise a remainder of 3 is the same as a remainder of –2. This produces yet another renumbering of the grid as seen in Fig. 7.7.

		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			
		-2	-1	0	1	2			

FIG. 7.7. Renumbering of the grid with negative remainders.

For the students, this renumbering speeds up the calculation of remainders even more because it allows them to "cancel out" numbers as Joanne explains:

> So, now I can place the pentominoes so that some of the squares cancel each other out. For example, I can place  with the longest column on the zero (Fig. 7.8). This way the 1 and the –1 cancel out leaving just the 2.

Looking to Diane, one of Joanne's group mates, shows where this idea came from:

> We were doing the adding up the remainders thing when I put the backwards L down on the grid. Everyone right away said that the remainder was 4, but I wanted to say it was one short. Then I had my AHA! Remainder of 4 is the same as remainder –1 and remainder 3 was the same as remainder –2. I knew right away that this was right, and I knew it was important. I explained this to my partners and we decided that we should try to renumber the grid *again*. It felt great. I was finally contributing to the group.

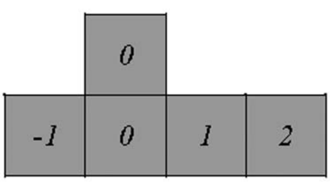


FIG. 7.8. Joanne's example of "canceling out."

It also allows them to very easily explain their earlier observations regarding symmetry, balance, reflection complements, and rotation complements presented in an earlier section. This is seen in Angela's comments:

> We noticed in the remainder zero group that there are always the same number of squares on the left as on the right. We now see that these cancel each other out because the ones on the right have a positive remainder and the ones on the left have a negative remainder.

Dynamic Summing of Remainder. The final, and perhaps ultimate discovery for the students occurs when they liberate themselves from the grid altogether and develop what I have come to call a "dynamic perception" of the pentominoes. Up until this stage, the students have worked with the pentominoes as static objects and they have manipulated everything else. They have played with the pentominoes position and orientation, and they have played with the contents of the 100s chart. In this final stage of the chain of discoveries, the students begin to manipulate the pentominoes themselves. This is well demonstrated in Colleen's words:

> Gary suggested that we start with the I, which we know has a remainder of zero. Then we give each step away from our starting position a value of 1 or –1 depending on which direction it moves. So, to get the remainder for the stairway we move one block to the left (–1) and two blocks to the right (+2). So, the remainder will be +1 (Fig. 7.9).

The connection of this very nonnumber-based explanation to the concept of division with remainder comes from the fact that every horizontal move of a single cell will change the dividend by only one, thereby changing the remainder by only one (positive or negative). At the same time, every vertical move of a block will change the dividend by a total of 10 and hence have no effect on the remainder. So, in essence the remainder can be calculated by simply tracking the horizontal moves.

This reshaping of the "I" into the various pentominoe shapes is an explanation for something that occurs in a much more innate manner. The constant exposure to the various shapes and the prolonged effort to solve the problem leads the participants to sponta-

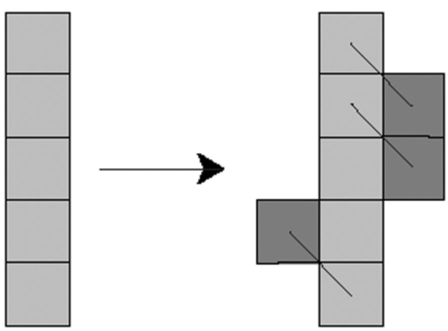
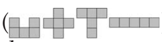
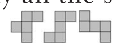


FIG. 7.9. Colleen's example of dynamic summing.

neously leave the workspace of the grid behind. An example of this can be seen in Sharon's journal:

> Laura suggested that we use the blocks to work on the pentominoes. She started making the shapes of the pentominoes with the cubes and she would move one block to make different pentominoe. As Laura moved the blocks Jen and I started calling out the remainders. At first we were just calling them out of memory, yet as we did this and Laura moved blocks we suddenly started to see a pattern or a system.

This discovery, like the previous discovery of moving the axis allows the students to very quickly explain the properties they discovered in the early parts of their exploration. This is nicely demonstrated by Elain's explanation:

> We are always counting how many moves we make to the left and to the right. So, if we make the same number of moves to the left as to right then the remainder will be zero. This explains why all the symmetric shapes () and all the balanced shapes () have no remainder.

DISCUSSION

The chain of discovery presented earlier can be summarized in the diagram presented in Fig. 7.10. It should be noted that although many of the participants traverse this chain exactly as presented here, not all do. Some of the participants bypass the discoveries represented in cells 2 and 4 as shown with the arrows. It should also be noted, however, that all participants make the discoveries summarized in cells 1, 3, and 5, and furthermore, they make the discoveries in this order.

The initial realization that position doesn't matter is more of an immediate observation than a discovery and hence is represented differently than any of the other discoveries in the diagram. Nonetheless, this realization leads to some number theory-related concepts regarding the contribution of the 10s part of a number to the remainder.

The first real discovery that is made pertains to the patterns that are observed with respect to the remainders of reflected and rotated pentominoes. Although this discovery remains unexplained until much later in the solution process, the property of complements is a result of a relative consideration of contribution to the remainder. That is, although all numbers contribute to the dividend, some numbers increase the remainder and some numbers decrease the remainder.

The next two discoveries that are made have to do with the realization that it is advantageous to consider the contribution that each number makes to the remainder instead of the contribution that each number

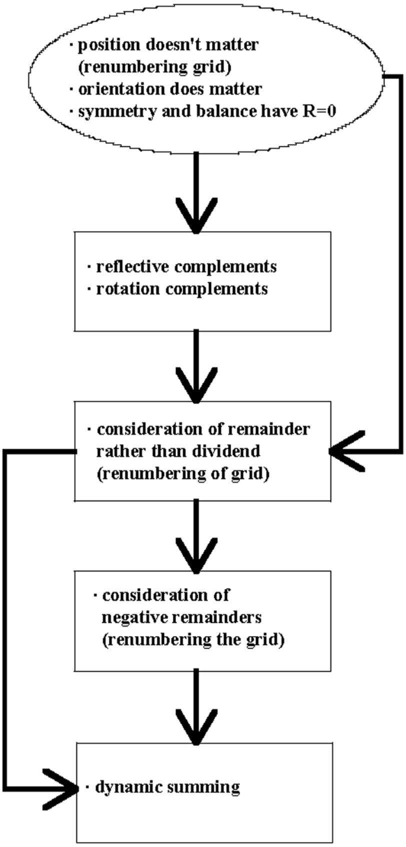


FIG. 7.10. Summary of the pantominoe chain of discovery.

makes to the dividend. This comes from the fact that the remainder of a sum will be the same as the sum of the remainders. The initial discovery brings to light this idea in the context of only positive contributions that a number can make to the remainder and is represented in the third box of Fig. 7.10. The secondary discovery of this concept comes when it is realized that contributions can be both positive and negative in nature and is represented in the fourth box of the diagram.

Finally, the last box represents the discovery that all discussions of remainders can be liberated from the 100s chart all together. This then leads to the dynamic construction of pentominoe shapes and tracking of the effect that each move of a block has on the final remainder.

The chain of discoveries detailed in the previous sections and summarized here is more than just a set of understandings that are achieved. Each new understanding is punctuated with a discovery. John sees the pattern that exists between rotated pentominoe shapes “jump out” at him when he suddenly saw how the property of reflection also applied to rotation. Jessica was “suddenly struck” by the realization that individual numbers contributed to the eventual remainder and thus warranted the renumbering of the grid to reflect these contributions. Alyssa “suddenly saw that [her group] was doing it wrong” and that the pentominoes could be placed more strategically for the calculation of remainders. Dianne had an AHA! and “knew right away” that remainders could be seen as being both positive and negative. Finally, Sharon “suddenly started to see a pattern” that the dynamic construction of pentominoes was producing. For the participants, the concepts that are presented to them at each link in the chain of discovery arrive in a flash of illumination on the heels of much deliberate effort and periods of incubation. These moments of discovery also bring with them all of the positive emotions that normally accompany such experiences and serve to motivate the participants to keep going. This state of constant motivation is nicely captured in Lisa’s reflective journal:

> Of all the problems that we worked on my favourite was definitely the pentominoe problem. We worked so hard on it, and it took forever to get the final answer. But I never felt like giving up, I always had confidence that we would get through it. Every time we got stuck we would just keep at it and suddenly one of us would make a discovery and we would be off to the races again. That’s how it was the whole time—get stuck, work hard, make a discovery—over and over again. It was great. I actually began to look forward to our group sessions working on the problem. I have never felt this way about mathematics before—NEVER! I now feel like this is ok, I’m ok, I’ll BE ok. I can do mathematics, and I definitely want my students to feel this way when I teach mathematics …

This journal entry also highlights the overall enjoyment that the participants felt regarding the problem as a whole. In fact, in surveying the various classes of students who were exposed to this problem between 75% and 90% of the students identified this problem as their favorite. Furthermore, between 85% and 95% of the students identified it as the most memorable problem they were exposed to, and 100% of the students identified it as a problem within which they experienced moments of discovery.

CONCLUSION

Research has shown that preservice teachers’ understanding of division with remainder is rooted in their procedural understanding of the long division algorithm (Campbell, 2002; Zazkis, 1999, 2000; Zazkis & Campbell, 1996a, 1996b). In particular, preservice teachers are often unable to determine the remainder without explicitly performing long division, even in cases where the remainder can be discerned from the nature of the dividend. In the study presented in this chapter, similar difficulties also presented themselves. However, these difficulties were overcome as the participants moved through a chain of discoveries occasioned by their work on the pentominoe problem. As they moved through the chain of discoveries, their understanding of remainder evolved, beginning with an understanding of remainder as one of the results of long division and ending with the understanding of how changes to the dividend contribute to changes in the remainder.

However, a chain of discovery is much more than just a cognitive experience. It is also an affective experience capable of changing even very robust negative beliefs and attitudes within a population of apprehensive and mathematically timid preservice elementary school teachers. The particular chain of discovery presented in this chapter is no different. Not only was the cognitive domain of the participants changed with respect to division with remainder but so too was the affective domain. The pentominoe problem engaged the participants, and it sustained that engagement through the changes in the attitudes and beliefs that it fostered.

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