

COMPUTER MATH SNAPSHOTS

REFLECTIONS ON REFLECTIONS

PETER LILJEDAHL

Simon Fraser University

E-mail: liljedahl@sfu.ca

This column will publish short (from just a few paragraphs to ten or so pages), lively and intriguing computer-related mathematics vignettes. These vignettes or snapshots should illustrate ways in which computer environments have transformed the practice of mathematics or mathematics pedagogy. They could also include puzzles or brain-teasers involving the use of computers or computational theory. Snapshots are subject to peer review.

This issue's snapshot explores some generalizations of the definition of geometric reflection. Dynamic geometry tools can facilitate generalizations such as those obtained by relaxing the requirement that the reflection be through a straight line. The author compares the families of curves obtained by reflecting thru circular arcs with the curves generated in response to a physical problem proposed by Wittgenstein. He suggests that the strategy of generalizing definitions is a good avenue for bringing students quickly to the activity of doing mathematics.

Computer Math Snapshots

Editor: Uri Wilensky

Center for Connected Learning and Computer-Based Modeling

Northwestern University, USA

E-mail: uri@northwestern.edu

INTRODUCTION

In this snapshot I use *The Geometer's Sketchpad* (KCP Technologies Inc., 2001) to explore some of the results of changing the definition of reflection to accommodate reflection across curved lines, in particular, circular arcs. My motivation for doing such exploration is two-



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fold. First and foremost is my wish to present students with mathematics that is ‘new’, both to the student and to the field. I use this strategy to help dispel the myths of mathematics as a collection of facts and skills long since established, and promote the view that mathematics is an emerging field that involves both ‘learning’ and ‘doing’. My second reason for engaging in this particular exploration comes from my desire to give students a richer set of experiences with regards to such taken for granted mathematical concepts as reflection. In this case my strategy is based on Brown and Walter’s (1974) method of using “what-if-not” to expand the context of established mathematics. What I present here are the results of one specific thread of these rich explorations; a thread that produced a set of plane curves that are both aesthetically pleasing and hauntingly familiar; and as such, the basis of further and deeper exploration.

The geometric principles behind the construction of the images are not difficult. However, in order to keep the article free of clutter I have taken a minimalist approach to the presentation of the images, and wherever suitable, opting for aesthetic appeal as opposed to tedious details. For the same reasons the derivations of the parametric equations that describe the curves I have created have likewise been left out.

WHAT IF NOT?

I begin with the definition of reflection.

A reflection R uses a fixed line l . Any point on l is its own image. Any other point P is mapped onto a point P' such that l is the perpendicular bisector of $\overline{PP'}$. As the name implies, a set of points and their images are reflections of each other; it is as if line l were a mirror (Smart, 1998).

Smart is capitalizing on the congruency of the mathematical reflection of an object across a line and the reflection of an object in plane mirror. That is, the mathematical definition of reflection creates an image, which I will refer to as the ‘reflected image’, in the same place as the conventional understanding of reflection, which I will refer to as the ‘mirror image’ (see Figure 1). Although these images are in the same place they are generated in very different ways. The reflected image (as indicated by the dotted line) is created, as defined above, through a mathematical reflection across the line of reflection. The mirror image (as indicated by the solid lines) is determined by the

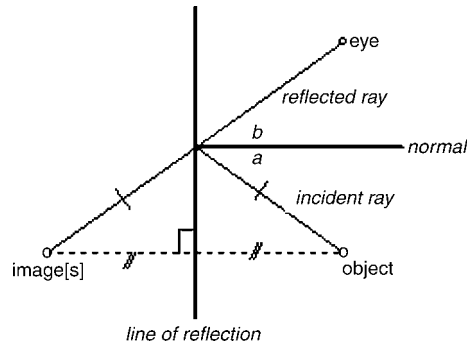


Figure 1.

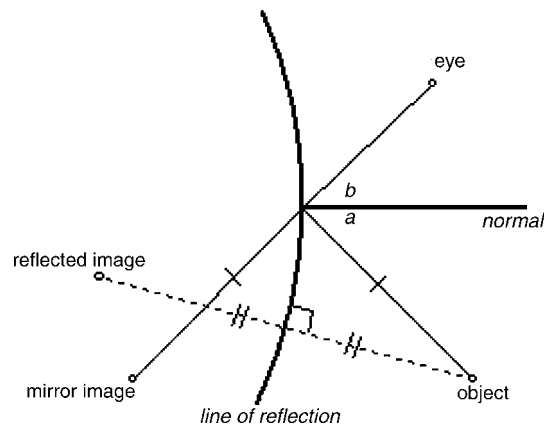


Figure 2.

congruency between the angle of incidence a (as measured from the normal) and the angle of reflection b (also measured from the normal) along with an extension along the reflected ray by a distance congruent to the incident ray.

If the line is not straight, however, then the notion of a mirror image still holds while the mathematical concept of a reflected image, as defined above, no longer has any meaning. This can be fixed if we simply adjust our definition of mathematical reflection to accommodate curved lines. Such a fix is most easily accomplished by allowing the tangent of a curve to replace (or act as) the line of reflection. However, this accommodation, although intuitively natural, destroys the general correspondence between the mirror image and the reflected image (see Figure 2).

Following either image leads to further interesting mathematics. For example, exploring the mirror image could lead one to the weird and wonderful world of anamorphic art; a form of art whereby seemingly incomprehensible pictures are ‘decoded’ by viewing them through curved mirrors, most often cylinders (Baltrusaitis, 1977). For this article, however, I follow the path of the reflected image.

REFLECTION ACROSS A CIRCLE

If the line of reflection is an arc as it is in Figure 2 then the original definition holds (with the understanding that the perpendicular bisector refers to the tangent of the curve, and not the curve itself). That is, it holds without the stipulation that “it is as if line l were a mirror”. But why limit ourselves to an arc? Let’s consider the whole circle. This will demand some further modifications to our definition. There are two arcs of the circle that a point P can be reflected across such that the tangent to the arc is the perpendicular bisector of $\overline{PP'}$ (see Figure 3).

Although each of these possible images will eventually produce nice results the particular curves I wish to present here comes from choosing the closest arc as the line of reflection.

The reflection R uses a fixed circle $c1$. Any point on $c1$ is its own image. Any other point $P (P \neq O)$ is mapped onto a point P' such that a tangent to $c1$ is the perpendicular bisector of $\overline{PP'}$ and $\overline{PP'}$ is the shortest segment possible.

These modifications, however, create an ambiguity for the point O . The solution to this is to leave the mapping of point O undefined.

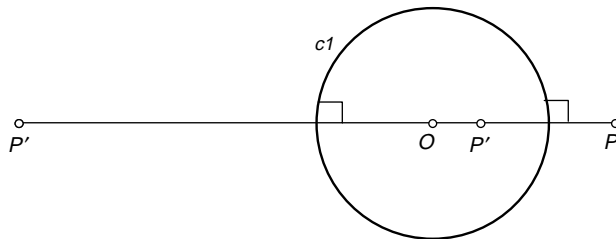


Figure 3.

It should be noted here that this is different from the classic method of reflecting across a circle called *inversion* (sometimes referred to as *reflection in a circle*). In inversion a point P is reflected across a circle of radius r centred at O to a point P' such that $(OP)(OP') = r^2$ (Eaves, 1969).

Because of the basic properties that result from the definition of reflection used there are two additional circles that are of great interest (see Figure 4). If the circle of reflection is defined as $c1$ (a circle with radius r centred at O) then the other two circles of interest are $c2$ (a circle of radius $2r$ centred at O) and $c3$ (a circle of radius $3r$ also centred at O).

In order to better facilitate a discussion of some of the results of using a circle as a line of reflection I have included (below) a list of some of the more basic and relevant properties that emerge from my definition.

1. Any point on $c1$ is its own image.
2. Any point P , not on $c1$, will map to a point on a straight line containing O and P .
3. Any point on $c2$ will be mapped to O .
4. Any point on $c3$ will be mapped to a point on $c1$.
5. Any point inside $c1$ will be mapped to a point that is between $c1$ and $c2$.
6. Any point between $c1$ and $c3$ will be mapped to a point inside of $c1$.
7. Any point outside of $c3$ will be mapped to a point outside of $c1$.

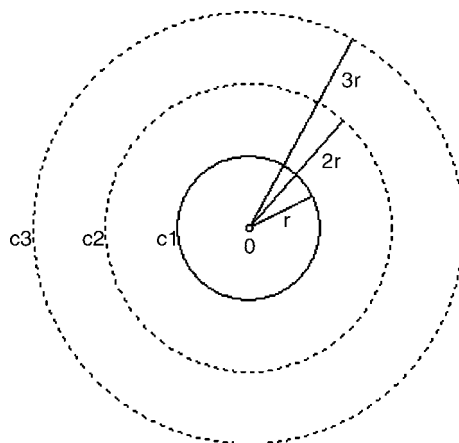


Figure 4.

8. Any point P' inside of $c1$ can be mapped to from two distinct points P , one between $c1$ and $c2$ and one between $c2$ and $c3$.
9. Any point P' between $c1$ and $c2$ can be mapped to from two distinct points P , one inside $c1$ and one outside $c3$.

REFLECTION OF A LINE ACROSS A CIRCLE

Although I used *The Geometer's Sketchpad* to explore the curves that resulted from performing the specified reflection on a straight line the same transformation can be performed simply by attending to the properties listed above. The software allowed me to define the transformation of a single point and then move this point along the straight line to produce the reflection. The software also allowed me to move the position of the straight line in order to observe the results as it cuts through the various circles $c3$, $c2$, and $c1$ (see Figures 5 and 8).

The self-crossing nature of the images in Figures 7 and 8 are as a result of property 3. The straight line cuts $c2$ in two places, each of which will map to O .

REFLECTION OF A CIRCLE ACROSS A CIRCLE

Similar approaches can be used to explore the results of reflecting a circle across a circle. In this more complex case, however, I recom-

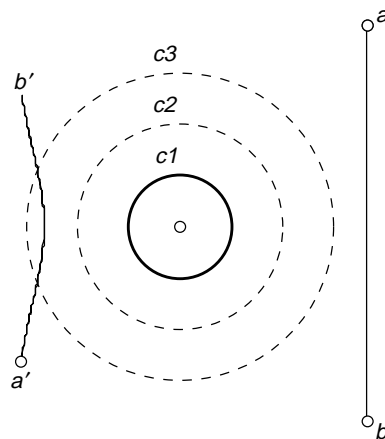


Figure 5.

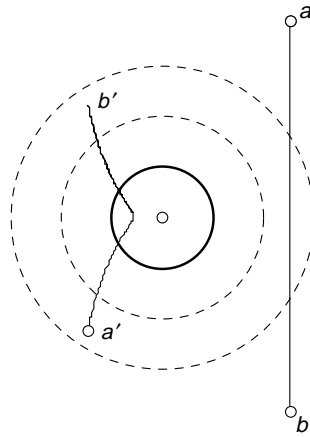


Figure 6.

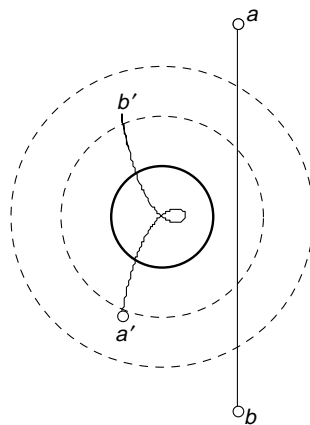


Figure 7.

mend the use of the software.¹ In order to facilitate an easier discussion of this process the circle that is acting as the line of reflection will be referred to as the reflective circle, the circle that is being reflected will be referred to as the object circle, and the resulting curve will be referred to as the image (see Figure 9).

For brevity, I introduce here some useful notation. If the reflective circle is of radius 1 and centred at the origin there would be no further loss of generality in defining the object circle as having a radius r centred at $(g, 0)$. Thus, the resulting image can be uniquely described

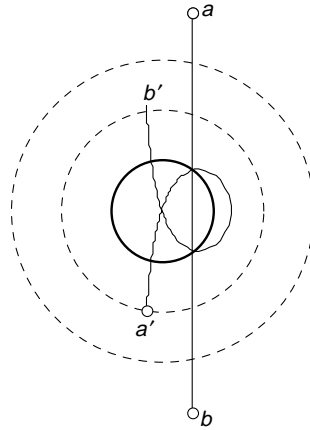


Figure 8.

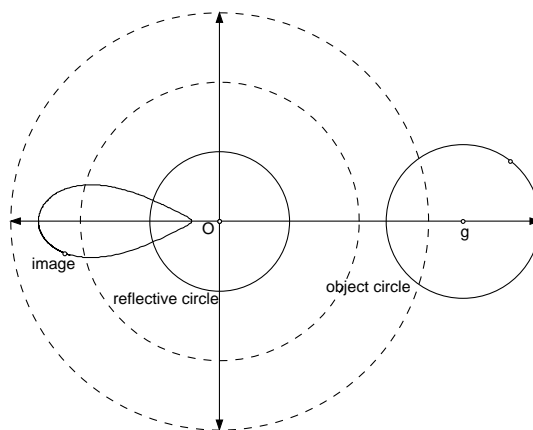


Figure 9.

by the parameters r and g of its object circle. The image in Figure 10 can, therefore, be referred to as $g = 3.6, r = 1.2$ or $(3.6, 1.2)$.

What follows is a collection of curves, labelled with the aforementioned notation, that result from the reflection of a circle across a circle.

These curves represent only a small portion of the possible images that can be produced. I left out the trivial results of reflecting an object circle that is centred at O . The selection of these curves was guided by my desire to present images produced by placing the centre of the image circle in each of the four domains (inside c_1 , between c_1 and c_2 , between c_2 and c_3 , and outside of c_3) and then modifying the


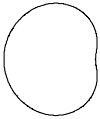
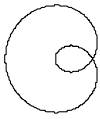
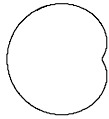

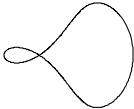

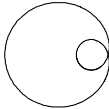



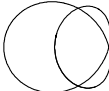




			
#1 (0.5, 0.4)	#2 (0.5, 1.1)	#3 (0.5, 1.8)	#4 (0.5, 2.6)
			
#5 (1.5, 0.4)	#6 (1.5, 1.1)	#7 (1.5, 1.8)	#8 (1.5, 2.8)
			
#9 (2.5, 0.4)	#10 (2.5, 1.1)	#11 (2.5, 1.8)	#12 (2.5, 2.8)
			
#13 (3.5, 0.4)	#14 (3.5, 1.1)	#15 (3.5, 1.8)	#16 (3.5, 2.8)

Figure 10.

radius to create a variety of intersections with each of the circles c_1 , c_2 , and c_3 .

Although it is onerous to generate these curves without the software it is not too difficult, and perhaps an interesting exercise, to analyse them using only the properties listed above. For example, upon inspection it quickly becomes apparent that curve 12 is quite different from all the others; there are three points where it crosses itself. One of the intersections comes from *property 3* (any point on c_2 will be mapped to O). The other two intersections come from *property 9* (any point P' between c_1 and c_2 can be mapped to from two distinct points P , one inside c_1 and one outside c_3).

DEEPER EXPLORATION

In seeing all the curves together I was struck by how aesthetically pleasing they were, while at the same time intrigued by how haunt-

ingly familiar some of them seemed. In sharing these curves with Nick Jackiw, he drew my attention to a set of curves that result from one of Ludwig Wittgenstein's thought experiments (Cuoco and Goldenberg, 1997; Wittgenstein, 1983). Wittgenstein posed the following problem:

A stick slides smoothly through a pivoting sleeve. As one end of the stick is moved in a circle, what shape does the stick's far end describe?

Using *The Geometer's Sketchpad*² to explore these curves it became clear that there were, indeed, similarities to some of the curves presented above (#9–11 and #13–16). To see this I have to generalize Wittgenstein's thought experiment to a circle of radius 1 centred at the origin and a stick of length L pivoting at $(p,0)$. Considering a general point on the circle to be traced and the resulting image of that point produces the following parametric equations:

$$\begin{aligned} x(t) &= \frac{L(p - \cos t)}{\sqrt{1 + p^2 - 2p \cos t}} + \cos t, \\ y(t) &= \frac{-L \sin t}{\sqrt{1 + p^2 - 2p \cos t}} + \sin t. \end{aligned} \quad (1)$$

Comparing these with the parametric equations I derived from my circle images:

$$\begin{aligned} x(t) &= 2\sqrt{1 - \frac{(r \sin t)^2}{r^2 + g^2 + 2rg \cos t}} - r \cos t - g, \\ &\quad -\pi + \cos^{-1} \frac{g}{r} \leq t \leq \pi - \cos^{-1} \frac{g}{r} \\ &\quad - 2\sqrt{1 - \frac{(r \sin t)^2}{r^2 + g^2 + 2rg \cos t}} - r \cos t - g, \\ &\quad \pi - \cos^{-1} \frac{g}{r} < t < \pi + \cos^{-1} \frac{g}{r}, \\ y(t) &= \frac{2r \sin t}{\sqrt{r^2 + g^2 + 2rg \cos t}} - r \sin t. \end{aligned} \quad (2)$$

further similarities (now at a more analytic level) presented themselves. In fact, if from (2) $r = -1$, from (1) $L = 2$, and from (1) and (2) $p = g$ then the equations will be identical. That is, Wittgenstein's

thought experiment is a special case of reflecting a circle across a circle. I was able to conclude that it is, in fact, possible to generate any curve produced by Wittgenstein's experiment through precise selection of radius and placement of the object circle.

CONCLUSION

The detour I have taken here has very quickly, and very accessibly, led to the creation of new mathematics. At the very least, such a detour can only serve to *enrich* students' experiences with such 'taken for granted' concepts as reflection. On a bigger scale, such mathematical encounters have the power to change students' perceptions of what mathematics is about – and perhaps even create some appeal for them.

NOTES

- ¹ For anyone interested in the process of doing this on The Geometer's Sketchpad send me an E-mail: liljedahl@sfu.ca and I will send the Sketchpad file.
- ² See Key Curriculum Press Thought Experiment 192. Available on line at: <http://www.keypress.com/sketchpad/javasketchpad/gallery/pages/wittgenstein.php> for a working applet on this.

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