PETER LILJEDAHL

PERSONA-BASED JOURNALING: STRIVING FOR AUTHENTICITY IN REPRESENTING THE PROBLEM-SOLVING PROCESS

Received: 13 October 2006; Accepted: 30 May 2007

ABSTRACT. Students' mathematical problem-solving experiences are fraught with failed attempts, wrong turns, and partial successes that move in fits and jerks, oscillating between periods of inactivity, stalled progress, rapid advancement, and epiphanies. Students' problem-solving journals, however, do not always reflect this rather organic process. Without proper guidance, some students tend to 'smooth' out their experiences and produce journal writing that is less reflective of the process and more representative of their product. In this article, I present research on the effectiveness of a persona-based framework for guiding students' journaling to reflect the erratic to-and-fro of the problem-solving process more accurately. This framework incorporates the use of three personas—the narrator, the mathematician, and the participant—in telling the tale of the problem-solving process. Results indicate that this persona-based framework is effective in producing more representative journals.

KEY WORDS: journaling, persona, problem solving

For mathematicians, problem solving is a complex activity that involves the confluence of both logical and extra-logical processes. Effective problem solving involves the oscillation between inductive and deductive logic while regulating the responses to aesthetic and intuitive sensibilities, and experiences with insight and illumination (Dewey, 1938; Fischbein, 1987; Hadamard, 1945; Poincaré, 1952). However, as creative a process as problem solving may be, the results of these processes are encoded in a linear textual format born out of the logical formalist practice that now dominates mathematics (Borwein & Jörgenson, 2001). This discordance between the process of problem solving and the presentation of its products is nicely summarized in the comments of Dan J. Kleitman, a prominent research mathematician:

> In working on this problem, and in general, mathematicians wander in a fog not knowing what approach or idea will work, or if indeed any idea will, until by good luck, perhaps some novel ideas, perhaps some old approaches, conquer the problem. Mathematicians, in short, typically somewhat lost and bewildered most of the time that they are working on a problem [sic]. Once they find solutions, they also have the task of checking that their ideas really work, and that of writing them up, but these are routine, unless (as often happens) they uncover minor

International Journal of Science and Mathematics Education (2007) 5: 661–680

> errors and imperfections that produce more fog and require more work. What mathematicians write thus bears little resemblance to what they do: they are like people lost in mazes who only describe their escape routes, never their travails inside. (Liljedahl, 2004, p. 157)

The discordance between process and product, however, is not a dilemma that is restricted to the domain of professional mathematicians. Students of mathematics also have a difficult time breaking away from the formalist practices of conventions as delivered to them in the form of curriculum, textbooks, and classroom instruction.

Journaling

Journal writing in mathematics education has a long and diverse history of use. Journaling helps students learn mathematical concepts (Chapman, 1996; Ciochine & Polivka, 1997; Dougherty, 1996). It has been shown to be an effective tool for facilitating reflection among students (Mewborn, 1999), as well as an effective communicative tool between students and teachers (Burns & Silbey, 2001). Journaling has also become an accepted method for qualitative researchers to gain insights into their participants' thinking (Mewborn, 1999; Miller, 1992). This is especially true of problem-solving journals, which can allow the researcher to enter into the otherwise private world of problem solving. In order for this to be effective, however, such problem-solving journals need to be representative of the problem-solving process. This is not always the case. Students' problem-solving experiences are fraught with failed attempts, wrong turns, and progress that moves in fits and jerks, oscillating between periods of inactivity, stalled progress, rapid advancement, and epiphanies. Without proper guidance, students may tend to 'smooth' out these experiences and, as a result, present stories in their journals that are less reflective of their 'travails inside the maze' and more representative of their 'escape route'.

A Model for a More Structured Method of Journaling

The literature that details mathematicians' problem-solving efforts is unrepresentative of the true process of 'doing' mathematics. One rare exception to this is an account (Hofstadter, 1996) that tells the story of a mathematical discovery with amazing sincerity. It is detailed and complete, from initiation to verification. It tells the story of being lost in a maze, searching for answers and, in a flash of insight, finding the path. Perhaps the reason that the account is so different is that Hofstadter is not a professional mathematician. He is a college professor of cognitive science and computer science and also an adjunct professor of history and philosophy of science, philosophy, comparative literature, and psychology. As such, he has a unique appreciation for tracking his own problem-solving processes.

In analysing Hofstadter's account, it becomes clear that one of the reasons it is so sincere is because of the way he incorporates the use of three different personas—a trinity of voices—in telling his tale. I have come to name these personas the *narrator*, the *mathematician*, and the *participant*. These personas are not explicit in Hofstadter's writing in that he does not introduce them, annotate them, or even acknowledge them. Instead they are implicit, emerging from active analysis more than from passive reading of his writing. Each persona contributes to the anecdotal account in a different way. The narrator moves the story along. As such, he often uses language that is rich in temporal phrases (e.g., *and then* or *I started*). He also fills in nonmathematical details seemingly for the purpose of providing context and engaging content. The mathematician is the persona that provides the reasoning and the rational underpinnings for why the mathematics behind the whole process is not only valid but also worthy of discussion. Finally, the participant speaks in the voice of a real-time, evolving present. This persona reveals the emotions and thoughts that are occurring to Hofstadter as he is experiencing the phenomenon.

Waywood (1992) found very similar voices in the journals of year 11 mathematics students. He also identified three types of journaling within his subjects, which he referred to as *recount, summary*, and *dialogue*. Recounting is very similar to what I refer to as the voice of the narrator, and summarizing is virtually identical to the voice of the mathematician. Dialogue, however, is only part of what I refer to as the voice of the participant. For Waywood, dialogue is the self-talk that goes on in the journals through which ideas are revealed. He does not, however, stipulate that dialogue contains any expressions of emotion. Both of these characteristics, presentation of ideas and emotions, make up the voice of the participant.

To demonstrate these personas, I present a portion of Hofstadter's chapter that contains all three voices. Before doing so, however, it would be useful to introduce the general context of his mathematical encounter. At the time of writing the chapter, Hofstadter had only recently become impassioned with Euclidean geometry and had never been introduced to the Euler line of a triangle. When he did learn about it, however, two things immediately struck him: the connectivity of seemingly different attributes and the exclusion of the incentre. Thus, he began a journey of trying to find a connection between the Euler line and the incentre. At the point in the passage presented below, Hofstadter has just discovered something about the incentre (see Figure 1 for an illustration of what Hofstadter is talking about).

> One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumcentre to the orthocentre, then, when your head has rotated exactly halfway between them, you will be staring at the incentre. More formally, the bisector of the angle formed by two lines joining a given vertex with the circumcentre and with the orthocentre passes through the incentre. (A more technical way of characterizing this property is to say that O and H are 'isogonic conjugates'.) It wasn't too hard to prove this, luckily. This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incentre back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as 'central' a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice. (Hofstadter, 1996, p. 4)

Even from this brief excerpt, it can be seen how the three personas interact with each other while at the same time presenting different aspects of the mathematical experience. It begins with *One day ...* - a clear indicator that the narrator is speaking:

> One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumcentre to the orthocentre, then, when your head has rotated exactly halfway between them, you will be staring at the incentre.

Hofstadter is telling us what he has found in an informal yet descriptive way. This is followed by his mathematician persona coming in and formalising this finding in a more precise and mathematical way:

> More formally, the bisector of the angle formed by two lines joining a given vertex with the circumcentre and with the orthocentre passes through the incentre. (A more technical way of characterizing this property is to say that O and H are 'isogonic conjugates'.)

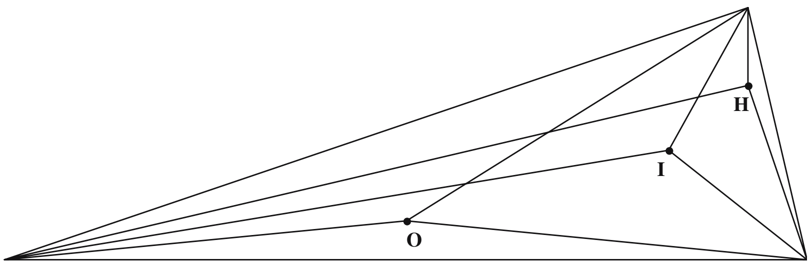

Figure 1. Triangle with Incentre (I), Orthocentre (H), and Circumcentre (O)

Finally, the participant reveals how he feels about his finding and what thoughts this find precipitates:

> This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incentre back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as 'central' a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.

The interplay between the three personas is further demonstrated in Figure 2 in which an entire page of text from Hofstadter's work is

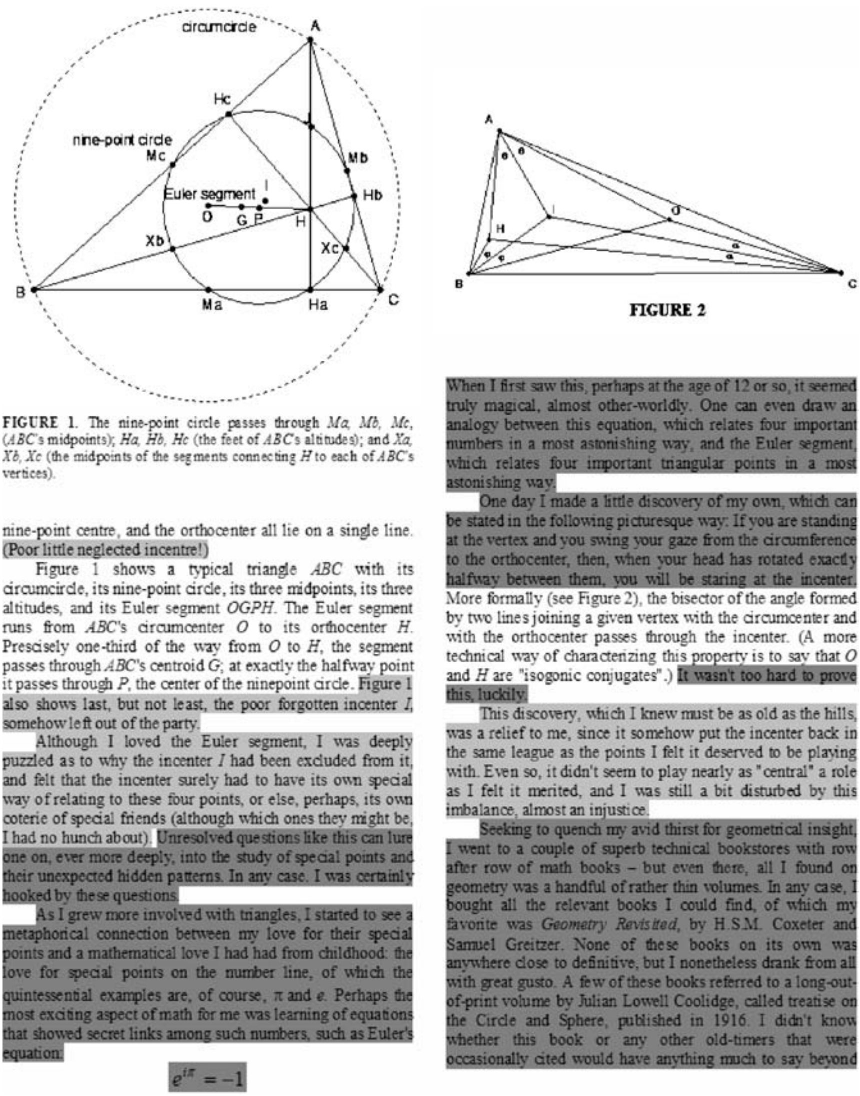

FIGURE 1. The nine-point circle passes through *Ma, Mb, Mc*, (*ABC*'s midpoints); *Ha, Hb, Hc* (the feet of *ABC*'s altitudes); and *Xa, Xb, Xc* (the midpoints of the segments connecting *H* to each of *ABC*'s vertices).

nine-point centre, and the orthocenter all lie on a single line. (Poor little neglected incentre!)

Figure 1 shows a typical triangle *ABC* with its circumcircle, its nine-point circle, its three midpoints, its three altitudes, and its Euler segment *OGPH*. The Euler segment runs from *ABC*'s circumcenter *O* to its orthocenter *H*. Prescisely one-third of the way from *O* to *H*, the segment passes through *ABC*'s centroid *G*; at exactly the halfway point it passes through *P*, the center of the ninepoint circle. Figure 1 also shows last, but not least, the poor forgotten incenter *I*, somehow left out of the party.

Although I loved the Euler segment, I was deeply puzzled as to why the incenter *I* had been excluded from it, and felt that the incenter surely had to have its own special way of relating to these four points, or else, perhaps, its own coterie of special friends (although which ones they might be, I had no hunch about). Unresolved questions like this can lure one on, ever more deeply, into the study of special points and their unexpected hidden patterns. In any case, I was certainly hooked by these questions.

As I grew more involved with triangles, I started to see a metaphorical connection between my love for their special points and a mathematical love I had had from childhood: the love for special points on the number line, of which the quintessential examples are, of course, π and e. Perhaps the most exciting aspect of math for me was learning of equations that showed secret links among such numbers, such as Euler's equation:

$$e^{i\pi} = -1$$

FIGURE 2

When I first saw this, perhaps at the age of 12 or so, it seemed truly magical, almost other-worldly. One can even draw an analogy between this equation, which relates four important numbers in a most astonishing way, and the Euler segment, which relates four important triangular points in a most astonishing way.

One day I made a little discovery of my own, which can be stated in the following picturesque way: If you are standing at the vertex and you swing your gaze from the circumference to the orthocenter, then, when your head has rotated exactly halfway between them, you will be staring at the incenter. More formally (see Figure 2), the bisector of the angle formed by two lines joining a given vertex with the circumcenter and with the orthocenter passes through the incenter. (A more technical way of characterizing this property is to say that *O* and *H* are "isogonic conjugates".) It wasn't too hard to prove this, luckily.

This discovery, which I knew must be as old as the hills, was a relief to me, since it somehow put the incenter back in the same league as the points I felt it deserved to be playing with. Even so, it didn't seem to play nearly as "central" a role as I felt it merited, and I was still a bit disturbed by this imbalance, almost an injustice.

Seeking to quench my avid thirst for geometrical insight, I went to a couple of superb technical bookstores with row after row of math books – but even there, all I found on geometry was a handful of rather thin volumes. In any case, I bought all the relevant books I could find, of which my favorite was *Geometry Revisited*, by H.S.M. Coxeter and Samuel Greitzer. None of these books on its own was anywhere close to definitive, but I nonetheless drank from all with great gusto. A few of these books referred to a long-out-of-print volume by Julian Lowell Coolidge, called treatise on the Circle and Sphere, published in 1916. I didn't know whether this book or any other old-timers that were occasionally cited would have anything much to say beyond

Figure 2. Excerpt from Hofstadter (1996)

presented. In this figure, each persona has been demarcated through the use of highlighting. The narrator has been identified with the use of dark highlighting, the participant with the use of light highlighting, and the mathematician with no highlighting. The presented interplay is typical of the first six pages of Hofstadter's chapter. At that point in the account, Hofstadter makes a profound discovery, which is revealed in his last use of the participant's voice. After this point, there is a brief interplay between the narrator and the mathematician and then the voice of the narrator also disappears. The last seven pages of the chapter are comprised of the mathematician articulating and proving his discovery.

As mentioned above, Hofstadter makes no explicit mention of these three personas, nor does he in any way acknowledge any intention to demarcate the different aspects of the problem-solving process. Indeed, Hofstadter may, himself, be unaware of the marked contrast between the voices that emerge from his writing. I speculate that what motivates his writing is his loyalty to preserving the integrity of the story of how he came to solve the problem while at the same time working in a paradigm in which logic and proof are the currency of practice. By existing within this tension, Hofstadter is, in part, paying close attention to the "different time dimensions - time of writing, time of the reader's progression through the text, time of events described" (Fauvel, 1988, p. 26). In so doing, he is writing in a genre that Fauvel would describe as Cartesian rhetoric, which Fauvel draws from the writing of Descartes. Fauvel characterized this form of writing as "quite another world of writer-reader relation" (p. 25), in which a "finely constructed story about the past persona (called 'I') of a narrator (also called 'I') [is] structured so as to bring out an imaginary intellectual journey" (p. 26). Although not intending to take the reader on a fictional journey, Hofstadter is certainly aware of the reader and is engaged, through his writing, in establishing a relationship among the writer, text, and reader (Fauvel), and through this relationship to convey the complexity of the mathematical problem-solving processes in general and the contributions of the extra-logical processes in particular.

Whether Hofstadter uses the personas as an explicit strategy to convey these processes or if these personas exist only as a result of a post hoc descriptive analysis of Hofstadter's writing is not clear. What is clear, however, is that the use of these personas is an effective mechanism for strengthening the relationship among Hofstadter's problem-solving process, the text, and the reader. As such, the question for this study was, can the description analysis of this piece of text be turned into a more prescriptive method for writing and, if so, how effective would such a method be? More specifically, could explicit use of the three personas improve problem-solving journaling in such a way as to make these journals more representative of the processes actually used while problem solving?

Method

Participants for this study were drawn from two different offerings of an elementary mathematics methods course (*Designs for Learning Mathematics: Elementary*) and two different offerings of a secondary mathematics methods course (*Designs for Learning Mathematics: Secondary*) taught by the author in two consecutive years. Each course ran for 13 weeks, with weekly, four-hour classes. During all four offerings of the course, participants were immersed in a problem-solving environment; that is, problems were used as a way to introduce concepts in mathematics, mathematics teaching, and mathematics learning. Problems were assigned to be worked on in class, as homework, and as a project. Each participant worked on these problems within the context of a group, but these groups were not rigid; as the weeks passed, the class became a very fluid and cohesive entity that tended to work on problems as a collective whole. Communication and interaction between participants was frequent, and whole-class discussions with the instructor were open and frank.

Throughout the course, the participants kept problem-solving journals in which they recorded their problem-solving processes. In one elementary methods course (E1, n = 34) and one secondary methods course (S1, n = 39), instructions regarding problem-solving journals were built around an insistence that the journals should reflect the problem-solving process—*Your efforts are to be recorded in a journal format detailing your progress, successes, failures, frustrations, thoughts, and reflections regarding the problem.* This was reiterated through a series of discussions on the nonlinear and collaborative nature of problem solving.

In the second elementary methods course (E2, n = 39) and the second secondary methods course (S2, n = 36), instructions regarding problem-solving journals were specifically focused on using personas in their writing. This was done over a period of four lessons, having students work on three different problems (see Appendices A and B). In the first lesson, problem #1 was to be solved, but only the solution was to be written up using as precise a mathematical language as possible (the voice of the mathematician). In the second lesson, problem #2 was to be solved, but only the story of how they arrived at the solution was to be written up (the voice of the narrator). In the third lesson, problem #3 was to be attempted, but only the feelings they experienced in attempting the problem were to be documented and subsequently presented (the voice of the participant).

These problems were carefully selected for their appeal to all three personas. For example, the Magic Addition problem used with the preservice elementary teachers in lesson #1 (Appendix A) appeals to the narrator because it has a clearly defined beginning, middle, and end. The beginning of this problem is different from other problems in that it is accompanied by the seemingly magical nature of the problem. Likewise, the ending (or solution) of this problem is very different from other problems in that it is immediately followed by the student performing the magic on someone else. As such, this problem is not only different from many other problems, but both the beginning and end are much more clearly demarcated from the middle. All this conspires to make for a good story—a story to be told. At the same time, there are affective aspects to this story that can be accentuated in the telling of the story. Finally, it is easy to articulate a solution to the problem. What is asked for, however, is that only the solution be presented. This serves to create a situation where the students feel that, although they are able to satisfy the requirements of the task, they are not satisfied with the narrow nature of the requirements. All of the problems used in this context have this same quality.

During the fourth class, the three journaling styles were discussed, and the students expressed their frustration with the constricting nature of the requirements and proposed that they should be allowed to use all three voices in their journaling. This proposal was formalized with an introduction to the three personas: the mathematician, the narrator, and the participant. From that point forth, it was explained that they were to write the remainder of their problem-solving journals using the voices of all three personas. No specifications were made as to how these voices were to be integrated or what proportions of voices were to be used.

All participants submitted their problem-solving journals for marking in week 9 or 10 of the course. These journals were not returned until the end of the course.

The Data

Aside from the problem-solving journals, all participants in this study were also asked to keep a reflective journal in which they responded to assigned prompts. These prompts varied from invitations to think about assessment to instructions to comment on curriculum. One set of prompts, given in either week 11 or 12 of the course, had them reflect on some of their problem-solving experiences. These prompts were as follows:

Reflect on your own problem-solving process.

- *How do you go about solving problems?*
- *Does it always work? If so, how often?*
- *For which problems in this course did this process work?*
- *For which didn't it, and what was it about those problems that made it so it didn't work?*

The reflective journals were submitted in week 13 of the course for marking.

Authenticity of Data

Hofstadter (1996) presented an account of his experience in solving a problem that he had set for himself. We ask: Is the account authentic? Is the experience as documented in his writing truly reflective of the process as Hofstadter experienced it? As already mentioned, problem solving is a complex activity that involves the confluence of both logical and extra-logical processes. Effective problem solving involves the oscillation between inductive and deductive logic while regulating the responses to aesthetic and intuitive sensibilities, moments of insight, and affective states. Many of these processes, by their very nature, are private phenomena and accessible only to the person experiencing it. Thus, research on these experiences must rely on post hoc accounts as data; and researchers of these experiences must live in the tension between relying on such accounts as their primary source of data and the realization that these accounts may not be authentic representations of the aforementioned phenomena.

Having stated this, however, there are ways to heighten the likelihood that the post hoc accounts of problem-solving experience are accurate representations of the process. These checks are no different than the checks that would be used with any first-person narrative data. The first of these is to check the account for verisimilitude, that is, its adherence to an understood reality. In the case of Hofstadter, for example, his account adheres closely to all aspects of mathematical discovery (Hadamard, 1945; Poincaré, 1952). Unfortunately, this could also be attributed to his familiarity with the discourse of mathematical discovery and hence does not necessarily lend itself to increasing the credibility of the account. However, the participants in this study were likely not aware of the discourse on mathematical discovery, and thus the adherence of their accounts to the discourse of discovery adds to the authenticity of the data.

A second measure of reliability comes from triangulation of the data. If different data sources correlate with each other, then there is greater likelihood that the data are authentic. In this study, triangulation comes from the redundancy of the data collected through two different forms of journaling (problem solving and reflective) at different times in the study.

The final reliability check is concerned with an analysis of what participants stand to gain through the fictionalization of their narrative accounts. For example, in this study, both types of journals (problem solving and reflective) had explicit value within the course. Initially, this was in the mark that was assigned to each journal. This, in itself, could potentially corrupt the data as participants may fabricate their journals to adhere to some perceived normative quality. However, in all cases (E1, E2, S1, S2), this value of the mark was de-emphasized in favour of the explicit and repeated instructions that the journals were to be used as authentic communicative and representational tools. Furthermore, it was emphasized that this standard of authenticity was the *only* standard by which the journals would be assessed.

Analysis

With both the problem-solving and reflective journals in hand, an analysis was conducted for each participant in which reflections on their problem-solving processes (see prompts 3 and 4 above) were compared with their relevant problem-solving journals. Of particular interest was if their after-the-fact reflections of specific processes (from their reflective journals) correlated with their in-the-moment documentation of problem-solving processes (from their problem-solving journals) and especially evidence of authentic aspects of the problem-solving process (such as collaboration, insight, intuition, and discovery) in both journals.

Results and Discussion

Given the significant difference in the nature of the participants enrolled in the elementary version from the participants enrolled in the secondary version of the course, the results have been disaggregated accordingly.

Comparison of E1 and E2

In general, participants enrolled in the elementary methods courses were quite adept at and equally receptive to journaling. There are many possible reasons for this, foremost of which is that they have experience with undergraduate courses in which writing in general and journaling in particular are more common. This includes their required prior enrolment in a *Foundations of Mathematics for Elementary School Teachers* course in which there is always a problem-solving journal assignment. Notwithstanding this comfort with journaling, the aforementioned analysis still revealed a difference between the two groups. This difference is presented in Table 1.

Although not remarkable, the results indicate greater correlation in the E2 group. At a more qualitative level, however, the difference between the two groups' journals is quite remarkable. The E2 group produced journals much richer in descriptions of the extra-logical (Dewey, 1938) processes of mathematics, such as instances of insight, intuition, and aesthetic sensitivities. This is nicely exemplified in one of Marie's (from E2) problem-solving journal entries.

> Yes! I think I have figured it out! The solution has to do with symmetry! I discovered that the cross is always divisible by 5, and I am pretty sure it is because it is symmetrical. Shoot! This doesn't necessarily work because there are other shapes like 'Tee' and 'Z' that are always divisible. Why! I really think that symmetry has something to do with it! But wait ... 'Tee' is only divisible when it is upright.

It is clear from this passage that something has occurred to her, even though it does not work out as nicely as she had hoped. From the exclamation of the participant as well as from the change in reasoning by the mathematician, it is reasonable to assume that she has had some sudden insight. However, because the voice of the narrator is missing,

TABLE 1

Correlation comparison between E1 and E2

	E1 (*n*=34)	E2 (*n*=39)
Number of participants whose reflective journals correlated with their problem-solving journals	24	33
Percentage of participants whose reflective journals correlated with their problem-solving journals	70%	85%

we need to read about this in her reflective journal to find out exactly how the idea came about.

> I had an interesting thing happen to me while I was trying to solve the Pentominoes puzzle. I was stuck on trying to figure out what the remainder was going to be just by looking at the numbers ... I couldn't possibly imagine that you could memorize all of the possible combinations. I had been working on the problem all day, and struggling with it, and had finally given up trying. I went out for the evening and came home and sat in the hot tub for about half an hour. Even though I wasn't consciously thinking about the problem, I think that the ideas were still in my head. I honestly don't know why the idea came to me ... perhaps it was because I was so relaxed and tired and not consciously struggling with the problem, but all of a sudden, I had the feeling that it wasn't about the numbers but rather about the specific configuration of the shapes ... obviously this discovery made me feel good because this idea eventually led me to the solution.

The journals of the E2 group were also much more reflective of the social and collaborative nature of the problem-solving process encouraged in class. Unlike the extra-logical processes, which are private phenomena and largely unobservable, the collaborative aspects of problem solving are visible. As such, the first level of analysis is the correlation between the problem-solving journals and what was observed in class. At this rudimentary level of analysis, the E2 group showed greater correlation as, in general, the events I read about in their journals resonated with my own recollections of their problem-solving efforts. Refining this analysis by seeking correlation with their reflective journals confirmed the initial analysis. The E2 group was more willing and/or able to articulate their collaborations within their problem-solving journals than the E1 group. This can be seen in some of Rebecca's (from E2) entries. In her reflective journal, Rebecca commented that she relied heavily on peer discussion as part of her problem-solving process.

> I have to say that the thing that works best for me is to talk problems over with other people in class. Even if they don't know the answer, just talking about it really helps. I think I did this for all of the problems, but the one it really helped on was the 'Treasure of Captain Bird' problem.

This was corroborated in her problem-solving journal in the documenting of her solution to the Treasure of Captain Bird problem.

> This problem had been driving me nuts all weekend. I tried talking to my husband about it, but he had no idea where to begin. Finally on Monday I saw Rita and we talked about the problem. She suggested that rather than look for a trick in the problem I just draw out a couple of examples. As soon as she suggested it, I knew it would lead to an answer - Rita always gives good hints. Anyway, after I drew three examples, I saw that it didn't matter where the oak tree was, the treasure ...

Comparison of S1 and S2

Participants enrolled in the S1 and S2 course offerings, by the very nature of the course, tended to have more courses in undergraduate mathematics. Thus, they had more exposure to a culture of presenting mathematical work logically rather than chronologically. The result of this exposure had been reflected in their previous problem-solving journal writings. This was very much the case for the S1 group. Their problem-solving journals were more reflective of an after-the-fact reorganization of what they had found into a mathematically sound explanation rather than an in-the-moment description of their process. Their only deviation from this was in response to my insistence that they *tell me the story of how they solved the problem*. This, more often than not, resulted in an overlay of narration on top of their logically organized solution. For example, Stan's (from S1) problem-solving entries about how he solved the Four Pocket problem reads more like a mathematical proof than a true process.

> First, I know that after a diagonal move and adjacent move I will always have an orientation of cups such as $XXXO$. I know this because, without loss of generality, and regardless of the original orientation ...

Stan's use of the word *know* as initial knowledge is immediately compromised by his ensuing explanation of how he knows it. Having watched Stan work on this problem, I know that it took Stan a very long time and a lot of dialogue with his group mates before he (they) arrived at this conclusion. This part of the process is not presented anywhere in his journals.

This propensity to write logically rather than chronologically was overcome through the use of the more structured, persona-based framework. The results presented in Table 2 show the difference in

TABLE 2

Correlation comparison between S1 and S2

	S1 (n=39)	S2 (n=36)
Number of participants whose reflective journals correlated with their problem-solving journals	14	26
Percentage of participants whose reflective journals correlated with their problem-solving journals	36%	72%

correlation between the two groups. Like the E2 group, the descriptions of problem-solving processes of the S2 group were much richer, including both the extra-logical and the social aspects of problem solving. Again, this can be exemplified by the journal entries of one S2 member. Stephan's problem-solving journal has the following entry with respect to the Pentomino Problem:

> I've got it! ... It's simple really and I've gotten it because of, believe it or not, golf! The explanation may be muddled but it makes perfect sense (in my head). In golf there are two values when keeping score: the number of shots actually taken and the number of shots relative to par ... How does this apply to the Pentominoes puzzle? ... When all the blocks are vertical, their sum divided by five will always be a whole number, no matter where they are on the number grid. These vertical blocks are par (E). If you then move a block to the right one, then your score changes to +1. If you move it left, then it changes to −1.

The nice interchange of voice among the participant, the narrator, and the mathematician continues into his reflective journal:

> The solution came right after I'd played a round of golf and I was watching golf on TV in the clubhouse. On the screen flashed a player's scorecard and I realized that the very notion of par was the solution to the Pentomino puzzle.

Comparison of Groups 1 and 2

Other than the aforementioned correlations between the two forms of journals, there were other interesting results that emerged out of the study. First, the extensive use of problem solving both as mathematical content and pedagogical context had a profoundly transformative effect on the participants' beliefs about mathematics and their beliefs about the teaching and learning of mathematics. This result was more pronounced with the preservice elementary teachers than the preservice secondary teachers. In part, these results have been reported elsewhere (Liljedahl, Rolka, & Rösken, 2007a; Liljedahl, Rolka, & Rösken, 2007b; Liljedahl, Rösken, & Rolka, 2006; Rolka, Rösken, & Liljedahl, 2006). Although the details of these results extend beyond the scope of this article, it is clear that a primary contributing factor is the role that the reflective journals played in the transformation.

More relevant to this article is the result that this transformation was also more pronounced in the groups that used persona-based journaling. That is, although both E groups had more profound transformations than the S groups, the E2 group had a more profound transformation than the E1 group. Likewise, the S2 group had a more profound transformation than the S1 group. This may be due to the fact that the persona-based method of journaling used in the problem-solving journals seemed to transfer into the reflective journals as well. This can be seen, for instance, in both Marie's and Stephan's journal entries above. These entries exemplify the intertwining that occurred between the problem-solving journal and the reflective journal in groups E2 and S2.

Conclusions

The mathematical problem-solving process is complex at best. It involves not only the logical processes of inductive and deductive logic but also the extra-logical processes of intuition, insight, aesthetics, and illumination. This process is again nicely summarized by Kleitman (Liljedahl, 2004, p. 34).

> Thus while you can turn the problem over in your mind in all ways you can think of, try to use all the methods you can recall or discover to attack it, there is really no standard approach that will solve it for you. At some stage, if you are lucky, the right combination occurs to you, and you are able to check it and use it to put an argument together. Thus in answer to your first question, deliberate endeavour is required, but it is rarely sufficient for important discoveries. ... You must try and fail by deliberate efforts, and then rely on a sudden inspiration or intuition or if you prefer to call it luck.

At the same time, problem solving involves the regulation of responses to these aforementioned phenomena.

Hofstadter (1996) offered more than mathematical insights; he offered insights into the process of mathematical problem solving. This results both from his mathematical prowess and his ability to articulate that prowess in his writing. Through an analysis of this work comes a framework for writing that is meant to help shape others' articulation of their own problem-solving processes. In this article, I examined the effectiveness of this framework in helping preservice elementary and secondary teachers articulate their problem-solving processes in general and their extra-logical and collaborative processes in particular. In this regard, the persona-based framework proved to be very effective in producing journals that correlated well with the participants' reflections on their problem-solving processes. This effectiveness was most visible within preservice secondary mathematics teachers. For both elementary and secondary preservice teachers, however, the persona-based framework facilitated the production of richer descriptions of problem-solving processes.

Implicit in this outcome is the implication that the persona-based method of journaling has for research on extra-logical processes. Methodologically, the persona-based framework could prove an invaluable tool for gathering data on these private and often unobservable phenomena. As such, it offers us a window into the very real, often tacit, but pivotal aspects of the problem-solving process. The use of such a method raises powerful questions for research on the problem-solving process. As a method, persona-based journaling is not inert. It does not quietly gather data without affecting that which it is intended to capture. Preliminary analysis of this framework indicates that it may be more than a descriptive tool—it seems to shape that which it meant to reflect. As such, further research is needed to ascertain its metacognitive influence on the problem-solving process.

If the persona-based framework does prove to be as effective a metacognitive tool as it seems to be, then further work will be needed to establish more effective pedagogical methods for developing this capacity in students of all ages. At the same time, there is also a separate, but related, line of research that can be pursued in the domain of discourse analysis. Such research would serve to add a finer layer of analysis to the research questions presented in this article.

Although the research presented in this article was motivated by questions concerning the presentation of authentic and often invisible aspects of the problem-solving process, there are other questions that can motivate this line of inquiry. In particular, questions around the assessment of problem solving have been raised by the use of this persona-based framework.

At the most basic level, assessment of any process needs to be aimed at that process. When that process is unobservable, as it often is with problem solving, then steps need to be taken to uncover the authentic process. Failure to do so will first result in a shifting of the focus of the assessment towards more concrete aspects of the process. This will be followed shortly thereafter by a shifting of the actual process to more concrete aspects. In the context of mathematical problem solving, this could manifest itself in a number of ways. For example, a teacher's inability to capture the extra-logical processes of problem solving may result in her/his over-emphasising the logical processes of problem solving in assessment instruments. This would likely materialize as a focus on correct answers or logically organized explanations. Such a focus would be perceived by students as a value statement; that is, the logically organized explanation is all that is valuable. This will result in a shifting of the students' problem-solving processes towards that which is valuable.

The persona-based method of journaling about problem-solving processes offers a way to break this cycle. Through its ability to help students articulate the authentic processes involved in problem solving, the persona-based framework can also help teachers to assess that which is authentic—they would be able to evaluate what they value.

Some research has already been done in this area and preliminary results indicate that the framework is also an effective instrument for authentic assessment. However, these results are preliminary, and much work needs to be done before any conclusions can be drawn. Such work is similar to the aforementioned work on the framework as a research methodology in that it seeks answers to many of the same questions. In particular, when used for assessment, how does the persona-based framework shape the problem-solving process?

Appendix A

Problems for E2

Problem #1 (Magic Addition). Write down any three 3-digit numbers as if you are going to add them up using column addition. I will then add two 3-digit numbers to this list and instantly tell you what the sum is. For example: you provide the numbers 271, 742, and 836. I will provide the numbers 728 and 257 and instantly tell you that the sum of all five is 2834. How do I do it so quickly?

Problem #2 (The Treasure of Captain Bird). The treasure of Captain Bird lies buried on the island of the parrot. Near the centre of this island three great trees form a triangle. The mightiest of the three is a great oak older than the treasure itself. Towards the west of the oak, some distance away there stands an elm tree, and towards the east of the oak there stands an ash. To find the treasure of Captain Bird, count out the paces from the oak to the elm. When you get to the elm, make a precise left turn and count out the same number of paces. Mark this spot with a flag. Return to the oak and count out the paces from the oak to the ash. When you get to the ash, make a precise right turn and count out the same number of paces. Mark this spot with a flag. The treasure lays buried midway between the two flags. So, you rent a boat and set out for the island. When you get there, however, you discover that the oak tree is missing without a trace. Where is the treasure? Why is it there?

Problem #3 (Psychic Math). Figure out how the Flash Mind Reader works. It can be found at http://www.albinoblacksheep.com/flash/mind.php.

Project (Pentomino Problem). A pentomino is a shape that is created by the joining of five squares such that every square touches at least one other square along a full edge. How many are there? Name them. If a pentomino is placed on the number grid, will the sum of the numbers it covers up be divisible by 5? When will it? When will it not? If not, what will the remainder be? Why?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Appendix B

Problems for S2

Problem #1 (Corner to Corner). An N×N array has a red marker in every cell except for two. One corner of the array is left empty and the corner furthest from this empty cell has a white marker in it. What is the smallest number of moves required to get the white marker into the initially empty cell given that the only valid move is to move a marker into an adjacent empty cell (that is, not kitty-corner)?

Problem #2 (Four Pockets). A round table has four deep pockets equally spaced around its perimeter. There is a cup in each pocket oriented either up or down, but you cannot see which. The goal of the game is to get all the cups facing up or all the cups facing down. You do this by reaching into any two pockets, feeling the orientation of the cups, and then doing something with them (you can flip one, both, or none). However, as soon as you take your hands out of the pockets, the table spins in such a way that you can't keep track of where the pockets you have visited are. If the four cups ever get oriented all up or all down, a bell rings to signal you are done. Can you guarantee that you will get the bell to ring in a finite number of moves and, if so, in how many moves?

Problem #3 (Student Moves). The desks of 25 students are arranged in a 5×5 array. The teacher comes in and tells the 25 students that everyone has to change to a different desk. The stipulation is, however, that they are only allowed to move into an adjacent seat (not kitty-corner) and everyone must move. Can it be done?

Project (Pentomino Problem). A pentomino is a shape that is created by the joining of five squares such that every square touches at least one other square along a full edge. How many are there? Name them. If a pentomino is placed on the number grid, will the sum of the numbers it covers up be divisible by 5? When will it? When will it not? If not, what will the remainder be? Why?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

REFERENCES

Borwein, P. & Jörgenson, L. (2001). Visible structures in number theory. *MAA Monthly, 2001*.

Burns, M. & Silbey, R. (2001). Math journals boost real learning. *Instructor, 110*(7), 18–20.

Chapman, K. (1996). Journals: Pathways to thinking in second-year algebra. *The Mathematics Teacher, 89*(7), 588–590.

Ciochine, J., & Polivka, G. (1997). The missing link? Writing in mathematics class! *Mathematics Teaching in the Middle School, 2*(5), 316–320.

Dewey, J. (1938). *Logic: The theory of inquiry*. New York: Henry Holt.

Dougherty, B. (1996). The write way: A look at journal writing in first-year algebra. *The Mathematics Teacher, 89*(7), 556–560.

Fauvel, J. (1988). Cartesian and Euclidean rhetoric. *For the Learning of Mathematics, 8*(1), 25–29.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht, The Netherlands: Kluwer.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Dover.

Hofstadter, D. (1996). Discovery and dissection of a geometric gem. In J. King & D. Schattschneider (Eds.), *Geometry turned on: Dynamic software in learning, teaching, and research* (pp. 3–14). Washington, DC: The Mathematical Association of America.

Liljedahl, P. (2004). *The AHA! experience: Mathematical contexts, pedagogical implications.* Doctoral Dissertation, Simon Fraser University, Burnaby, British Columbia, Canada.

Liljedahl, P., Rösken, B. & Rolka, K. (2006). Documenting changes in pre-service elementary school teachers' beliefs: Attending to different aspects. In S. Alatorre, J. L. Cortina, M. Sáiz, A. Méndez (Eds.), *Proceedings of the 28th international conference for Psychology of Mathematics Education - North American Chapter* (vol. 2, pp. 279–285). Mérida, Mexico: Universidad Pedagógica Nacional.

Liljedahl, P., Rolka, K. & Rösken, B. (2007a). Affecting affect: The re-education of preservice teachers' beliefs about mathematics and mathematics learning and teaching. In M. Strutchens & W. Martin (Eds.) *69th NCTM Yearbook* (pp.319–330). Reston, VA: National Council of Teachers of Mathematics.

Liljedahl, P., Rolka, K. & Rösken, B. (2007b). *Thinking about belief change as conceptual change.* Paper presented at the 5th congress of the European Society for Research in Mathematics Education, Larnaca, Cyprus.

Mewborn, D. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for Research in Mathematics Education, 30*(3), 316–341.

Miller, L. (1992). Teacher benefit from using impromptu writing in algebra classes. *Journal of Mathematics Teacher Education, 23*(4), 329–340.

Poincaré, H. (1952). *Science and method*. New York: Dover.

Rolka, K., Rösken, B. & Liljedahl, P. (2006). Challenging the mathematical beliefs of preservice elementary school teachers. *Proceedings of the 30th international conference for Psychology of Mathematics Education* (vol. 4, pp. 441–448). Prague, Czech Republic: Charles University.

Waywood, A. (1992). Journal writing and learning mathematics. *For the Learning of Mathematics, 12*(2), 34–43.

Peter Liljedahl
Faculty of Education
Simon Fraser University
8888 University Drive, Burnaby, BC V5A 1S6, Canada
E-mail: liljedahl@sfu.ca