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4. IN THE WORDS OF THE CREATORS

In 1943 Jacques Hadamard gave a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City. These talks were subsequently *published as The Psychology of Mathematical Invention in the Mathematical Field* (Hadamard,1945). In this chapter I present a study that mirrors the work of *Hadamard. Results both confirm and extend the work of Hadamard on the inventive process, mathematical creativity, and the phenomenon of the AHA! experience. In addition, the results also speak to the larger context of 'doing' and learning mathematics.*

INTRODUCTION

What is the genesis of mathematics? What mechanisms govern the act of mathematical creativity? This "is a problem which should intensely interest the psychologist. It is the activity in which the human mind seems to take the least from the outside world, in which it acts or seems to act only of itself and on itself" (Poincaré, 1952, p. 46). It should also intensely interest the mathematics educator, for it is through mathematical creativity that we see the essence of what it means to 'do' and learn mathematics. In this chapter I explore the topic of mathematical creativity in general and the AHA! experience in particular. I begin with a brief synopsis of the history of work in the area of mathematical creativity. This is then followed by the results of a study designed to elicit from prominent mathematicians ideas and thoughts on their own encounters with mathematical creativity and the AHA! experience.

HISTORICAL BACKGROUND

In 1908 Henri Poincaré (1854–1912) gave a presentation to the French Psychological Society in Paris entitled 'Mathematical Creation'. This presentation, as well as the essay it spawned, stands to this day as one of the most insightful, and thorough treatments of the topic of mathematical creativity and invention. Inspired by this presentation, Jacques Hadamard (1865-1963) began his own empirical investigation into this fascinating phenomenon. The results of this seminal work culminated in a series of lectures on mathematical invention at the École Libre des Hautes Etudes in New York City in 1943. These talks were subsequently published

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as The Psychology of Mathematical Invention in the Mathematical Field (Hadamard,1945).

Hadamard's treatment of the subject of invention at the crossroads of mathematics and psychology is an extensive exploration and extended argument for the existence of unconscious mental processes. To summarize, Hadamard took the ideas that Poincaré had posed and, borrowing a conceptual framework for the characterization of the creative process in general, turned them into a stage theory. This theory still stands as the most viable and reasonable description of the process of mathematical invention. In what follows I present this theory, referenced not only to Hadamard and Poincaré, but also to some of the many researchers whose work has informed and verified different aspects of the theory.

MATHEMATICAL INVENTION

The phenomenon of mathematical invention, although marked by sudden illumination, consists of four separate stages stretched out over time, of which illumination is but one part. These stages are initiation, incubation, illumination, and verification (Hadamard, 1945). The first of these stages, the initiation phase, consists of deliberate and conscious work. This would constitute a person's voluntary, and seemingly fruitless, engagement with a problem and be characterized by an attempt to solve the problem by trolling through a repertoire of past experiences (Bruner, 1964). This is an important part of the inventive process because it creates the tension of unresolved effort that sets up the conditions necessary for the ensuing emotional release at the moment of illumination (Davis & Hersh, 1980; Feynman, 1999; Hadamard, 1945; Poincaré, 1952; Rota, 1997). Following the initiation stage the solver, unable to come to a solution stops working on the problem at a conscious level (Dewey, 1933) and begins to work on it at an unconscious level (Hadamard, 1945; Poincaré, 1952). This is referred to as the incubation stage of the inventive process and it is inextricably linked to the conscious and intentional effort that precedes it. After the period of incubation a rapid coming to mind of a solution, referred to as illumination, may occur. This is accompanied by a feeling of certainty (Poincaré, 1952) and positive emotions (Barnes, 2000; Burton 1999; Rota, 1997). Illumination is the manifestation of a bridging that occurs between the unconscious mind and the conscious mind (Poincaré, 1952), a coming to (conscious) mind of an idea or solution. The experience that surrounds this rapid coming to mind of a solution is referred to as the AHA! experience. What brings the idea forward to consciousness is unclear, however. There are theories of the aesthetic qualities of the idea (Sinclair, 2002; Poincaré, 1952), effective surprise/shock of recognition (Bruner, 1964), fluency of processing (Whittlesea and Williams, 2001), or breaking functional fixedness (Ashcraft, 1989). Regardless of the impetus, the correctness of the emergent idea is evaluated during the fourth and final stage – verification.

HADAMARD RESURECTED

For the study presented here a portion of Hadamard's original questionnaire was used to elicit from contemporary mathematicians ideas and thoughts on their own encounters with the phenomenon of mathematical discovery. Hadamard's original questionnaire contained 33 questions1 pertaining to everything from personal habits, to family history, to meteorological conditions during times of work (Hadamard, 1945). From this extensive and exhaustive list of questions the five that most directly related to the phenomena of mathematical discovery and creation were selected. They are:

> Would you say that your principle discoveries have been the result of deliberate endeavour in a definite direction, or have they arisen, so to speak, spontaneously? Have you a specific anecdote of a moment of insight/ inspiration/illumination that would demonstrate this? [Hadamard # 9]
>
> How much of mathematical creation do you attribute to chance, insight, inspiration, or illumination? Have you come to rely on this in any way? [Hadamard# 7]
>
> Could you comment on the differences in the manner in which you work when you are trying to assimilate the results of others (learning mathematics) as compared to when you are indulging in personal research (creating mathematics)? [Hadamard # 4]
>
> Have your methods of learning and creating mathematics changed since you were a student? How so? [Hadamard # 16]
>
> Among your greatest works have you ever attempted to discern the origin of the ideas that lead you to your discoveries? Could you comment on the creative processes that lead you to your discoveries? [Hadamard # 6]

These questions, along with a covering letter, were then sent to 150 prominent mathematicians in the form of an email.

Hadamard set excellence in the field of mathematics as a criterion for participation in his study. In keeping with Hadamard's standards, excellence in the field of mathematics was also chosen as the primary criterion for participation in this study. As such, recipients of the survey were selected based on their achievements in their field as recognized by their being honoured with prestigious prizes or membership in noteworthy societies. In particular, the 150 recipients were chosen from the published lists of the Fields Medalists, the Nevanlinna Prize winners, as well as the membership list of the American Society of Arts & Sciences. The 25 recipients, who responded to the survey, in whole or in part, have come to be referred to as the 'participants' in this study. Of these 25 participants all but one agreed to allow their name to appear alongside their comments.

After these participants supplied their responses to the aforementioned five questions they were sent a further two questions, again in the form of an email. These additional questions were designed to specifically focus on the phenomenon of the AHA! experience2. They are:

> How do you know that you have had an AHA! experience? That is, what qualities and elements about the experience set it apart from other experiences?
>
> What qualities and elements of the AHA! experience serve to regulate the intensity of the experience? This is assuming that you have had more than one such experience and they have been of different intensities.

The responses were initially sorted according to the survey question they were most closely addressing[3]. In addition, a second sorting of the data was done according to trends that emerged in the participants' responses, regardless of which question they were in response to. This was a much more intensive and involved sorting of the data in an iterative process of identifying themes, coding for themes, identifying more themes, recoding for the new themes, and so on. In the end, there were 11 themes that emerged from the data, each of which can be attributed to one of the four stages presented in the previous section. Some of these themes serve to confirm the work of Hadamard while others serve to extend it. Some of these themes such as 'sleep' and 'persistent work' are obvious and self-explanatory within the context of the data itself. Other themes such as 'significance', however, are more subtle and require a heightened level of analysis and explanation to elucidate this subtlety. As such, the presentation of the various themes is accompanied by varying degrees of discussion and analysis as necessary to bring them into sharper focus.

Sleep

The first theme has to do with sleep and the role it plays in doing mathematics. Many of the participants commented at one time or another about the phenomenon of waking up to a solution, either in part or even fully formed. Although not part of the survey used for this study Hadamard found this phenomenon sufficiently interesting to include it as part of his survey:

> Have you ever worked in your sleep or have you found in dreams the answers to problems? Or, when you waken in the morning, do solutions which you had vainly sought the night before, or even days before, or quite unexpected discoveries, present themselves ready-made to your mind? (Hadamard, 1945, p. 138)

As such, it was not altogether surprising to see comments of this nature cropping up in the responses to the survey.

> While at a meeting in Philadelphia, I woke up one morning with the right idea. (Dick Askey)
>
> I'm convinced that I do my best work while asleep. The evidence for this is that I often wake up with the solution to a problem, or at least with a clear idea of how to proceed to solve it. (Charles Peskin)

> One of the most intense such experience I had actually turned out to be nonsense. It occurred in a dream in which I really thought that I got insight into a really hard problem. When I got up, I rushed to my desk to think and after an hour realized that it was all gibberish. But it was quite intense. I find, in general, ideas that come to you "in the shower" are more reliable than those that come to me in my sleep (which does not happen very often). (Jerry Marsden)

While Askey and Peskin discuss the fruitfulness of the process of solving mathematical problems in their sleep Marsden comments that awakening with an answer, although very tempting, is often quite misleading. Christodoulou presents a slightly different role of sleep.

> I do not attribute much to chance; I would attribute all to insight and illumination. In regard to illumination, I would like to add that in my case the best instances have been at night when I am lying in bed, somewhere between consciousness and sleep. It is during these times that I have the greatest power of concentration when all else except my subject lose reality. (Demetrios Christodoulou)

The suspension of conscious thought in that state between being asleep and being awake is, for him, a fertile venue for new ideas to form.

In general, discussions of creativity and sleep are reflective of Hadamard's discussion of unconscious work. Sleep, and the period just before sleep, are times during the day where the unconscious mind is unhampered by the distractions of conscious thought. It seems logical, therefore, that such times would produce more such instances of insight. However, Marsden's warning should be heeded, unconscious (or sleep) work does not mean good work.

Metacognition

Metacognition is the awareness and regulation of one's own thinking processes. In some cases this manifests itself as thinking strategies.

> I try to build/find structure and cohesion in which I am looking at. I think math is a language; one sees things with some internal eye and needs to find a language to express this. (Dusa McDuff)

> Rather, I stop and ask myself: What did I really find? What is next? Sometimes this is the first step for real progress. (Enrico Bombieri)

> A shorter analog: after a partial step, wondering: what does this argument really mean. (Pierre Rene Deligne)

Each of these mathematicians – McDuff, Bombieri, and Deligne – uses their respective strategies in solving problems. McDuff looks for structures and cohesion of these structures. This is indicative of visual thinking and is consistent with her later comment that her "imagination is quite visual (though not as much as some others I know of). Other mathematicians have a feel for algebraic structure, actual equations turn them on – not me...” McDuff thinks with pictures. Bombieri and Deligne’s strategies are very different from that of McDuff, but very similar to each other. They use their strategy to stop their progress and question what they have found, or what they are constructing. Schoenfeld (1985) refers to this as ‘control’. What they all have in common, however, is that they are all aware of how they think, and they use it.

Other mathematicians’ reflections are more of a commentary of their thinking process than a description of a thinking tool.

> That is, unless you are aware of a problem and have given it some thought, it is very unlikely that you will solve it: and if you did, how would you know you did, if you were really unaware of the problem. (Dan J. Kleitman)

> My passion for the history of ideas is boundless and I go to endless lengths, not to hide the influences from which I benefited, but to understand and express them thoroughly. To me, the value of a thought combines its novelty and difficulty with the depth of its roots. The greatest thrill is to add to streams of ideas that already have a long and recognizable past. (Benoit B. Mandelbrot)

> I have become more suspicious than I used to be about the originality of my ideas. (Connor4)

Kleitman philosophises about the role of conscious work in the problem solving while both Connor and Mandelbrot reflect on the origin of their ideas.

Mathematical Landscape

The participants often turned to very rich descriptors and metaphors in their comments. In several cases these descriptions refer to the “mathematical landscape.

> My approach to research consists in looking to the mathematical landscape, taking notice of the things I like and judge interesting and of those I don’t care about, and then trying to imagine what should be next. If you see a bridge across a river, you try to imagine what lies on the other shore. If you see a mountain pass between two high mountains, you try to imagine what is in the valley you don’t see yet but secretly know must be there. (Enrico Bombieri)

Bombieri not only uses the metaphor of building a bridge as but he also imagines what lies on the other side. For Bombieri this is more than just a metaphor, it is also a thinking tool that he uses in research. He continues with the metaphor in later comments where he explains his Platonic view of mathematics.

> My attitude towards mathematics is that most of it is lying out there, sometimes in hidden places, like gems encased in a rock. You don’t see them on the surface, but you sense that they must be there and you try to imagine where they are hidden. Suddenly, they gleam brightly in your face and you

> don't know how you stumbled upon them. Maybe they always were in plain view, and we all are blind from time to time. (Enrico Bombieri)

Connor also mentions the landscape, but he does so from a descriptive perspective, where he views it rather than traverse it like Bombieri did.

> However, all during that process, 'insights' do appear seemingly spontaneously. However, this seems to me to be akin to artists looking at a landscape and being amazed by how interesting this or that view is – an amazement that ignores the fact that their artistic pattern recognition will only make them aware of certain views, namely those that are striking. (Connor)

Burton (1999), in her work with mathematicians, also found that the mathematical landscape was a prevalent metaphor, both descriptively and prescriptively.

Although the mathematical landscape was the most common metaphor in the mathematicians' responses, it wasn't the only one. Art provided inspiration for the formation of metaphors for both Brad Efron and Enrico Bombieri.

> At first I'm terribly confused, but after awhile I chip away at my wrong ideas until I'm left with an answer. So I think I'm working in the sculptor mode, rather than the inspired painter. (Brad Efron)

> I would say that my attitude towards mathematics is more that of a problem solver than of a builder of theories. I can paraphrase this by saying that I am not an architect or urban planner, rather more of a painter working small paintings depicting what the inspiration leads him. (Enrico Bombieri)

Gaps

Along with a mathematical landscape comes gaps in knowledge; places in the terrain that have not yet been explored.

> Certain gaps in knowledge needed to be filled and my main role was to feel that these gaps could be filled. (Dick Askey)

However, this is not the only context where mention of gaps appeared in the mathematicians' responses. There were also several comments pertaining to gaps or holes in thinking, reasoning, and arguments.

> In my discoveries the general direction or topic has been planned, of course. However after that one has to search the terrain until one finds an opening (or gives up), and where that is cannot be planned. (Gerd Faltings)

> I go round arguments time and again seeking for a hole in my reasoning, or for some way to formulate the problem/structures I see. (Dusa McDuff)

> You look in your mind for something and come back with something else which may, with enough looking and some luck, be just what you need to fill the gap in your argument. (Dan J. Kleitman)

The existence of these gaps is indicative of a related theme that emerged, the theme of failure and wrong ideas.

Failures and Wrong Ideas

Although none of the questions posed to the mathematicians dealt explicitly with the notion of incorrect or fruitless ideas many of the mathematicians chose to comment them anyways.

> Also I should mention that these experiences are not so uncommon, but many of them do not last long because often the new insight later turns out to be false. (Gerd Faltings)

> My usual mode is to jump in and compute (I cannot really think without a pen in hand). Then having computed fast and probably wrong, I find that this particular calculation would not have done what I wanted anyhow, so I throw it out and start over. Sometimes I simply repeat what are at bottom the same stupidities for weeks, and though this looks useless on the face of it, I get familiar with the question and learn a few tricks. Of course I know already what I want to come out, mostly by analogy with old things of my own or others, and I'm looking for the mathematical mechanism that makes it work … [the intensity of the AHA! experience] depends how long one has worked, how many silly mistakes one made. (Henry McKean)

> And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, (many of these ideas turn out not to work very well) or even when you are driving. You must try and fail by deliberate efforts, and then rely on a sudden inspiration or intuition or if you prefer to call it luck. (Dan J. Kleitman)

> But I should say that my subconscious usually would present a novel way of attack; if it presented a ready-made "solution", it often was quite wrong. (Peter J. Huber)

> Also, one must expect to consider many ideas which turn out later to be failures. (Wendell Fleming)

> At first I'm terribly confused, but after awhile I chip away at my wrong ideas until I'm left with an answer. (Brad Efron)

A qualitatively different type of mistake is the one pointed out by Connor in which he discusses the false assumption that one's ideas are original.

> I imagine that a mathematician's brain is similarly engaged in a ceaseless search for striking patterns in the ever-changing stream of ideas, making the mathematician aware of only the patterns found striking by the mathematician's pattern recognition process. I am certainly aware of the very moment of such `insight' on more than one occasion. In fact, this moment can be so striking that it has led people (me included, I am sorry to say) to believe they are the

> first to have a certain idea even when it can be established (afterwards) that they heard or read that idea earlier; the moment of 'seeing' it, i.e., of recognizing the pattern, is much more powerful than the moment of hearing or reading some fact to which one has, at that point, no immediate connection. (Connor)

This is an interesting and important statement for two quite different reasons. The first is that it highlights the impact that the sudden appearance of an idea can have on otherwise rational thought. It is perhaps for just such a reason that Deligne and Bombieri have developed a thinking strategy that gets them to question exactly what it is they have found (see section on Metacognition). The second reason this statement is of interest is because of Connor's explicit mention of the fact that "the moment of 'seeing' it, i.e., of recognizing the pattern, is much more powerful than the moment of hearing or reading some fact". Although it is not certain what the 'it' that he is referring to, such a statement could speak both to the cognitive and the affective aspect of the AHA! experience. The cognitive dimension refers to the depth of the understanding that is gained and the affective dimension speaks to the emotions that are felt at 'seeing' it.

Persistent Work

Persistence is a crucial element in creative work. Persistence is what describes the initiation phase of Hadamard's four stage process. One must persist through this period of unresolved effort in order to move onto the incubation and eventual illumination phases. This contribution of persistence was articulated in one way or another by many of the participants.

> I don't believe that any true progress arises spontaneously. I believe it is always the result of lots of hard work, covert or overt, with the understanding that old work will sometimes come into a new focus so that you get something, if not for free, then at no extra cost. Such ''inspiration'' is the outcome of covert work and so can be surprising, but the work has to have been done, even if invisibly. (Henry McKean)

> Obviously you work like hell and once in while you notice something really unexpected. (David L. Donoho)

> Namely one has to spend much time on the subject before one gets inspiration. Gerd Faltings)

> I found that in order to do creative work, I had to be at it without interruption for at least a week at a time. (Peter J. Huber)

> In my principle discoveries I have always been thinking hard trying to understand some particular problem. Often it is just a hard slog, I go round arguments time and again seeking for a hole in my reasoning, or for some way to formulate the problem/structures I see. Gradually some insights builds and I get to "know" how things function. (Dusa McDuff)

The persistent effort can also occur after the initial inspiration as one works to verify and extend the idea. This was especially evident in the anecdote provided by Bombieri where he highlights both the perseverance as well as the obsessive intensity with which he applies it. Huber and Efron also offer very concise comments on the topic.

> I worked three days and three nights never taking a rest save for eating a little and drinking coffee. (Enrico Bombieri)

> Inspiration starts things, but only hard work really gets anywhere. (Brad Efron)

> When I had a successful idea, I could not let loose and worked furiously. (Peter J. Huber)

Persistence is also an important element of ‘doing’ mathematics in general as attested to by the following three mathematicians.

> The main reason for occasional success is perseverance; never give up on a problem, continue day after day, week after week... also when it looks hopeless. (Ulf Grenander)

> First, you can only do something worthwhile by devoting an embarrassingly extreme amount of time preparing yourself both within a specialty and by reading voraciously and very broadly outside the specialty as well. If they only knew the amount of dedicated concentrated effort involved, most people would be shocked and repelled at the sacrifice involved. (Of course, a few people will have exactly the kind of obsessive personality that drives them to this kind of effort, the rest would find it an unimaginable deprivation). I think this is true even of the greatest mathematicians, and I’ll bet it is true of greatness in many other fields as well. [..] In olden times, where there was a heavier emphasis on gruelling hard work in school, the tendency to “see flashes” was probably very muted. So that the concept of ‘genius’ that was so revered previously is, I think, simply people being jealous of what they do not have. I would not valorize something simply because there was a “flash”. (David L. Donoho)

> It requires persistence and ability of high orders which are rare separately. (Dick Askey)

Deriving and Re-Creation

A theme that emerged out of questions three and four, but also resonated with comments from responses to other questions, was the importance of deriving mathematics for oneself. In part, this theme reflected the personal practices of assimilating the work of others through active re-creation of the mathematics rather than passive reading of it.

> I find it very hard to learn the results of others these days unless they are very close to my own research interests. I used to be able to absorb things rather passively, just reading and doing over the chains of ideas. Now I need to work out examples, specific instances of the new ideas to feel that I have any real understanding. Anything one creates oneself is much more immediate and real and so harder to forget. (Dusa McDuff)

> Learning work of others of course means following their thoughts, as opposed to thinking oneself. However I usually try whether I can find my own proofs for the assertions before I dig into their details. (Gerd Faltings)

> When there is a need to fully assimilate something, I must redo everything in my own way. (Benoit B. Mandelbrot)

> I've forgotten what Hadamard had to say on this, but for me there's no difference, – in order to `understand', I have to (re)create. To be sure, it's much easier to follow someone else's footsteps, i.e., it is much easier to prove a result one knows to be true than one that one merely guesses to be true. (Connor)

> I have always studied results discovered by others, but frequently reworking the material in my own way. [..] I am impatient when reading work by others and usually try to work things out myself. (Dick Askey)

However, the theme also manifested itself as advice to young mathematicians about how to best approach doing mathematics in the expansive responses of both Kleitman and Huber.

> You cannot tell when a successful idea comes to you whether it is luck that it did. On the other hand you can position yourself to be lucky by thinking hard about the problem, and by practicing. I do think that if someone wants to do this they should try to train their minds by exercising this skill on problems whose answer is known. That is, they should try to figure them out themselves. I try to train students to read a paper by first reading enough to find out what the author is trying to accomplish, and then put the paper down and try to think out an approach of your own to accomplishing it. If a student succeeds immediately in seeing what to do, the paper could not have been very good. More likely, he or she will fail; after thinking for a certain amount of time, one should go back to the paper and find a clue from it, and try again with this clue. This process can be iterated until the student can solve the problem. When I started on thesis research as a graduate student, my advisor gave me and a number of other prospective advisees a practice problem. Now it so happened that several weeks before I had heard a lecture about this very problem and read a paper in which it was solved. In giving the advisor my solution to this problem I of course made extensive reference to that paper. His reaction was: why did I want to look at a reference rather than trying to do the problem myself? I was too embarrassed to explain that I read the paper

> first; but I took his comment to heart and from then on have attempted to learn as much as I can myself, at least in part. (Dan J. Kleitman)

> If you have an idea, develop it on your own for, say, two months, and only then check whether the results are known. The reasons are: (1) If you try to check earlier, you won't recognize your idea in the disguise under which it appears in the literature. (2) If you read the literature too carefully beforehand, you will be diverted into the train of thought of the other author and stop exactly where he ran into an obstacle. This happened to a friend of mine, who started three Ph.D. theses in totally unrelated fields, before he finished one. (Peter J. Huber)

This result speaks in support of such methods of instruction as discovery learning (Bruner, 1961; Dewey, 1916), constructivism (Steffe, Cobb & von Glasersfeld, 1988), teaching through problem solving (Cobb, Wood & Yackel, 1991), teaching through problem posing (Boaler, 1997; Brown & Walter, 1983), and the Moore Method of instruction (Jones, 1977).

The Role of Significance

Several of the mathematicians mentioned significance as one of the characteristics that, both sets the AHA! experience apart from other mathematical experiences, and regulates the intensity of the experience. However, their usage of the term is problematic. Consider the following three excerpts:

> It is, in my experience, just like other AHA! experiences where you suddenly "see the light". It is perhaps a little more profound in that you see that this is "important". I find that as one gets older, you learn to recognize these events more easily. When younger, you often don't realize the significance of such an event at the time. (Jerry Marsden)

> What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place. (Wendell Fleming)

> The depth of the experience depends on how profound the ultimate result is. (Michael Atiyah)

Each of these three comments uses significance (or important, or profound) in a post-evaluative sense. That is, they speak of how significant a new idea will turn out to be once verified. However, such a thing cannot be known in the instance of illumination. It is only through verification, a potentially lengthy process, that this can be truly ascertained. If age has an effect on this, as Marsden claims, it would only serve to shorten the process of verification, not eliminate it as would be necessary for the significance to truly be known in the instance of illumination. This is particularly pertinent to Fleming's comment that the AHA! occurs "when something mathematically significant falls into place".

However, such statements cannot be ignored. In order for the mathematicians to be associating the AHA! experience with the significance of the idea that reveals itself at the time of illumination then one of three things must be occurring. First, they are consciously suspending any evaluation of an experience as to whether or not it is an AHA! experience until after all the results have been checked. This may, in fact, be what Atiyah is doing in ascertaining the depth of the experience. Secondly, their recollection of the experience is being influenced by the outcome of the eventual evaluation of the idea that was presented to them during their AHA! experience. In psychology this is referred to as memory reconstruction (Whittlesea, 1993) and has a large amount of empirical data to support it. The third option, and the one that I find most likely, is that although the absolute significance of a find may ultimately be verified, at the time of illumination what they are, in fact, experiencing is a sense of significance. This sense of significance is not too dissimilar from the sense of certainty that they also experience, and like certainty may in the end prove to be unfounded. This possibility is displayed in Fleming's complete passage.

> What I think you mean by an AHA! experience comes at the moment when something mathematically significant falls into place. This is a moment of excitement and joy, but also apprehension until the new idea is checked out to verify that all the necessary details of the argument are indeed correct. (Wendell Fleming)

Although Fleming begins the passage with an absolute usage of significance, stating that the AHA! occurs when something "mathematically significant falls into place", he softens this stance in his second sentence where he acknowledges that he is filled with a feeling of apprehension until the results are verified. Fleming's statement clearly indicates that regardless of how he uses the term significance, he sees it as temporary and tentative. The same theme is also nicely demonstrated by Huber's response to question six where he makes a distinction between the AHA! experience and the EUREKA! experience.

> First I would have a promising, brilliant(?) idea (the AHA! event) which would induce me to drill. But the eureka event ("I found it!") at best would come hours or days later, if and when the oil would begin to gush forth. That the idea had been brilliant and not merely foolish would be clear only in retrospect, after attempts to verify and confirm it. And later on one tends to suppress and forget foolish ideas because they are embarrassing (but they are indispensable companions to the brilliant ones!). (Peter J. Huber)

The Contribution of Chance

There are two types of chance, intrinsic chance and extrinsic chance. Intrinsic chance deals with the luck of coming up with an answer, of having the right combination of ideas join within your mind to produce a new insight. This was discussed by Hadamard (1945) as well as by a host of others under the name of "the chance hypothesis". Extrinsic chance, on the other hand, deals with the luck associated with a chance reading of an article, a chance encounter, or a some other chance encounter with a piece of mathematical knowledge, any of which contribute to the eventual resolution of the problem that one is working on.

> But chance is a major aspect: what papers one happens to have read, what discussions one happens to have struck up, what ideas one's students are struck by (never mind the very basic chance process of insemination that produced this particular mathematician). (Connor)

> As my anecdote indicates, it is not just chance, but rather inspiration in the presence of lots of surrounding information. The surrounding information is really crucial, I believe. (Jerry Marsden)

> Do I experience feelings of illumination? Rarely, except in connection with chance, whose offerings I treasure. In my wandering life between concrete fields and problems, chance is continually important in two ways. A chance reading or encounter has often brought an awareness of existing mathematical tools that were new to me and allowed me to return to old problems I was previously obliged to leave aside. In other cases, a chance encounter suggested that old tools could have new uses that helped them expand. (Benoit B. Mandelbrot)

> Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together. (Dan J. Kleitman)

The idea that mathematical discovery relies, at least in part, on the fleeting and unpredictable occurrences of chance encounters is starkly contradictory to the image projected by mathematics as a field reliant on logic and deductive reasoning. Extrinsic chance, in particular, is an element that has been largely ignored in the literature.

The De-Emphasis of Details

A particularly strong theme that emerged from this study was the role that detail does NOT play in the learning of mathematics. Many of the participants mentioned how difficult it is to learn mathematics by attending to the details, and how much easier it is if the details are de-emphasized. Solomon Feferman nicely captures this idea in his comment.

> Understanding others is often a painful process until one suddenly goes beyond the details and sees whole what's going on. (Solomon Feferman)

In some cases, this also manifested itself as a strategy for problem solving and research.

> There is not much difference. More precisely, I seldom study or learn mathematics in detail. (Benoit B. Mandelbrot)

> Get the basics of the problem firmly and thoroughly into the head. After that, an hour or two each day of thinking on it is all that's needed for progress. [..] For that reason, I started some 20 years ago to ask students (and colleagues) wanting to tell me some piece of mathematics to tell me directly, perhaps with some gestures, but certainly without the aid of a blackboard. While that can be challenging, it will, if successful, put the problem more firmly and cleanly into the head, hence increases the chances for understanding. I am also now more aware of the fact that explaining problem and progress to someone else is beneficial; I am guessing that it forces one to have the problem more clearly and cleanly in one's head. (Connor)

In presenting his strategy for getting "the basics of the problem firmly and thoroughly into" his head, Connor has come up with a strategy that de-emphasizes the details by forcing the transmission of the problem through a medium wherein details are impossible. In his comment, he also introduces a subtly different role for talking that I alluded to in the next section. More than simply moving information around, talking de-emphasizes details and, as a result, will "put the problem more firmly and cleanly into the head, hence increases the chances for understanding." This has pedagogic implications that are not so obvious.

The Role of Talking

Further to this previous theme, it is also clear from the mathematicians' responses that, while working in the initiation phase, they have a much higher regard for transmission of mathematical knowledge through talking than through reading. This is best summarized in the comments of Marsden and Papanicolaou.

> I assimilate the work of others best through personal contact and being able to question them directly. [..] In this question and answer mode, I often get good ideas too. In this sense, the two modes are almost indistinguishable. (Jerry Marsden)

> I get most of my real mathematical input live, from (good) lectures or one-on-one discussions. I think most mathematicians do. I look at papers only after I have had some overall idea of a problem and then I do not look at details. (George Papanicolaou)

Considering these last two themes (de-emphasizing of details and the role of talking) it becomes clear that the painstakingly rigorous fashion in which mathematical knowledge is written, both in journals and in text-books, as well as the detailed fashion of over-engineered curriculums stand in stark contrast to the methods by which mathematicians claim they best come to learn new mathematics.

Context of AHA! Experiences

Moments of illumination and insight are purported in the literature to occasionally occur during times of non-mathematical activities. Anecdotes spread throughout this chapter support this. Instances of illumination while showering, walking, sleeping, talking, cooking, driving, eating, waking, and riding the subway are reported among the mathematicians surveyed. To accentuate this I gather together these anecdotes here.

> This can occur while I am officially working. But it can also occur while I am doing something else, having a shower, doing the cooking. I remember that the first time I felt creative in math was when I was a student (undergrad) trying to find an example to illustrate some type of behavior. I'd worked on it all the previous evening with no luck. The answer came in a flash, unexpectedly, while I was showering the next morning. I saw a picture of the solution, right there, waiting to be described. (Dusa McDuff)

> While at a meeting in Philadelphia, I woke up one morning with the right idea. (Dick Askey)

> And relevant ideas do pop up in your mind when you are taking a shower, and can pop up as well even when you are sleeping, (many of these ideas turn out not to work very well) or even when you are driving. (Dan J. Kleitman)

> In regard to illumination, I would like to add that in my case the best instances have been at night when I am laying in bed, somewhere between consciousness and sleep. (Demetrios Christodoulou)

> I distinctly remember the moment early in our collaboration when I saw how to get past one of the major technical difficulties. This happened while walking across campus after teaching a class. (Wendell Fleming)

> It may have been in the shower that it just occurred to me that the work of some of the classical authors could be generalized in a certain way [..] I can be talking to a colleague or my wife or eating breakfast and suddenly, like a voice from the blue, I get told what to do. (Jerry Marsden)

> The overt work is much the same as it always was. The covert work (in bed, on the subway, in dreams) is harder now. (Henry McKean)

> I'm convinced that I do my best work while asleep. The evidence for this is that I often wake up with the solution to a problem, or at least with a clear idea of how to proceed to solve it. (Charles Peskin)

These contextualized anecdotes add richness to the descriptions of the AHA! experience. Together they convey an understanding that the AHA! may happen, not just in the absence of mathematical work, but in the context of very different activities and, as such, its occurrence is not only unexpected but also untimely. This untimeliness may act to heighten not only the sense of suddenness, but also the senses of certainty and significance.

CONCLUSIONS

When Hadamard embarked on the research that eventually led to the writing and publication of his book he hoped to unlock some of the mysteries of the secret workings of the unconscious mind. When I resurrected his survey, however, I did so for the purposes of explicating some of the elements of mathematical creativity. In particular, I wanted to establish what it was about the AHA! experience that sets it apart from other mathematical experiences. With regards to this issue I have come to several conclusions.

First and foremost, the understanding of the essence of AHA! experience has been extended with regards to the role of the suddenness, certainty, and significance with which a solution presents itself. The literature on invention and discovery treats the first two of these as defining characteristics of the phenomenon – suddenness has been used to describe the rapid appearance of an insight, while certainty describes the accompanying sense that the insight is correct. In this study significance joins the ranks of these defining characteristics as a second feeling that accompanies the sudden appearance of an insight. That is, an AHA! is marked both by a sense of certainty AND a sense of significance. It should be noted, however, that these senses of certainty and significance are just that, senses. In the immediacy and brevity of an AHA! experience neither the certainty nor the significance of an insight can be checked.

Suddenness, itself, also takes on greater meaning through the interpretation of the data in this study. Classically, the characteristic of suddenness has been used in decontextualized isolation as a descriptor of rapid appearance of a solution. This is the case in the literature, and this was the case with some of the responses in this study. However, it was also the case that many of the responses were embedded in the larger context of the AHA! experience. From these cases of contextualized suddenness it becomes clear that the feeling is not always produced by the rapidity with which an idea appears, but on the untimely nature of its appearance.

> When, after some considerable, quite non-productive effort, usually while not at all consciously working on the problem, there appears, for no apparent reason, in your brain the answer to that problem – that's the AHA! experience. (Connor)

Examples of showering, walking, sleeping, talking, cooking, driving, eating, waking, and riding the subway appear within the anecdotal data and speak to the contextual dependency of the feeling of suddenness. That is, suddenness is defined by what the person is doing when the solution appears – or rather, what the person is not doing (i.e. mathematics).

But suddenness and unexpectedness are not the only defining characteristics of an AHA! experience. Sudden illumination is also accompanied by a strong emotional response.

> Certainly it is standard among mathematicians to enjoy these AHA! moments while they last and postpone for a bit (e.g., until the next day) the necessary checking of the insight. (Connor)

The prevalence of anecdotal comments pertaining to this strong emotional response along with the uniform absence of any mathematical detail in these same anecdotes leads to a final conclusion that the AHA! experience is primarily an affective experience. That is, what helps to set it apart from other mathematical experiences is not the ideas, but rather the affective response to the appearance of the ideas.

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NOTES

1. Hadamard based his survey on a similar survey that was published in the pages of L'Enseignement Mathematique, the journal of the French Mathematical Society in 1902 and 1904. Édouard Claparède and Théodore Flournoy, two French psychologists, who were deeply interested in the topic of mathematical creativity, authored this survey.
2. For the participants in this study (and others) this is a self-defining term obvious in meaning to all.
3. For a more comprehensive analysis of the data see Liljedahl (2008).
4. Connor is the pseudonym for the one mathematician who preferred to remain anonymous.

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