Repeating Pattern or Number Pattern: the Distinction is Blurred[1]

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"Life itself is a creator of patterns." (Piaget, 1950, p. 167)

1.0 Introduction

Many argue that patterns are the cornerstone of mathematics. They are the foundation that the whole of the subject is built on. From the earliest tally systems to the development of differential calculus to modern mathematics patterns were and are the genesis, the motivation, and the foundation of mathematical knowledge. As such mathematics is often referred to as the science of patterns (Borwein & Jörgenson, 2001; Resnick, 1997). Steen (1988) articulates this relationship between patterns and mathematics thus:

> *Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind on type of patterns to another to yield lasting mathematical structures. Applications of mathematics use these patterns to "explain" and predict natural phenomena that fit the patterns. Patterns suggest other patterns, often yielding patterns of patterns. (p. 612)*

Even the very description of what it means to do mathematics can be defined in the context of patterns - "mathematicians observe patterns; they conjecture, test, discuss, verbalize, and generalize these patterns" (National Council of Teachers of Mathematics - NCTM, 1994).

However, the role of patterns in mathematics is an ironic one. While much of mathematics has its roots in patterns, there is no place for patterns in the formal representation of mathematics. The contemporary view is that mathematics is axiomatic in nature. As such, convention dictates that mathematics is presented in a linear and deductive argument, in the form of theorems and proofs. Patterns, on the other hand, are not axiomatic, nor are they necessarily linear. By nature, a pattern is inductive, and thus has no place in mathematical proof. "Much of what is "pattern" in the knowledge of mathematics is instead encoded in a linear textual format born out of the logical formalist practice that now dominates mathematics." (Borwein & Jörgenson, 2001, p.897).

In the teaching and learning of mathematics this irony is extended. While a mathematical proof, with its unfaltering deductive logic, contains within its structure the truth about a mathematical concept, it is often inappropriate for conveying mathematical concepts in that it may, in fact, be conveying the wrong message to our students – that full rigor is the core of mathematical (Hanna, 1989). Furthermore, it is often the use of a pattern that unlocks that truth and both presents it

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to the student and convinces them of it (Harel & Sowder, 1998; Mason, 2002; Rowland, 2002; Tahta, 1980). Consider, for example, the properties of negative exponents. Although they can be shown to be true using deductive reasoning these concepts initially defy students' intuition. It is the use of patterns, which most often facilitates the conceptual change necessary to create the understanding.

$2^4 = 16$ $2^3 = 8$ $2^2 = 4$ $2^1 = 2$ $2^0 = 1$ $2^{-1} = ½$ $2^{-2} = ¼$	While the exponent decreases by one as you move down the left column, the value of the exponential expression is divided by two as you move down the right column.

Unfortunately, the way in which patterns are used in the teaching of mathematical concepts can create a whole new set of misunderstandings – not of the mathematical content but of the patterns themselves. If through the pedagogical use of patterns in the teaching and learning of mathematics due care is not taken to preserve distinction between the types of patterns used then there is a risk that students' understanding of patterns can become blurred. This article examines how the lack of explicit attention to the distinction between repeating patterns and number patterns leads to difficulties for students engaged in problem solving activities that involves the investigation of patterns and offers a pedagogical solution to the prevention of this blurring.

2.0 In Pursuit of Pattern

What is a pattern? This question, although simply stated, is not so simple to answer. To see this I must first ask the question – to what ends is it useful to ask what is a pattern? That is, what logical purpose would an answer to the question serve? A precise definition of what a pattern is would help facilitate the process of discernment; it would help us distinguish patterns from non-patterns and thus form the basis of a classification scheme. "Definition can thus be seen as a way of adding precision to the boundaries of a concept, once formed; and of stating explicitly its relation to other concepts" (Skemp, 1971). The definition of prime number is what helps us discern primes from composites, the definition of multiples is what helps us discern multiples from non-multiples, etc.

Skemp makes a further distinction between primary concepts and secondary concepts. A primary concept - such as the colour red – can be conveyed through the use of examples. Given a set of suitable examples the variability of the examples and the constancy of the concept within the examples will cause the invariant property to emerge and become associated with the concept. So for the example of the concept red, a set of objects all of which are red would be gathered together as exemplars of the colour.

A secondary concept, alternatively, is one that can only be explained through the bringing together of words or symbols – that is, definitions. The concept of prime number is one such secondary concept. Skemp also acknowledges, however, that in defining and naming concepts the scope of their meaning can become limited. Take, for instance, his discussion of the concept of chair. We all have a well-developed ability to discern chairs from non-chairs. However, if we attempt to define what is a chair in some precise fashion (for example – a four legged seat with a backrest) we may exclude, by the use of this definition, things that are very clearly chairs (for example – a rocking chair). The same is true for the concept of pattern. A meaningful definition of what it means to be a pattern may exclude things that we would otherwise include in the set of patterns. To see this, consider what a constructed definition of pattern would have to encompass. It would have to capture all visual patterns – in one-dimension, two-dimensions, and three-dimensions – as well as auditory and movement patterns. It would also need to include any pattern that was a combination of these aforementioned patterns – like a dance or a movie clip. Now consider the types of words that would be used to create such a definition: sequenced, ordered, predictable, regular, etc. Each of these words either limit the scope of what things are patterns or provide nothing more capable of capturing the essence of pattern than the word pattern is.

Where does that leave the question 'what is a pattern?' Based on the discussion above it would appear as though patterns are a primary concept and thus describable (not definable) by the set it belongs to. That is, a pattern is that which we perceive to belong to the set of patterns. However, there are sets of patterns that act as secondary concepts in that they are well-defined. Scales, for example, are a well-defined form of auditory pattern, tessellations are a well-defined form of two-dimensional patterns, and a line dance is a well-defined form of movement pattern. What follows is a discussion on two such patterns - repeating patterns and number patterns.

2.1 Repeating pattern

A repeating pattern is a pattern in which there is a discernable unit of repeat (Threlfall, 1999). That is to say, the pattern has a cyclic structure that can be generated by the repeated application of a smaller portion of the pattern. This would include patterns such as A, B, A, B, …, the days of the week, or a tessellation. For the purposes of this article the unit of repeat will be defined as the smallest subset of elements of the pattern that can generate the pattern through successive application. That is, although ABAB… or even ABABAB… can both generate the repeating pattern ABABAB… the unit of repeat will be considered to be AB.

ABABAB…	has a unit of repeat of length 2 – AB
ABcABc…	has a unit of repeat of length 3 – ABc

The underlying principle of repeating patterns is their cyclic nature. Given a repeating pattern with a unit of repeat of length n the determination of the next element can be accomplished in two ways.

- There is an equality between every element in the pattern and one of the first n elements.
- There is an equality between every element in the pattern and the element n positions prior to it.

The length of the unit of repeat creates an isomorphism between repeating patterns. Thus, ABAB… is isomorphic to clap, stomp, clap, stomp, ... This is often referred to as the transfer of a repeating pattern (Burton, 1982) and does not change the crucial property of the pattern.

2.2 Number pattern

It could be argued that a number pattern is any pattern constructed on number, but I propose that the definition of number pattern be limited to those patterns in which the numerical value of the elements is important. That is, the pattern cannot be transferred (as defined above) to a non-numeric pattern without loss of some crucial property of the pattern. For example, the pattern 1, 2, 3, 4, 3, 2, 1 is transferable to abcdcba and thus is not a number pattern but a pattern using numbers as individual elements. A pattern such as 1, 2, 1, 1, 2, 1, 1, 1, 2, … is also not a number pattern as it can be transferred to abaabaaab… without loss of the nature of the pattern. Furthermore, repeating patterns constructed from numbers are not considered number patterns as they can be transferred to a non-numeric representation. What remains are patterns such as:

1, 2, 3, 4, 5, …
3, 7, 11, 15, 19, …
1, 1, 2, 3, 5, 8, …
1, 4, 9, 16, 25, …
1, 3, 6, 10, 15, …

In all of these cases the (numerical value of the) element is dependent on the numerical value of the previous element(s) or the numerical value of the position. That is, for each of these patterns the numerical values of its elements create the pattern and as such are a significant attribute would be lost through any form of transfer.

3.0 The Teaching and Learning of Patterns

The value of patterns to the teaching and learning of mathematics is well understood. “When we involve or appeal to pattern in teaching mathematics, it is usually because we are trying to help students to extract greater meaning, or enjoyment, or both, from the experience or learning environment with which they are occupied, and perhaps also to facilitate remembering.” (Orton, 1999, vii). As a general skill it is thought that the ability to discern a pattern is a precursor to the ability to generalize and abstract (Burton 1982, Threlfall, 1999). Scandura (1971) classifies pattern recognition (the detection of regularity) as one of the six basic processing skills (the others are the abilities to: particularize, interpret, describe, make logical inferences, and axiomatize). Sinclaire (2001) claims that patterning is an aesthetic mode of cognition – “by which we constantly and successfully make sense of our environment” (p.26). In a more specific context the NCTM (1994) identifies competency with patterns as being necessary in the ability to

- solve problems;
- develop understandings of important concepts and relationships;
- investigate the relationships among quantities (variables) in a pattern;
- generalize patterns using words or variables;
- extend and connect patterns;
- construct understanding of function.

The role of patterns as pedagogical tools cannot be overlooked. It has already been mentioned that it is often through the use of patterns that a teacher is able to unlock the truth within mathematical theorems and proofs. It is also through the use of patterns that students are able to explore new ideas. “The role of work with repeating patterns is as a useful basis for teaching about other matters, with the pattern-making acting as a concrete and familiar experience, which can be meaningfully referred to in talking about new ideas.” (Threlfall, 1999, p. 20).

3.1 Teaching and learning of repeating patterns

The formal treatment of patterns in primary years is focused initially on repeating patterns. A repeating pattern, as already mentioned, is a pattern that has a discernable unit of repeat. In many ways this is a formalization of the patterns that students have begun to experience in the cyclic nature of the days of the week, months of the year, and hours of the day (Charlesworth, 2000).

In these early school years children are exposed to a variety of repeating pattern tasks. They are: reproduce – copy, identify – express the unit of repeat, extend – continue, extrapolate – fill in gaps, transfer – change to a different modality, and create – make their own repeating pattern (Burton, 1982; Greeno and Simon, 1974; Threlfall, 1999). Reproduction of a pattern is the simplest of these activities and is achievable by most four year olds. The identification of the unit of repeat, as well as the extension, extrapolation, and transfer of a pattern is usually achieved by first grade, but this is largely dependent on the complexity of the repeating pattern (Burton 1982; Vitz and Todd, 1967).

3.2 Teaching and learning number patterns

Treatment of repeating patterns quickly moves to the introduction of number patterns. By third grade the majority of the patterns that students have experienced are built on number. The processes that are most often involved in the handling of number patterns can be summarized into four basic tasks: solve – provide a rule, expressed informally or formally, by which the elements of the pattern are produced, extend – continue the pattern, identify – determine the existence of a number pattern, and create – construct their own number pattern (Heargreaves, Threlfall, Frobisher, & Shorrocks-Taylor, 1999).

Much of early number pattern work is done in the context of coming to know the base ten numeration system. This is seen in the exploration of even and odd numbers, the use of skip counting to learn multiplication facts, and the search for number patterns within the multiplication tables. In later years students are exposed to number patterns in the form of arithmetic and geometric sequences during which time they are taught explicit tools for the treatment of such number patterns.

4.0 The Common Ground

Although repeating patterns and number patterns as defined are disjoint, there are several venues within which they are treated indiscriminately. As a result the boundary that distinguishes them from one another becomes blurred. In what follows I present several such venues and discuss how they contribute this blurring.

4.1 Task based blurring of distinction

As demonstrated above, the tasks associated with repeating patterns are in many cases the same as those associated with number patterns. In both cases students work towards being able to identify, extend, and create the patterns. Within these three tasks are two very distinct ways of thinking. Watson (2000) distinguishes between patterning activities as either reading with the grain or across the grain. Reading with the grain means to read with the pattern – to determine the next term by previous terms. Reading across the grain involves looking across the direction of the pattern and determining the term based on the position the term is occupying.

Reading with the grain is more commonly associated with the task of extending a pattern while reading across the grain is associated with generalizing a pattern. However, there is no exclusivity here. For example, the pattern 2, 5, 10, 17, … can either be extended by seeing it as an increasing arithmetic sequence of the differences (+3, +5, +7, …) which is reading with the grain or as 12 + 1, 22 + 1, 32 + 1, 42 + 1, … which is reading across the grain.

In the primary years both repeating patterns and number patterns are treated almost exclusively with reading with the grain tasks. As a result the transition from repeating patterns to number patterns is seamless and therefore there is an implication that there is no distinction between the two types of patterns. In later years the tasks tend towards reading across the grain in order to foster generalizing skills needed for algebra and functions (Clemson and Clemson, 1994 – cited in Threlfall, 1999). However, in these later across the grain type activities repeating patterns are not used. In the case where patterns are used then they are number patterns, in the form of sequences, and as such, do not provide an opportunity to be distinguished from repeating patterns.

4.2 Pedagogical blurring of distinction

Perhaps one of the first pedagogical uses of patterns is in the teaching of multiplication. One popular means by which basic multiplication facts are taught is through skip counting (Burton, 1982). Skip counting is where the multiples of a number are emphasized by the clap of the hands as the students chant the natural numbers. For example, the multiples of three would be brought forth by 1, 2, 3-clap, 4, 5, 6-clap, ... The rhythmical nature of the clapping is a repeating pattern while the natural numbers themselves are not.

4.3 Mathematical blurring of distinction

The convolution of repeating patterns and number patterns begins even before skip counting, however, with the chanting and writing of the natural numbers. The base ten numeration system that we use is the most fundamental of number patterns, yet it has repeating elements to it. Consider the written sequence of counting numbers 1,2,3,4,5… The unit digit of these numbers forms a repeating pattern with a unit of repeat that is ten elements long. This is a visual pattern that is especially apparent when the numbers are seen on the hundred chart so commonly used in elementary schools. Also apparent in the hundreds chart is the preservation of the units digit within each column. This property is further accentuated through the use of this pattern in teaching the addition of tens property of unit preservation (Burton, 1982; Threlfall and Frobisher 1999). The sequences of even (2,4,6,8,10) and odd numbers (1,3,5,7,9) also display a similar repeating pattern. This time, however, the repeating pattern formed by the unit digits is 5 elements long.

The repeating nature of the decimal system is less apparent in the chant of the numbers primarily because the unit of repeat is too long to maintain a rhythm and also because of the way in which the numbers from 11 to 19 have been named. Incremental counting by fives, however, creates a repeating pattern whose chant has an easily perceivable unit of repeat. For example, 5, 10, 15, 20, 25, 30, 35, 40, … contains in its unit digit a pattern of 5,0,5,0,5,0, … and in its spoken form (once you get over 20) the rhythmical endings of –ty,-five,-ty,-five,-ty, … A repeating pattern is even more apparent in the arithmetic sequence with a common difference of five but with a starting element not being a multiple of five: 23, 28, 33, 38, 43, 48, ... In this case the visual and the auditory are synonymous. That is the auditory endings of the numbers (-three, -eight, -three, -eight, …) are identical to the written unit digits 3, 8, 3, 8, 3, 8, ...

What follows is an examination of how the blurring of the distinction between repeating patterns and number patterns leads to difficulties for students engaged in problem solving activities that involves the investigation of patterns.

5.0 The Synthesis of Prior Studies

Over the last two years I have been involved in a number of research studies on pre-service elementary school teachers understanding and use of arithmetic sequences and other patterns (Liljedahl, 2002, 2001; Liljedahl & Zazkis, 2001; Zazkis & Liljedahl, 2001, 2002a, 2002b). None of these previous endeavours, however, dealt with the lack of distinction between repeating patterns and number patterns. This article is an exploration of the common theme that emerged from these previous studies – that the lack of distinction between repeating patterns and number patterns in instructional activities creates an obstacle (cognitive obstacle if you will) for the student. As a result, the data comes from a number of different sources: informal surveys, clinical interviews, and various written assessments.

5.1 The participants

Participants for all the studies were preservice elementary school teachers who were, at the time of the respective studies, enrolled in a course "Foundations of Mathematics for Teachers", which is a core course in the elementary teacher education program. Within the course students are exposed to activities that involve repeating patterns and sequences (both arithmetic and geometric). In addition, they are exposed to a variety of other number patterns – such as sequence of squares, Fibonacii sequence, etc. – in the context of problem solving activities. However, no efforts were ever made to draw attention to the distinction between the different types of patterns.

5.2 The tasks

Create a Pattern

This task was an informal survey administered in a class of 67 students at the end of the semester.

(a) Create a pattern.
(b) Create a pattern that is somehow fundamentally different from the first pattern you wrote down.
(c) Create a number pattern.

Train Problem

This task was given on a number of occasions. It appeared on two course final exams and as part of an informal class survey at the beginning of the semester. The number of participants for each was 106, 98, and 76 respectively.

(a) A toy train has 100 cars. The first car is red, the second is blue, the third is yellow, the fourth is red, the fifth is blue, the sixth is yellow, and so on. What is the colour of the 80th car? What is the number of the last blue car?
(b) Imagine a toy train with 1000 cars, following the 7 colour repeating pattern:
1 - red, 2 - orange, 3 - yellow, 4 - green, 5 - blue, 6 - purple, and 7 - white. What is the colour of the 800th car? What is the number of the last blue car?

These questions can be answered through the use of either division with remainder or counting up/down from a multiple.

Division with remainder

This is a very powerful strategy in that it partitions the natural numbers into sets of like attributes. For the Train Problem that common attribute is the colour of the cars. For example, the red cars are all in positions whose remainder in division by three will be one, the blue cars have a remainder of two, and the yellow cars have no remainder. Therefore, the 80th car (80/3 = 26 with a remainder of 2) will be blue. If the same strategy is applied to a number pattern like 1, 5, 9, 13 the like attribute is the remainder itself. For example, every element of the number pattern 1, 5, 9, 13 has a remainder of one when divided by four.

Counting up/down from a multiple

This strategy relies on the fact that every element of the pattern is some distance from a multiple of the length of the unit of repeat. For the Train Problem that means that every car that is in a position that is a multiple of three will be yellow, every car that is one beyond a multiple of three will be red, and two beyond will be blue. Thus, the 80th car which is two beyond 78 (78 = 26 x 3) will be blue. Similarly, for the number pattern 1, 5, 9, 13 every element is one more than a multiple of four.

The Calendar Problem

This problem was given in an audio taped clinical interview setting to 12 participants.

> I've chosen a calendar page, October 2000, and I'm going to place a red marker on the 1, a blue on the 2, a green on the 3, and a yellow on the 4. Now, I'm going to repeat this pattern; red on the 5, blue on the 6, green on the 7, and yellow on the 8.
>
> (a) What colour will number 13 be?
> (b) What colour will number 28 be?
> (c) If the calendar continued on forever, what colour would 61 be?
> (d) What colour would 178 be?
> (e) What colour would 799 be?
> (f) If there were five colours (red, blue, green, yellow, and black), what colour would 799 be?
> (g) If there were six colours, what colour would 799 be?

The specific questions asked varied as the interviewer followed up on the participants' comments. Again, the use of either division with remainder or counting up/down from a multiple can answer these questions.

The Sequence Problem

This problem was given as part of two different audio taped clinical interviews – one with 20 participants, the other with 12.

> Consider the sequence 1, 5, 9, …
>
> (a) What will the next few numbers in the sequence be?
> (b) Will the number 48 be in this sequence?
> (c) Will 63 be in the sequence?
> (d) Can you give me a big number that you know for sure will be in the sequence?
> (e) Consider the sequence 2, 5, 8, … Is 48 going to be in the sequence?
> (f) Can you give me a big number that you know for sure will be in the sequence?
> (g) Consider the sequence 8, 15, 22, … Can you give me a big number that you know for sure will be in the sequence?
> (h) Consider the sequence 15, 28, 41, … Is 1302 going to be in this sequence?

The specific questions asked varied as the interviewer followed up on the participants' comments. Either division with remainder or counting up/down from a multiple can help to answer these questions.

The Snake Problem

This problem was given as a project to a group of 36 students to be completed in a journal displaying all their work.

> Consider the following pattern:

1	2	3	4		
		8	7	6	5

9	10	11	12			
		16	15	14	13	

If this pattern continues, where will the numbers 86, 151, 1151 be?

This problem presents a pattern that has both a repeating pattern and a number pattern component to it. The numbers themselves forms a number pattern while the position of the numbers forms a repeating pattern. Thus, either division with remainder or counting up/down from a multiple will determine the position of any specified number.

5.0 Data and Analysis

What will be presented here are excerpts from the data as exemplars of some of the difficulties that students had in working with the aforementioned tasks. Because The Common Ground section dealt extensively with the inherent lack of distinction between repeating patterns and number patterns the data has been organized to reflect the emerging difficulties that such a lack of distinction produced. Each subsection will present a difficulty along with exemplars drawn from a variety of the tasks/studies as well as an analysis of the strategy used. Quantitative results will be given to emphasize the robust nature of some of these strategies.

5.1 Last digit patterns

This difficulty is centred both around students' inappropriate attention to the pattern of last digit of the elements of a number pattern and their propensity towards it. Attention to the last digit is an explicit effort to find a repeating pattern in a number pattern. This is no different than the (repeating) pattern-spotting activities that children engage in the defining of odd and even numbers, incremental counting, and the multiplication tables (Threlfall & Frobisher, 1999)

Attention to the last digit was used by six students (n = 32) while working on the Sequence Problem. In some cases it is helpful to the student, as can be seen by the Mike's efforts.

Interviewer: The sequence is 1, 7, 13, 19, 25 and so on I'll pick a number, a big number, let's say 360, and my question is, if I continue this sequence, will the number 360 be one of the elements?

[..]

Mike: Well no, it wouldn't be because, (pause) well I was just looking at these numbers, 1, 7, [13] the final digit is 3 and the final digit is 9, 5 and, oh wait a minute, yeah, no that would be 31 and there's the, and 31 and then 37, so the pattern, you're seeing a period of the pattern there, 1, 7, 3, 9, 5, 1, 7, 3, uh 9 and uh 5 and so on, so 360 wouldn't be in this sequence.

In other cases, however, it doesn't help. If the term that is being checked for isn't so easily excluded then uncertainty prevails. This can be seen by the difficulty that Helen has.

Helen: The pattern [of last digits] is 1, 5, 9, 3, 7 and then it start over again. So, no … 360 is not in the sequence.

Interviewer: Interesting. How about 63? Is 63 in the sequence [1, 5, 9, …]?

Helen: Um, (pause) I can't say. It might be. My instinct is telling me that it is…

Interviewer: Um hm . . .

Helen: I can't say for certain. 62 is not in the sequence …

5.2 Unit-attribute link

Unit-attribute link is the term I've given to strategies that involve the mapping of the unit preservation property of addition of ten from the natural number system to a problem where such a property does not apply. That is, adding ten (or some multiple of ten) to a natural number does not change the unit digit. Thus, the unit digit is an attribute that does not change if ten is added to the number. This leads to an overgeneralization that adding ten in a different circumstance will also result in the attended attribute remaining static.

The use of unit-attribute link was prevalent in many of the tasks mentioned. It is most likely an artefact left over from students initial introductions to the natural numbers and the emphasis that was placed on the preservation of the unit digit when adding tens. This is typically exemplified by the columns in the hundreds chart used in the classroom in primary years.

In the Snake Problem Melissa makes the assumption that the number 86 and 186 would be in the same column. In total seven participants (n = 36) in this study mentioned this assumption somewhere in their problem-solving journal.

> Since 86 in column D then 186 will also be in column D. So 187 is in column C or E depending on the direction that the pattern is running.

Megan was one of five (n = 32) who made the same assumption in the Sequence Problem.

Interviewer:	Ok. So 63 is not in the sequence [1, 5, 9, …]. Can you think of a big number, a three digit number, that you are sure is here or you are sure isn't here?.
Megan:	Okay. 61 would be the next number in this sequence and then um … 91 would be in the sequence.
Interviewer:	Um hm. . .
Megan:	Um, (pause) and 391 would be in the sequence.

On the Calendar Problem, Greg is one of two (n = 12) who initially believes that because 13 is red then 23 will also be red.

Interviewer:	Okay, so 13 is red. How about number 28?
Greg:	Okay. Um…23 will be red, 24 blue, green, yellow, red, blue. 28 will be blue.
Interviewer:	Okay…let's see if that is right.
[…]	
Greg:	Um…no its, its yellow…

In the Train Problem Jennifer is certain that adding ten to the number of a car does not change the colour of the car. She uses this strategy to answer each of the questions.

> The last blue car is 98 because any number ending with the digit 8 will be blue.
> The last blue car will be 995 because all numbers ending in the digit 5 will be blue.

Over the three studies that this problem was used 9 of the students (n = 280) used this logic on at least one of the four questions.

5.3 Tens patterns

This is very similar in nature to the unit-attribute link in that it deals with multiples of ten. The difference, however, is that instead of assuming that the attribute remains constant the students focus on the pattern that is produced by examining the multiples of ten. In some cases this can be a viable, yet cumbersome, strategy. While attending to the pattern of last digits is a strategy that involves creating a repeating pattern within a number pattern, tens multiples is a strategy of looking for a number pattern within a repeating pattern.

In the Train Problem Shirley was one of 23 (n = 280) who notice a pattern in the multiples of 10.

> The 80th car will be blue. This is because there are three different coloured cars and each multiple of 10 will be the next colour. As the 10th car is red, the 20th is blue, 30th yellow, 40th red, 50th blue, 60th yellow, 70th red, 80th blue.

Cheryl was the only one (n = 12) who noticed a pattern of 10's in the Calendar Problem.

Interviewer:	If the calendar continued on past 31 what colour would number 61 be?
(pause)	
Cheryl:	I'm just thinking…10 is blue, and 20 is yellow, 30 is blue again…
Interviewer:	Hmm…
Cheryl:	I think 60 will be yellow.
Interviewer:	What about 61?
Cheryl:	Oh, yes, blue … I mean red. 61 will be red.

In the Snake Problem most of the students also noted a pattern of 10's. While for some it served as a shortcut for counting up to a desired number for John it was a pattern with little usefulness.

> 10 in column B, 20 and 30 in column D, 40 and 50 in column B, 60 and 70 in column D, etc., an interesting pattern that will not help.

In all there were 24 students (n = 36) who commented on this pattern. This strategy also appeared once (n = 32) in the Sequence Problem as an exploration of whether or not a specific multiple of ten is in the sequence or not as can be seen by Bob's efforts.

Bob:	10 is in [the sequence 2, 6, 10, …] but 20 is not. 30 is in, 40 is not.
Interviewer:	Okay, so what about 360?
Bob:	Its in.

5.4 Pattern of primes

Although much can be said on this one issue alone I will focus on the repeating pattern nature of these exemplars. There were several cases where students looked for a pattern in the location of primes. Although no one makes an explicit statement as to the search for a repeating pattern it is implied by the purpose for searching in the first place. Like attention to the last digit, this strategy is a case of trying to spot a repeating pattern in a number pattern.

Lisa was the only one (n = 12) who looked for the placement of primes on the Calendar Problem.

Lisa:	No it won't. (pause) Or, oh, would it be green because its prime?
Interviewer:	Green is prime?

Lisa:	3, 7, 11, …
Interviewer:	Okay, where is the next green one?
Lisa:	(pause) 16, no. (pause)

There were five students (n = 36) who made mention of looking for a pattern of primes on the Snake Problem. Stephanie notices the placement of the primes.

> I looked at where the primes were and I noticed that they always land in column A, C, or E. Except 2. 2 lands in column B. I can't see a pattern…this is so frustrating.

All five of the students who explored the placement of primes abandoned this strategy quickly.

5.5 Repeating pattern as pattern

My feeling at seeing these difficulties emerging over and over again in the data from various studies was that students have a propensity towards repeating patterns. This would explain the robust tendency to look for repeating pattern and units of repeat in all patterns. To test this I constructed the Create a Pattern survey and used it in a class of 67 students during the last lesson of the course. This is the only instrument implemented specifically for this study.

Of the 67 responses, 41 of them used a repeating pattern as an example of pattern (question 1). The second question was used to probe if they indeed understand that there are patterns that are not repeating patterns. By and large, they do. Of the 41 who responded to question 1 with a repeating pattern, 29 used a non-repeating pattern in response to question 2.

On its own this survey would mean little. There are many shortcoming regarding the method of sampling and lack of ability to discern what the students took 'fundamentally different' to mean. However, in conjunction with the four previous sections it does confirm a propensity towards repeating patterns in the pattern work of the participants.

5.6 Discussion of results

The blurring between these two types of patterns begins, most likely, in the primary years with the seamless transition between repeating patterns and number patterns. The seamless-ness is facilitated on two fronts: the explicit search for repeating patterns in number patterns and the constancy of the with the grain tasks that both types of patterns are treated with. The search for repeating patterns in the natural numbers is an inherent and effectual method of coming to understand the base ten numeration system and cannot be avoided. The constancy of with the grain tasks used to explore the patterns also cannot be avoided as the primary age students are not developmentally ready to start working with across the grain tasks. The blurring of the boundaries between repeating patterns and number patterns is not improved in later years primarily because the focus of patterns moves from content to pedagogy. That is, patterns cease to be the content matter and instead become the pedagogical tools by which new content matter is taught. The focus is less on relationship of the pattern to other patterns and more on the relationship of the pattern to the new content at hand.

6.0 Pedagogical Implications

The diagnosis is clear, but the treatment is not trivial. The main obstacle in treating the blurring of distinction between patterns is that the conventional use of patterns in the teaching and learning of mathematics are very powerful and effective pedagogical strategies. To eliminate, or even limit, the implementation of any of these teaching strategies would be detrimental to the learning of the mathematical concepts they support. As such, any treatment of the problem needs to be from a constructive and integrated perspective. That is, solutions need to be found that build on, and work with the existing use of patterns in the teaching of mathematics. In what follows I propose one solution for the remediation and mediation of the tension between the inherent benefits of using patterns as a pedagogical tool and the inherent problems of using patterns as a pedagogical tool.

6.1 Definitions

In the early part of this article I laid out a clear set of definitions for repeating patterns and number patterns (a repeating pattern is a pattern in which there is a discernable unit of repeat, and a number pattern is a pattern with numbers as elements that cannot be transferred to a non-numeric pattern without loss of some crucial property of the pattern). Although the definition of repeating pattern is a conventional one my definition of number pattern is not. I felt that in order for the distinction between the two patterns to become clear their definitions needed to create distinct sets. That is, I wanted to create a definition of number pattern in such a way that no pattern could be both a repeating pattern and a number pattern. Such distinction through definition would give us – mathematicians and teachers – a language with which to discuss and teach the concept of patterns. The definition I created accomplishes this. However, I have not created anything new – the patterns existed before the definition and they remain after the definition. What I have done is to make explicit the distinction between the two types of patterns. I offer up one such set of definitions. I make no claim that others do not exist, or that others are not better, but only that they are necessary.

6.2 Language

Once the definitions that provide distinction are in place the terms themselves need to be incorporated into the language of mathematics and mathematics instruction. "The need to make drastic changes to the prior thinking may not even occur to the students unless the needed change is made very explicit in the teaching." (Pehkonen and Merenluoto, 2001, p.264). That is, the terms repeating pattern and number pattern need to be used to accentuate the patterns being encountered – whether those encounters are in the investigation of the patterns themselves or in the use of the patterns in the investigation of mathematical content. The repeated usage of the language of patterns will help to facilitate students' conception of patterns.

In simple terms, the solution to the problem is to create and use a language of distinction – a classification scheme. However, merely stating that this is the solution is an innocuous treatment of the situation. The much greater task is to implement these changes.

6.3 Teacher education/re-education

The misconception of what is a pattern and the blurring of distinction discussed in this article have all been done so in the context of preservice elementary school teachers' responses. The misconceptions and lack of understanding that I have shown exists among these participants will follow them out into their teaching careers. This indicates that the bigger problem lies not with the students, but with the teachers themselves. In order to make changes for students there first needs to be a process of change for teachers.

What would this change look like? I propose that the issue at hand is an issue of conceptual change. That is, they need to be re-educated – not just educated. For many teachers, the conceptual understanding of patterns that they have is not strained by the curriculum they teach. This is because the lack of distinction created by the pedagogical use of patterns is not immediately problematic to the student or the teacher. As a result the teacher may be satisfied with their current understanding and usage of patterns. In order to create a conceptual change two things are required – dissatisfaction with their current understanding and the presentation of a favourable alternative.

For preservice teaching this would best be facilitated in the treatment of the topic of patterns in both their mathematics preparation (Foundations of Mathematics courses) and their mathematics teaching preparation (Methods of Teaching Mathematics courses). In the foundation course, dissatisfaction with their current way of thinking can be facilitated with the use of pattern-based problem solving activities such as the ones presented in this article. For many students, these questions would be enough to strain their understanding of patterns enough to reveal to them that they need a stronger conceptual knowledge base of the pattern content. Once they have reached this stage of dissatisfaction they are ready to be presented with a favourable alternative. I propose that this alternative be in the form of a unit on patterns – from primary curriculum topics to secondary topics – initiated with the definitions suggested and taught with a heavy emphasis on the language of patterns.

The treatment of patterns as content in the foundations course can then be supported in the teaching methods course with the presentation of data – from this article or others – on the effects of not creating a clear distinction between repeating patterns and number patterns. This can be facilitated in the context of a case study, a research assignment, or simply the reading of a journal article.

Unfortunately, the luxury of course work is not available for the re-education of practicing teachers. The best vehicles for dissemination of information to this population are workshops, professional development, teaching journals, and curriculum.

6.4 Curriculum

One of the best vehicles by which new mathematical concepts can be delivered to in-service teachers is through the curriculum. Although the changes discussed are small, a treatment of them in the K-12 curriculum – with explicit attention to the fact that there has been a change –

will greatly improve the dissemination of the information. However, the changes in the curriculum are more in the rational of how to teach than in the content matter. Other than the presentation of a definition all other prescriptions are pedagogically based. It needs to be made clear that teachers need to start using the language of patterns with their students. Only then will they be helping to facilitate the conceptual distinction necessary to help their students to choose appropriate strategies for the treatment of patterns in the future.

7.0 Conclusion

Patterns are the ether of mathematics. They fill the space between ideas. From the time a child enters schooling (pre-kindergarten) until the time they leave high school they will be exposed to countless patterns. In some cases this will involve the formal treatment of patterns as content and other times it will involve the utilization of patterns as pedagogical tools. In both cases, however, the concept of pattern is dealt with implicitly. That is, the students are not provided with any sort of explicit definitions.

Although the elusive nature of patterns makes definition difficult, the creation of precise definitions of repeating patterns and number patterns is not only possible, but necessary. There is no shortage of examples of students' inappropriate use of repeating patterns in the solving of number pattern problems. In general, there is a propensity on the part of students towards repeating patterns. This may be a result of the students' conceptualization that patterns are repeating patterns. Thus, when faced with a situation where either a pattern is clearly presented, or there is indication that a pattern may be a useful problem-solving tool, there is a tendency to default towards a repeating pattern. Lee (1996) dealt with a similar problem in the teaching of algebra and concluded that it was not the inabilities of the students to spot patterns that prevented their success but their inability to spot algebraically useful patterns. I propose that the students in this study often failed to spot the useful patterns for the task at hand. I further propose that this need not have been the case. Appropriate re-education of preservice and in-service teachers along with slight changes in curriculum can help reduce – if not eliminate – such difficulties.

8.0 References

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