Experiencing Mathematics through Problem Solving Tasks

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Approval

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Abstract

Learning through problem solving is an old concept that has been redeveloped as a valuable strategy to teach mathematics. Many teachers feel a tension between the value of teaching through problem solving and the necessity of teaching a prescribed curriculum[1], often resulting in minimizing the time students spend on genuine problem solving. The purpose of this thesis was to investigate the extent that a mathematics student encounters curriculum while working freely on problem solving tasks. A student in a Pre-Calculus and Foundations Math 10 course, which already had a culture of thinking and problem solving, was observed for a 1-month period to see what mathematical content they engaged with through problem solving. Observations, photographs, and notes were taken about the tasks and the mathematics that the student encountered during problem solving each day. The variety of tasks was very broad to prevent students from assuming a problem solving strategy based a current unit of study. Through analysis of the content one student engaged with, it was found that almost the entirety of the FPM10 prescribed learning outcomes was encountered[2] in addition to both a review of some curricular content from Math 6 through Math 9, as well as exposure to curricular content from Math 11 and 12.

Keywords: mathematics, education, thinking classroom, problem based learning

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[1] The term curriculum has a variable meaning for both myself and the ministry of education depending on the context. In some cases it refers to the specific list of learning outcomes for a particular course, and sometimes it also includes the big ideas of mathematics such as problem solving and communicating mathematics.

[2] A learning outcome that a student has seen, used, or experienced in some way.

To my parents, who have always encouraged life-long learning,

and my sister, Nelleke.

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List of Acronyms

NCTM	National Council of Teachers in Mathematics
FPM10	Pre Calculus and Foundations of Mathematics 10
LO	Learning Outcome
WAM10	Workplace and Apprenticeship Mathematics 10
WNCP	Western and Northern Canadian Protocol

Chapter 1. Introduction

1.1. Fascinated by Mathematics …Or Not?

Stepping out of the circle, I watched as each team of students sat cross-legged on the ground and started passing oranges back and forth. I was working as a *ScienceRocks!* camp leader for kids from ages 9 to 12 at the University of the Fraser Valley. Each camp week had a different focus, and that week was all about Mathematics. The Orange Game[3] had them working together to solve a sorting problem. It was unbelievable that these children were not only engaging in mathematical activities for a whole week in the middle of their summer holidays, but they absolutely loved it. Some of them had their heels dug in and scowls on their faces when their parents dropped them off at the university on the first day of Summer Math Camp. By the end of the day, however, these same children had smiles on their faces and, at the end of the week, they were begging their parents to let them stay for another week! "I wish school math could be this much fun!" one boy said to me during lunch near the end of the week. "I know, you're right!" I agreed with him, "I wish that too!"

It became clear to me that nine to 12-year-old children could be completely fascinated by mathematics. This opened my mind to huge possibilities! What changes could be made in mathematics education so that every student could experience that same fascination during math lessons in school? Before my teacher training, I had a part-time position teaching Grade 9 mathematics. This was my chance to show students how entertaining and amazing mathematics could really be. Each lesson started with a

[3] The Orange Game: https://www.ncwit.org/sites/default/files/resources/computerscience-in-a-box.pdf

mini problem-solving question. I alternated lessons between abstract and hands on. I collaborated with the art teacher for a major problem-based learning assignment that had students using an imaginary million dollars to spend on designing their dream home on an actual plot of land available for sale. The students rolled their eyes and complained every step of the way. Why were they not enjoying this? I was confused. Here I had developed a way to cover the curriculum in the most enjoyable and meaningful manner I could dream up, and all they did was whine about it. It must be me, I figured. Maybe I am not meant to be a teacher. That experience almost killed my passion for teaching before I even started the teacher-training program.

1.2. Search for the key

The thought still fizzled in the back of my head as I finished my degrees and completed my teacher training. *Students are still sitting in mathematics classrooms, hating it and fearing it.* Something needed to be done to change this sentiment! I remembered back on my first day in university when my calculus instructor told the class, 'Learning mathematics is like learning to play piano: the only way to improve is to actually practice'. I started thinking that this analogy could be extended much further. A student can learn to play piano beautifully without knowing any music theory. Can students do mathematics without all the theory? What would that look like? Studying music theory without ever playing music is quite purposeless; so is learning about mathematics theoretically, without doing mathematics, also purposeless? Is music theory even important at all? An avid violin player, my niece told me that, while studying music theory, concepts would often come up that were directly connected to songs she was learning, and she found it helped her better understand and perform the music! She recognized the importance of theory. This is what seemed to be missing in my mathematics classes. While I taught my students facts, algorithms, and conventions in mathematics, they were rarely engaging in mathematical tasks where they genuinely needed to apply the skills they learned to solve mathematical problems. Maybe the students were never getting to experience the joy of making connections between concepts and discovering solutions when the lessons were focused on the facts and algorithms of mathematics.

Once I completed my Bachelor of Education and started teaching mathematics to both high school and middle school students, I started exposing students to the deeper world of mathematics whenever I could, through history of mathematics, fractals, graph theory, puzzle mathematics, Fibonacci and classic probability games! I experimented with idea after idea in my mathematics classes: changing the assessment, incorporating projects, learning through games, writing in mathematics journals, and anything I could glean from books and professional development sessions. While I did everything I could on my ‘teacher stage’ to unveil the amazing-ness of mathematics, students were still walking away with little appreciation for mathematics. They still came in asking, “Are we going to do something fun today?” It was all about the entertainment because they were not being entertained by the mathematics itself. The incredible mathematics was only seen as incredible through my eyes. Students were not seeing or enjoying activities because mathematics itself intrigued them, but because they liked puzzles and games that made it feel less like mathematics. I think this is because they were not creating or discovering any mathematics themselves! Every engaging activity I planned for them was much like a trip where I did all the planning and they simply went along for the ride. Or, like a music teacher who performed beautiful music for her students or got them VIP tickets to incredible concerts, but never gave them the opportunity to play any music themselves besides practicing their scales. Although my students preferred hands on activities and games more than a lecture, something significant was still missing.

Amazing discoveries are not quite as amazing when they are not discovered personally. I wanted them to experience a sense of wonder, fascination, and perplexity about big ideas and concepts in mathematics.

During teacher education, designing well-organized course overviews, unit plans and lesson plans that were carefully aligned to the curriculum and ensured differentiation of abilities seemed to be the main goal. Timing in lessons was considered critical, because it was important to be sure that every concept was covered, nothing was missed, and that students were engaged. But now, no matter how well I planned my lessons, I could never stick to the plans I had made while I was actually working with the students. No matter how well I planned, I also started noticing that the best learning moments, and the most successful tasks, were the spontaneous and unplanned tasks! Could this perhaps be the key to authentically engaging students in mathematics?

1.3. A New Approach

As I started out in the Secondary Mathematics Master's Program at SFU, I began toying with a new idea that grew out of the readings and discussions. What would happen if I just let go? If I were to provide students with learning opportunities and just see where it would take them mathematically? Would students really learn mathematics if it wasn't forced? Would it be possible to teach the whole course this way? Throughout the first year of the program, I started changing my classroom from a teacher-centered environment into a more learning-centered environment using ideas from Peter Liljedahl's research on *Building Thinking Classrooms* (Liljedahl, 2016). The purpose was to have the students engage in mathematics in order to obtain a deeper understanding of mathematics. I started putting students into random groups every lesson to work on non-curricular problem-solving tasks. Although some students in my smaller classes already used whiteboards regularly to work out solutions to mathematics and physics problems, I started getting all my students to work on whiteboards more regularly for problem-solving questions. Students showed mixed feelings about the approach and were often frustrated as I shifted back and forth between a teacher-centered and student-centered environment. My motivation to increase the level of my students' engagement in mathematics was founded on my desire to have students understand and enjoy the mathematics they are required to learn by the government. I felt that the solution to this problem was to completely convert my classroom into a problem-based learning environment. A problem-based learning environment should give room for student autonomy, deeper understanding in mathematics, and an atmosphere that fosters problem solving.

1.4. Starting Over

With the start of a new school year coming up and a new group of students that I had never taught before, I decided to jump in with both feet and just see what happened. Instead of plowing through the curriculum with the students that could keep up, while making the ride as tolerable as possible for those who couldn't keep up, I would allow all the students to experience the mathematics. During the first month, I worked on enacting Liljedahl's (2016) Building Thinking Classroom framework[4]. The students worked in random groups every day on a non-curricular mathematical task. To prevent them from settling strongly into old norms of a traditional mathematics classroom, I regularly changed my furniture layout and the tools they could use to investigate and solve a problem. I alternated between allowing them to use vertical whiteboards,

[4] Further details about the Thinking Classroom framework will be discussed in Chapter 2 Section 2.3.3.

portable table-top whiteboards, pen and paper, or only oral communication to solve a problem. A few times students asked me, “So, when are we going to start doing math?” to which I always responded, “This is mathematics! Isn’t it great?!” Although they may have rolled their eyes a little, my students started to realize that whatever was going on in mathematics class was there to stay, and not just a fun introduction week.

Eventually I started incorporating mathematics tasks that related to the curriculum into the classes. Over the first few months of the year, the students increased their problem-solving endurance. As the school year progressed, they learned to just experiment and play with the mathematics and to start asking their own questions. Within a couple months, they were able to work on a single challenging mathematics problem for an entire block. I learned how to incorporate the required curricular mathematics concepts into their problem-solving ideas through mini lessons which I taught to small groups of students or even the whole class. These mini lessons became tools to help them move forward in their problem solving.

This class was becoming like none I had ever had before. The students were actively engaged in mathematics and problem solving every day. They were learning mathematics through problem solving. It was as though the mathematics curriculum supported the problem-solving tasks rather than the tasks or activities supporting the mathematics curriculum. In previous years, I had incorporated rich mathematical problems and engaging mathematical activities into my lessons, but they were often disconnected from the mathematics and the students perceived problems as miserable hard work and the activities as non-mathematical games. The mathematics itself wasn’t appreciated. Now, as I let students discover more on their own and as they developed their problem solving and communication skills, it became clear to me that the students were starting to recognize that mathematics itself could be interesting, without the façade of a game.

1.5. Will it work?

As I journeyed with my students through this new way of teaching and learning, I wondered if I could keep it up. When I posed a problem to solve, I never knew exactly what to expect when students started solving it. I was surprised at their ingenuity when solving problems where they had never learned the theory or the most efficient procedure for that problem type. Could I justify teaching my students an entire school year through problem-solving tasks? Mathematics lessons were much more enjoyable, but would it really work for them to meet the curricular requirements from the Ministry of Education? If they solved only one or two problems during class time, I was concerned that we would not be able to finish the entire course before the end of the school year. Although I could see the students were actively working towards meeting the main goals of mathematics education (See Figure 1.1), I was not sure about the feasibility of them learning the entire curriculum for the course. Still, I was convinced that teaching with a problem-based approach would be more effective than teaching the students more traditionally. I was interested to know if an entire curriculum could be taught through problem solving. The purpose of the research in this thesis was to determine the amount of mathematical content from the curriculum that a student uses, sees, learns, or experiences while problem solving.

Mathematics education must prepare students to use mathematics confidently to solve problems.

The main goals of mathematics education are to prepare students to:

- solve problems
- communicate and reason mathematically
- make connections between mathematics and its applications
- become mathematically literate
- appreciate and value mathematics
- make informed decisions as contributors to society.

Students who have met these goals:

- gain an understanding and appreciation of the role of mathematics in society
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical problem solving
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics.

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through:

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history.

Figure 1.1 WNCP Common Curriculum Framework for Grades 10–12 Mathematics

Chapter 2. **Literature**

The only way to learn mathematics is to do mathematics.

-Paul Halmos (1982, p. vii)

The only way to learn mathematics, as Halmos (1982) stated, is to do mathematics. This seems to be a fairly obvious statement, true in learning almost anything. How can one learn to play tennis, play a piano, or even ride a bike without ever doing any of these activities? Observing others performing a task can help in the learning process, but ultimately, it is by doing or participating in the task that one learns and becomes more proficient with the task. A few questions come to mind on reading this quote by Halmos: What does it mean to do mathematics? What does it mean to learn mathematics? And lastly, how can mathematics be taught so that students learn mathematics?

In what follows I address these questions by looking at related literature. First, I address what it means to do mathematics, with a focus on mathematical problem solving. Next, I review how students can learn mathematics by doing mathematics, with a focus on learning through problem solving. I then describe some of the research about how and why mathematics can or should be taught through problem solving. Finally, I discuss how the literature leads into my research question.

2.1. Doing Mathematics

2.1.1. What do mathematicians do?

According to the Merriam-Webster Dictionary, Mathematics is "*the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, transformations, and generalizations" (*mathematics, Merriam-Webster, 2017). A mathematician is by definition, a specialist or expert in mathematics (mathematician, Merriam-Webster, 2017). According to Schoener (2016), "mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engage in the science of patterns – systematic attempts, based on observation, study; and experimentation to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics")" (p. 60). He considers the operations and notations of mathematics referred to in the definition above to be the 'tools of mathematics' (Schoener, 1992).

Just using the tools of mathematics does not make a person a mathematician. Schoener (1992) makes an analogy similar to the piano analogy referred to in the introduction. It is essential for a craftsman in carpentry to know how to use a hammer and other such tools, but just knowing how to use a hammer does not make a person a craftsman in carpentry (Schoener, 1992). In addition to learning the tools and rules of mathematics, a mathematician works actively in the science of patterns. This would be evident in a mathematician exploring patterns and mathematical ideas, which build on the knowledge-base of mathematics, or to engage in problem solving. The scope of work in the field of mathematics generally lies between the spectrum of pure mathematics and applied mathematics. But, regardless of where on the spectrum a mathematician lies, his or her work can be summarized as problem solving.

> Imre Lakatos, mathematician and philosopher, describes mathematical work as 'a process of conscious guessing about relationships among quantities and shapes.' Problem solving is at the core of mathematics, and it starts with making a guess. … After making a guess, mathematicians engage in a zigzagging process of conjecturing, refining with counter-examples, and then proving (Boaler, 2008, p. 25).

2.1.2. What is a mathematical problem?

> A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide post on the mazy paths to hidden truths, and ultimately a reminder of our pleasure in the successful solution. (Hilbert, 1900, p. 241)

Although not a definition, David Hilbert's description of a mathematical problem gives insight into the possibilities of a rich problem in mathematics. This description seems to state that richness of a mathematical problem is somewhat dependent on the individual encountering the problem. There is much that can be said about this relation between the mathematical problem and the process of solving it, as what is considered a great problem to one person, is of no consequence at all to another. A *problem* by definition, according to the Merriam Webster Dictionary, is as follows:

1. *a question raised for inquiry, consideration, or solution; or a proposition in mathematics or physics stating something to be done.*
2. *an intricate unsettled question; a source of perplexity, distress or vexation; or difficulty in understanding or accepting.*

Either of these definitions could be accurately used in terms of teaching mathematics. In mathematics, the first definition for a problem can be applied to anything from a basic arithmetic exercise to a complex unsolved situation. The second definition is more in line with Hilbert's description because it suggests that there should be an element of challenge for a mathematical task to be a problem. Although this official definition is related to any sort of problem, not specifically mathematics, people use the two definitions interchangeably in mathematics.

Wilson, Fernandez, and Hadaway (1993) bring up this challenge about defining a mathematics problem in their research on problem solving. They explore the idea that two people in a discussion about mathematical problem solving are not always on the same page as to just what exactly it is (Wilson et al, 1993). Even earlier, Henderson and Pingry (1953) discuss the definition of a mathematical problem. Although it was common for a teacher to assign students to 'work out the problems' from a textbook where 'the question in each problem was either implicitly or explicitly "What is the answer?"' (Henderson & Pingry, 1953, p. 228). This is very much in accordance to the first definition given above for a problem. They came to conclude that a second concept, that of the problem in relation to the ability of the individual facing the problem, would also come into play. Through an analysis of the problem-solving process for a particular individual, they determine three conditions that make a task, process or question into a problem for an individual. These conditions are:

1. *"The individual has a clearly defined goal and desires to attain it.*

2. *Something blocks the path towards the goal, and the individual has fixed patterns of behavior or habitual responses that are not sufficient for removing the block.*
3. *Deliberation takes place. The individual becomes aware of the problem, defines it more or less clearly, and identifies various solutions, which are then tested for feasibility." (Henderson & Pingry, 1953, p. 230)*

Given these conditions for problem solving, Henderson and Pingry (1953) choose to define a mathematical problem as a problem if and when it requires the individual solving it to be faced with those conditions. So, a 'textbook problem' is not necessarily a real mathematical problem for a student unless the student understands and desires to attain the solution and also faces some challenge in solving it. A problem is a problem only when it is a problem!

In 1980, the National Council of Supervisors of Mathematics said that the whole principle of mathematics is to solve problems so, if that is the case, it is rather important to clarify that it can only mean to solve a problem! To be clear, I consider a task to be a mathematical problem when it has elements of both definitions. This is in line with both Henderson and Pingry (1943), and Wilson et al (1993), and ties together well with Hilbert's description of a problem. Therefore, for the purposes of this study, I define a problem as a question or proposition in mathematics raised for inquiry, consideration or solution, in addition to having an element of difficulty or perplexity. If the mathematical task has no element of difficulty or perplexity to the individual working it out, the task will be referred to as an exercise.

2.1.3. What is mathematical problem solving?

With the definition of mathematical problem in place, what it means to problem solve becomes clearer. Henderson and Pingry (1953) settled on three necessary conditions for problem solving, as listed above. These conditions are in line with what Polya (1957) and Mason, Burton and Stacey (2010) discuss in “How to Solve it” and “Thinking Mathematically” respectively. The first condition demands that the individual has a desire to solve the problem and has a clear goal. Polya (1957) begins with advising a student to begin with understanding what the problem is asking, and Mason et al (2010) suggests ways to make some sort of start on any required task or problem of interest. This is followed by a period of being stuck. Polya and Mason et al, have a series of possible strategies that can help an individual or student to overcome the challenges of the problems and make a fresh attack towards a solution or towards a new block in the problem. Mathematical problem solving is the process of actively working towards a goal or solution and deliberating about a different strategy of solving when faced with a period of being stuck.

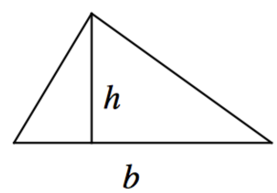
2.1.4. What does it mean for a student to do mathematics?

The phrase ‘to do mathematics’ uses the verb ‘do’ as an action verb rather than an auxiliary verb. Verbs such as accomplish, achieve, conclude, create, determine, perform, and create appear in the list of words that are synonymous to *do*. For a student to do mathematics, the student would be required to actively participate in solving mathematical problems and making senses of mathematical ideas. In *Thinking Mathematically,* Mason et al (2010) demonstrate how a student can do mathematics strategically. The entire book is organized around the phases of work in doing mathematics and thinking mathematically. Each strategy for thinking mathematically is closely connected with problem solving and shows how these processes of thinking are at the heart of mathematics. Given a problem or task, a student would be doing mathematics by specializing or generalizing the situation to get a better understanding of the situation and finding ways to make and prove conjectures that can be applied in more contexts and situations. Doing mathematics, in essence, the act of solving problems.

The three phases of work in doing mathematics to improve one's skills in mathematical thinking ability are entry, attack, and review (Mason et al, 2010). During the entry phase, students should make sense of a problem, asking questions about what they know, what they want, and what they can introduce into the question. During this stage of problem solving, Mason et al (2010) recommend that students specialize the problem into a simpler case if they are stuck, so that they can better understand the mathematics. Then the students would move towards an attack stage, where the students would be making generalizations and conjectures about the problem that they are working on, by proving and disproving. The final stage of working on a mathematics problem is to review. This stage includes checking if conjectures are reasonable, comparing solutions to the original task, and extending the task to more general cases. During each of the stages, students working on a problem would be thinking and working actively on the task and therefore would be doing mathematics.

Mason's (2010) strategies for improving one's ability to think mathematically are similar to Polya's (1957) problem solving method in *How to Solve it*. Since problem solving is at the core of mathematics, a student doing mathematics would be actively problem solving on a regular basis. Such a student would be learning and participating in the art of problem solving, not as an algorithm or recipe to obtaining answers, but in a process that engages them in the various phases of work leading towards general solutions and sense making.

A mathematics student would need to be actively engaging in mathematical thinking and problem solving. If a student is presented with mathematical problems and solutions, and not does not engage mindfully in the solving process (e.g. rote calculations), that student would only be passively experiencing mathematics. Although a student could learn a little about mathematics by passively watching another person do mathematics, that student would not actually be doing mathematics. This is similar to how a student could learn about music by watching an expert pianist perform, but would not actually be playing piano in that moment. In *A Mathematician's Lament,* Paul Lockhart (2009) writes of passive mathematics when students are not given any mathematics to do.

> "In place of a simple and natural question about shapes, and a creative and rewarding process of invention and discovery, students are treated to this:
>
> ***Triangle Area Formula:***
>
> $A = 1/2\ bh$
>
> 
>
>
> "The area of a triangle is equal to one-half its base times its height." Students are asked to memorize this formula and then "apply" it over and over in the "exercises." Gone is the thrill, the joy, even the pain and frustration of the creative act. There is not even a *problem* anymore. The question has been asked and answered at the same time— there is nothing left for the student to do." (Lockhart, 2009, p. 5)

As Wilson (1993) says, problem solving is a major part of mathematics. It is the sum and substance of our discipline and to reduce the discipline to a set of exercises and skills devoid of problem solving is misrepresenting mathematics as a discipline and shortchanging the students. A solution to moving students from being passive in mathematics to becoming actively engaged in mathematics is given by the National Research Council.

> "Mathematics is a living subject which seeks to understand patterns that permeate both the world around us and the mind within us. Although the language of mathematics is based on rules that must be learned, it is important for motivation that students move beyond rules to be able to express things in the language of mathematics. This transformation suggests changes both in curricular content and instructional style. It involves renewed effort to focus on:
>
> - Seeking solutions, not just memorizing procedures;
> - Exploring patterns, not just memorizing formulas;
> - Formulating conjectures, not just doing exercises.
>
> As teaching begins to reflect these emphases, students will have opportunities to study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized. They will be encouraged to see mathematics as a science, not as a canon, and to recognize that mathematics is really about patterns and not merely about numbers." (National Research Council, 1989, p. 84)

Here it can be seen that there is more to doing mathematics than just completing practice exercises and memorization, but doing mathematics is the act of seeking solutions, exploring patterns, and formulating conjectures. This is exactly what Mason et al (2010) and Polya (1957) explain in their respective books about problem solving.

2.2. Learning Mathematics by Problem Solving

2.2.1. What does it mean to learn mathematics?

"To learn is to gain knowledge or understanding of or skill by study, experience, or being taught" (learn, Merriam-Webster, 2017). A student in mathematics would be gaining knowledge about numbers and their operations, making connections and generalizations, and would improve their skill and understanding of shapes and space to make abstractions out of their configurations, measurements, transformations, and generalizations. Since problem solving is at the heart of mathematics, a student learning mathematics would be gaining an understanding of and skill in solving problems. Everything about learning mathematics would be to increase the students understanding about all things connected to number and space; and to improve their skill in solving mathematical problems and communicating solutions. A mathematics classroom should in some way encapsulate the art of problem solving in mathematics in order for students to learn about the true nature of mathematics.

2.2.2. What does it mean to learn mathematics by problem solving?

In the definition for learning given above, it states that mathematics can be learned through study, experience, or by being taught. It is possible for mathematics to be learned more passively, by observing another problem solving, by listening to explanations and instructions, or by mimicking algorithms. But in learning mathematics only through rote memorization and completing mathematical algorithms and exercises, a student does not get to experience the nature of mathematics.

> "Students who are taught using passive approaches do not engage in sense making, reasoning, or thought (acts that are critical to an effective use of

> mathematics), and they do not view themselves as active problem solvers. This passive approach, which characterizes math teaching in America, is widespread and ineffective" (Boaler, 2008, p. 40).

For a student to learn mathematics by doing mathematics would mean they would learn the skills and knowledge of mathematics while actively working on a mathematical task or problem. The mathematical task would be a genuine problem where they would need to make decisions and use mathematical tools. The students would learn to use the tools of the mathematics while working on bigger problems without mimicking or observing a teacher's procedure or strategy as an exercise.

2.3. Teaching Mathematics through problem solving

2.3.1. What is problem-based learning?

Problem-based learning is a method of instruction where students learn new concepts during the process of actively working on problems. As Barrows (1980) writes, 'problem-based learning can be defined best as the learning that results from the process of working toward the understanding or resolution of a problem' (p. 18). The problem is generally not a task that has students practicing a skill they have already learned about, but is an authentic problem that refers to 'an unsettled, puzzling, unsolved issue that needs to be resolved' (Barrows, 1980). This method of instruction was first developed in medical education in the 1950s, and was further developed and officially integrated into a medical program by the McMaster University Faculty of Health Sciences in the 1970s where the problem-based approach was used throughout the three-year curriculum (Barrows, 1996; Hung, Jonassen, & Liu, 2008). With its success in the medical field, the problem-based learning model has been implemented throughout various fields of higher level education and in K-12 education (Hung et al, 2008). The core model of problem-based learning in the field of medicine developed at McMaster has the following characteristics:

1. **Learning is Student–Centered**. Under the guidance of a tutor [or teacher], the students must take responsibility for their own learning, identifying what they need to know to better understand and manage the problem on which they are working and determining where they will get that information. This allows each student to personalize learning to concentrate on areas of limited knowledge or understanding, and to pursue areas of interest.
2. **Learning Occurs in Small Student Groups**. In most of the early PBL medical schools, groups were made up of five to nine students. Generally, the students are re-sorted randomly into new groups with a new tutor at the end of each curricular unit to give them practice working intensely and effectively with a variety of different people.
3. **Teachers are Facilitators or Guides**. At McMaster, the group facilitator was referred to as a *tutor*. It was someone who did not give students a lecture or factual information, did not tell the students whether they were right or wrong in their thinking, and did not tell them what they ought to study or read. The tutor asks students the kinds of questions that they should be asking themselves to better understand and manage the problem. Eventually the students take on this role themselves, challenging each other.
4. **Problems Form the Organizing Focus and Stimulus for Learning**. In PBL for medicine, a patient problem or community health problem is presented in some format, such as a written case, computer simulation, or videotape. It represents the challenge students will face in practice and provides the relevance and motivation for learning. By attempting to understand the

problem, students realize what they will need to learn from the basic sciences. The problem thus gives them a focus for integrating information from many disciplines, which facilitates later recall and application to future patient problems.

5. **Problems Are a Vehicle for the Development of Clinical Problem-Solving Skills.** For this to happen, the problem format must present the patient problem in the same way that it occurs in the real world, with only the patient presenting complaints or symptoms. The format should also permit the students to ask the patient questions, carry out physical examinations, and order laboratory tests.
6. **New Information is Acquired Through Self-Directed Learning**. As a corollary to the characteristics already described (the student-centered curriculum and the teacher as facilitator of learning), the students are expected to learn from the world's knowledge and accumulated expertise by virtue of their own study and research, just as real practitioners do. During this self-directed learning, students work together, discussing, comparing, reviewing, and debating what they have learned. (Barrows, 1996, p. 5-6)

These characteristics of problem-based learning in mathematics education would only look slightly different, in that the problems would relate to content and problem-solving skills in mathematics, rather than patients and clinical problem-solving skills.

2.3.2. What is the value of problem-based learning?

The value of problem-based learning in mathematics is that students experience mathematical problem solving in a more meaningful way, and are more actively engaged in increasing their understanding of mathematical content. "Polya takes it as a given that for students to gain a sense of the mathematical enterprise, their experience with mathematics must be consistent with the way mathematics is *done"* (Schoenfeld, 2009, p. 339). Schoenfeld argues in *Learning to Think Mathematically: Problem Solving, Metacognition, and Sense-Making in Mathematics* that 'students develop their sense of mathematics – and thus how they *use* mathematics—from their experiences with mathematics (largely in the classroom)' (2009, p. 339) and therefore 'that classroom mathematics must mirror this sense of mathematics as a sense-making activity, if students are to come to understand and use mathematics in meaningful ways' (p. 340). The National Council of Teachers of Mathematics made recommendations to make problem solving the focus of school mathematics in the 1980s for a few reasons, including the fact that it is the sum and substance of the discipline and to reduce mathematics to a set of exercises and skills misrepresents mathematics and shortchanges students (NCTM, 1980, 23-24). As Wilson states, the art of problem solving is the heart of mathematics, thus mathematics instruction should be designed so that students experience mathematics as problem solving (1993).

Researchers such as Boaler have noticed that students can spend years learning mathematics in the classroom, but are still unable to apply the mathematical skills they learned to solve mathematical problems outside of the classroom setting (2002). This calls teachers to consider what sense their students are actually making of the mathematics they are being exposed to in the classroom. Mason asks us to notice what students are really attending to while they are working on a mathematical task (1993). The students might not be aware of the connections between the problem and the situation, or between different solutions or methods to solve the same problem because they are not attending to the general idea but particular examples or procedures. 'What is going on inside their heads?' Mason says, is endemic to teaching (p. 76, 1993). It is suggested that students who are expected to solve concept rich problems, where failure is a part of the learning process, are more likely to be successful when learning and applying mathematics (Ben-Hur, 2006). A concept rich problem is a mathematical problem solving task that results in students using a variety of mathematical concepts and problem solving strategies to determine a solution. Concept rich problems require students to understand the mathematical concepts they use since the tasks are too complex to just mimic an algorithm.

Boaler (2002) conducted a three-year longitudinal study on students learning mathematics in England, in two schools, to compare the impact of a more traditional learning environment and a reform problem-based learning environment. She concluded that, although the students in the traditional non-constructivist school worked hard, they were disadvantaged when it came to real world mathematics. They had more difficulty applying the mathematical concepts than the students in the reform school who had learned mathematics through problem solving. Their belief of mathematics was that it was very procedural and the concepts and solution strategies needed to be memorized for them to be successful in mathematics. The students at the reform school worked on open-ended problems and had the freedom to work with partners. At first glance, the class often had the appearance of chaos with the students off task and chatting. Yet, after three years, the students not only enjoyed mathematics more, had a better understanding of mathematics and were better at problem solving; they were equally or more successful on standardized testing.

Barrows (1996) responds to the question '*Is Problem-Based Learning Worth the Trouble?' i*n his article on about using problem-based learning in higher education. He states that it is usually teachers who have never seen PBL in action that raise this question and responds that without researching the value of PBL it seems that anyone who has the opportunity to be involved with PBL usually becomes a convert to this methodology (Barrows, 1996).

> Faculty members can see how students think, what they know, and how they are learning. This allows teachers to intervene early with students having trouble before it becomes a more difficulty issue. Faculty members work with alert, motivated, turned-on minds in a collegial manner that has no equal. This is quite different from lecturing to a passive and often bored array of students whose understanding of the subject the teacher can only deduce indirectly from their answers to test questions. (Barrows, 1996, p. 9)

Using the problem-based learning methodology in a mathematics classroom gives students the space and time to think and learn on their own. It is important that the problems given in this type of instruction are authentic problems that give students the opportunity to construct content knowledge and develop both problem-solving skills and self-directed learning skills (Hung, et al, 2008). Looking into the results of various studies about the benefits of PBL, Hung, Jonassen and Liu (2008) discovered that although "there is consensus that PBL curricula result in better knowledge application and clinical reasoning skills but perform less well in basic or factual knowledge acquisition than traditional curriculum," there are still studies such as that done by "McParland et al (2004) [which] 'demonstrated that undergraduate PBL psychiatry students significantly outperformed their counterparts in examination" (p. 490). When looking at retention of content, Hung et al found some interesting results. In short-term retention PBL students recalled slightly less than or equal to that of those in traditional classrooms, but consistently outperformed students that learned in a traditional setting in long-term retention assessments (2008). Tans and associates found that physiotherapy students' ability to recall the concepts studied was 5 times greater in a PBL setting compared to those in a more traditional setting only 6 months after the completion of a physiology course (Tans et al., 1986, as cited in Norman and Schmidt, 1992, p. 560).

While the problem-based learning methodology promotes an increased level of thinking, problem solving, and application of knowledge among students learning mathematics, it's a real challenge for traditional teachers to adopt this practice in their classrooms.

2.3.3. How can a traditional classroom be converted into a problem-based classroom?

The purpose of the problem-based approach is to actively engage students in their learning of knowledge and skills in a way that simulates future problems they may need to solve in that field of knowledge. In mathematics, the problem-based approach of teaching fits in well with the nature of mathematics as discussed earlier. It has the potential to build a learning-culture in the classroom that gives students the opportunity to learn mathematical content in an environment that is structured around problem solving – the heart of mathematics. 'Mathematicians who maintain that problem solving is the heart of mathematics also take the position that mathematics instruction is best organized as a set of problem-solving experiences' (Ben-Hur, 2006, p. 75). Problem solving as an instructional means can be incredibly valuable in the classroom, because when students actually 'do' mathematics, they also learn mathematics as well as enjoy mathematics. It is said that students with the opportunity to problem solve, actively engage in mathematics and gain a deeper understanding of the concepts, while thinking both actively and creatively during the problem-solving process (Ben-Hur, 2006; Boaler, 2008). The book *Concept-Rich Mathematics Instruction* by Ben-Hur (2006) convinces teachers that using problem solving as an instructional technique in a constructivist approach will have very positive outcomes. This book is not so much a guide to teaching through problem solving, as *Thinking Mathematically* by Mason can be used, but evidence that having students doing mathematics is ultimately a necessity for students to learn and understand mathematical concepts. Students must work with their intuition and battle with misconceptions and become more adept at moving through the stages of problem solving. Novice problem solvers will simply read the problem, and then explore the ideas or try immediately to come up with a solution, whereas more experienced problem solvers will move with affluence between reading, analyzing, exploring, planning, implementing, verifying and extending (Ben-Hur, 2006). These skills can be taught, but largely come through experience with problem solving. Making the change to a more problem-based approach can be an overwhelming challenge to teachers who have gone through a traditional education system.

In *Building Thinking Classrooms: Conditions for Problem Solving*, Liljedahl (2016) coins the concept of a 'thinking classroom' and the set of teaching practices that are conducive to transforming a traditional classroom into a thinking classroom. He defines a thinking classroom as follows:

> A *thinking classroom* is a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion. It is a space wherein the teacher not only fosters thinking but also expects it, both implicitly and explicitly. (Liljedahl, 2016)

In his research, he determined nine key elements that fostered a classroom culture of thinking and problem solving in mathematics. These elements can be implemented in stages, where the first stage is easiest to implement with the greatest impact, and the 3rd stage is more challenging and least impactful on its own.

Stage One:

1. **Begin lessons with problem-solving tasks**. Initially these tasks should be highly engaging and collaborative. As the thinking classroom becomes more established and students learn to problem solve authentic problems, these tasks will merge into the curriculum more fully as students learn content through problem solving.
2. **Vertical Non-Permanent Surfaces**. Students should work on vertical surfaces such as whiteboards, black-boards, and windows so that both the teacher and other groups of students can easily see the work done. When the work is easily erasable, or *non-permanent*, such as that written with whiteboard markers or chalk, students are more likely to start working quicker. Giving each group only one marker or piece of chalk during problem-solving tasks facilitates discussion.
3. **Visibly random groups.** Students work together in groups of two to four students that they know are random, and not obviously manipulated by the teacher. The groups should be changed randomly at the start of every lesson, or even multiple times per lesson. Over time, the students learn to work better with all their peers, and knowledge, skills, and strategies are shared amongst the entire group as a whole.

Stage Two:

4. **Oral instructions.** The teacher presents the task and instructions to students orally, so that student groups can immediately begin with problem solving, rather

than using time trying to decode written instructions. Diagrams and data necessary to complete the task can be given to the students on paper.

5. **De-fronting the room**. The thinking culture in the classroom is stronger when there is not just a single 'teaching wall' in the room that all students face. Ideally, the teacher should move around and speak to the class from various locations in the room and the desk groups should be placed in random directions around the room.
6. **Answering Questions**. The teacher should only answer *keep-thinking* questions that the students ask. These are questions that they need to have answered to continue working on the task. The teacher should acknowledge but not answer *stop-thinking* questions, such as "Is this the right answer?", or *proximity questions*, which are questions they only ask because the teacher is nearby.

<u>Stage Three</u>

7. **Levelling**. Once each group has passed a minimum threshold for the task, the teacher should engage a discussion about the understanding the class has about the task. The teacher should level to the bottom, formalizing and confirming the understanding to the minimum threshold that the student groups may have reached. This process of leveling can be considered the formal portion of the lesson.
8. **Assessment**. In a thinking classroom, assessment should value the learning process and problem-solving process that occurs during the class activities and tasks and more than just focusing on correct solutions or final products. Assessment should also communicate to the student where they are at and

where they are going with their learning. If group work is a significant part of the tasks, then it should also be a component of assessments.

9. **Managing Flow**. Nurture a thinking classroom by finding a good balance between giving hints and extensions to prevent students with greater ability from becoming bored and students that are struggling from becoming frustrated. (Liljedahl, 2016)

Incorporating the elements on this list into their teaching practice is a practical way for teachers to begin transforming their classroom into a thinking classroom.   Developing a classroom culture that fosters thinking and collaboration results in an ideal environment to teach using the problem-based learning methodology in mathematics.

When students are first exposed to the expectations in a thinking classroom, Liljedahl recommends starting with problem solving tasks that are highly engaging and not connected to curriculum. This allows the students to learn the routines, to break down any cliques in the classroom culture and build a more supportive culture of learning, and to learn strategies that will help them once they begin solving curricular tasks. After the classroom culture shifts from a more traditional and passive learning into a more learning centered environment, the teacher can start including tasks and problems that meet the curriculum requirements. If this shift occurs, once the thinking culture is established, students are more likely engage with the task as they would a non-curricular problem or puzzle. The challenge for teachers at this point is to develop concept rich tasks that allow students to learn the curriculum through problem-based learning.

2.3.4. How can the curriculum be problematized?

Although there are many resources with interesting and rich mathematical problems and tasks available to keep students both engaged and learning mathematics, most mathematics courses have a very specific curriculum that the students need to master. When the collection of fantastic problems does not overlap with the required curriculum, then teachers in the traditional setting tend to give the curriculum priority when they are short on time. Therefore a shift to problem-based learning can be extremely challenging. A solution to this issue is to problematize the curriculum. Researchers such as Stephen Brown and Marion Walter, Marion Small, and Dan Meyer share many strategies for problematizing the curriculum.

In *The Art of Problem Posing*, Brown and Walter (2005) explore different techniques that mathematicians at all levels can use to explore new ideas and to gain a better understanding of concepts. Considering an equation or mathematical strategy, a student or teacher can turn it into a problem for inquiry and deeper understanding by investigating some of the following questions:

1. What are some answers?
2. What are some questions?
3. What if not?

By searching for *some* answers, the student can begin to get a sense of the equation and is not limited by the pursuit of only searching for a single answer (Brown & Walter, 2005). This strategy for posing a problem puts students into Polya's (1957) first step in problem solving and Mason's (2010) attack strategy referred to earlier. Each of those first steps in problem solving has the student experimenting with the task and just getting a start that can lead into further investigation. By looking at a static statement, equation, or scenario, such as $x^2 + y^2 = z^2$ and asking oneself what are some questions or what-if-not can open a whole new investigation or set of problems (Brown & Walter, 2005). Exploring these questions or variations can lead into interesting extensions that are personalized to the solver. Teachers can use these same strategies to problematize both knowledge and skills in the curriculum that students need to learn about. Brown and Walter (2005) also suggest that the teacher not ask a question, but ask students to study some data and simply start with making statements about what they notice before moving on to generalizing, making conjectures and even proving a conjecture. Even in a very traditional classroom, problem posing and solving can be integrated into the curriculum by asking students to pose questions that challenge the formulas and procedures they use. Students can gain a deep understanding of mathematics they are learning through the discussions, debates, and articulation of thoughts that come with problem posing. (Brown & Walter, 2005)

Dan Meyer developed a way of mapping mathematical tasks to a storytelling framework and coined the term *Three Act Tasks*. In Act 1 of the task, the central conflict of the story or problem is introduced as clearly as possible and with few words. During this stage of the problem, the problem should be clear to the student although the solution is still unknown. This is followed by Act 2, where the students search for the solution, determine what information or resources they will need, and develop tools to solve the problem. The task is concluded with Act 3 when the problem is resolved. Here the students receive some sort of validation about their solution and can then extend the task into a follow-up problem. (Meyer, 2011). A textbook exercise can be adapted into a rich problem using this three-act, story-telling style or Brown and Walter's problem-posing strategies. A teacher can begin developing a rich task that relates directly to the curriculum with a word problem exercise from the textbook by removing all the details given. This leaves just a diagram or photograph with a simple question or scenario. The students can then put forward questions to consider and request extra information they determine would be necessary for the task. Once students begin posing their own questions and exploring solutions, a much deeper investigation can occur than the original textbook question would have intended. This style of creating problem-solving tasks from short story scenarios, photographs, or diagrams fits in well with Liljedahl's 4th element in a thinking classroom of presenting problems orally without written instructions (Liljedahl, 2016).

2.4. Research Question

While creating a learning environment that fosters thinking and collaboration in mathematics may be an attainable and even a desirable goal for a teacher, this can also feel a little idealistic. Teachers may find it challenging to meet the requirements of a given curriculum while still nurturing the environment of a thinking classroom. One of the challenges is likely to be time. Is there enough time in a course to teach an entire curriculum through problem-based learning in a thinking classroom? Another concern is if the students will learn the required content through the given tasks and activities. Ben-Hur (2006) brings up many reasonable excuses that these teachers may have, such as, problem solving is too difficult for my students, the tasks take too much time and, before they can solve problems, the students must master the facts, procedures, and algorithms.

Problem solving is at the heart of mathematics, and there clearly are advantages to learning mathematics through problem solving. The purpose of this study is to capture the extent of the mathematics curriculum that students can experience in a thinking classroom where problem solving is given precedence over the curriculum.

Combining the open-ended problem posing technique from Brown and Walters (2005), which Jo Boaler (2002) observed in her research at the reform school, with Liljedahl's (2016) strategies of *Building Thinking Classrooms,* I designed a new classroom culture of learning and teaching. As a teacher, I was especially interested in experiencing what would happen in my classroom if I gave my students the opportunity to problem-pose and problem-solve in mathematics without having a predetermined plan as to what curriculum content I expected them to be delving into during a task. I decided to let go of unit planning and lesson planning and let the learning happen around mathematical puzzles, problems, and tasks. This led to my key research question:

To what extent can a curriculum be encountered through problem-based learning in a thinking classroom?

Chapter 3. Methodology

To understand the extent of learning that takes place in a mathematics classroom through free-style teaching and problem-based learning, the research question has been investigated through a case study of one student. There will be a focus on the development of tasks, and the mathematics that the participant encounters will be linked to the prescribed learning outcomes from the provincial government of British Columbia for mathematics education. The word encounter will be used to describe any moment where the student experiences, uses, speaks about, experiences, or engages with the prescribed learning outcomes. The purpose is to get a better understanding of the diversity of content that the students come into contact without considering the depth of understanding or engagement. To describe the methodology used in this research, what follows are details of the course, the participant, the classroom environment and procedures, the data, and analysis.

3.1. Setting: Pre-Calculus and Foundations of Mathematics 10

Students entering Grade 10 in British Columbia have the choice of taking Workplace and Apprenticeship Mathematics 10 (WAM10) or Pre-Calculus and Foundations of Mathematics 10 (FPM10). To receive a British Columbia Certificate of Graduation (Dogwood Diploma), students must complete one of these two courses followed by a Grade 11 mathematics course. Both the FPM10 and WAM10 require that the students take a provincial final exam that is worth 20% of their final grade. On completion of FPM10, students typically continue to take either Pre-Calculus or Foundations of Mathematics for their Grade 11 Mathematics credit. While learning about the mathematical learning outcomes listed below, the curriculum requires that the following 7 mathematical processes should be used in teaching and learning mathematics: communication, making connections, mental mathematics and estimation, problem solving, reasoning, technology, and visualization (WNCP Common Curriculum Framework, p. 6). The prescribed learning outcomes of FPM10 can be summarized as follows.

Students are expected to

- Solve problems involving linear measurement, surface area and volume of 3D objects
- Convert between SI and imperial units of measure
- Demonstrate an understanding of Primary trigonometric ratios
- Demonstrate an understanding of factors of whole numbers by determining the prime factors, greatest common factor, least common multiple, square root, and cube root.
- Demonstrate an understanding of irrational numbers
- Demonstrate an understanding of powers with integral and rational exponents
- Demonstrate an understanding of multiplication of polynomial expressions
- Common factors and trinomial factoring
- Interpret and explain relationships among data, graphs and situations.
- Demonstrate an understanding of relations and functions.
- Understand and solve problems involving linear equations, functions, and systems both graphically and algebraically.

3.2. Setting: Participant

Foundations and Pre-Calculus Math 10 was taught at a small rural private Christian school in South West British Columbia from September 2014 until June 2015. Students in the course attended class for 55 minutes, four days per week, over the course of the school year. The majority of students attend this school from Kindergarten until Grade 12, so by the time they reach Grade 10, they know each other very well. Most students reside in the countryside with two parents who work hard to cover the tuition costs. Although parents generally support the school and their child's education, there are only a few families that really encourage their children to strive for academic excellence and to go on to university. By the time they reach Grade 10, most of the students have afterschool and weekend jobs, help at home with household chores in large families, or help out in the family business.

There were five male students and 11 female students enrolled in FPM10 during the course of this research. This case study focuses on the mathematical content that one female student, Caroline, encountered during the period of one month approximately half way through the school year. I chose Caroline for this case study because she was rarely absent and she was generally engaged in the mathematical tasks.

3.3. Setting: Classroom

To help build a learning environment that would be conducive to collaborative work and critical thinking for students solving problems, I made several key changes. First, I put in a request to the school administration for as many vertical white boards as they were willing to let me have in my classroom. This was approved and at the start of the school year, there were six permanent whiteboards distributed around the four walls of my classroom. Since the computer lab was being updated, I managed to claim six of the old computers for my students to use for some basic researching or graphing. I arranged these computers around the room so that each whiteboard area had access to one of the computers (Figure 3.1, Figure 3.2).

Figure 3.1 Setting up the classroom: Flexible classroom arrangement for group work

Figure 3.2 Setting up the classroom: Whiteboards and computers

The next major change was the classroom furniture. To allow students to move easily through the classroom, I removed as many desks and tables as possible to make the room as spacious as possible. I kept a few folding tables against the wall in case I needed more table space. Since folding tables are not comfortable to work around and the shape is not conducive to discussions, I found some kitchen tables on Craigslist to improve the working environment.

For the most part, we used folding chairs that could be stacked away when we needed a large open space; and there were a few benches and boardroom chairs that I rescued from the pile of furniture that was about to be sent to the garbage dump. These chairs were actually falling apart, but it encouraged some students to come to class early because they wanted a ‘cushy’ chair, so I chose to keep them at the students’ request.

In general, my classroom furniture was a mish-mash assortment and all fairly tacky. The students often commented that they liked how my room didn't really feel like a classroom, with all desks in rows. This helped me establish a new classroom culture at the start of the year because the students were not sure what to expect when they entered the room. The minimalistic furniture gave me a lot of freedom to change things around as well. On days that I wanted to reset the expectation, I would make a dramatic change in the set up. Sometimes the tables were all pushed to the side or folded up and only a circle of chairs would be set up. Other days, there would not be any chairs, but students would stand around tables. Generally, the students could move freely through the classroom and would choose to work at a whiteboard or table, based on the type of problem they were solving.

Figure 3.3 Textbooks stored in classroom, grid and lined paper pads always available to students, and whiteboard markers and erasers always accessible to students.

One of the final major changes I made to my classroom environment were the supplies available to students. Instead of having the students purchase their own school supplies, such as notebooks, for math, they brought $5 to contribute to the math supplies. I purchased note pads with grid and ruled paper, whiteboard markers, and erasing cloths (Figure 3.3). The students also left their textbooks in the classroom. The purpose for this was to ensure that forgetting supplies would never get in the way of learning or problem solving. It also gave me the opportunity to choose what I wanted them to work with. I wanted students to just show up for mathematics, without needing to worry about taking anything with them. In addition, I really enjoyed the fact that no large binders cluttered my classroom tables and floors as they usually did (Figure 3.4).

Figure 3.4 Students using a variety of tools as they work on a problem solving task.

Since students were not bringing binders, I gave them each a file folder that was stored in the classroom where they could choose to save any scraps of work or notes that they made for future reference. This idea came from the reform school that Jo Boaler writes about in her book *Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning*.

3.4. Setting: Procedures

When I started collecting data on Caroline in January, my classroom procedures were well in place. As students entered the room at the start of a class, I would welcome them and let them know what I expected them to do. Sometimes I asked them to sit or stand in a circle, and other times I would tell them to just find a spot anywhere in the room to get comfortable. I usually introduced the task through discussions, demonstrations, or stories and then the students would start working on the problem in random groups of 2-4 students.

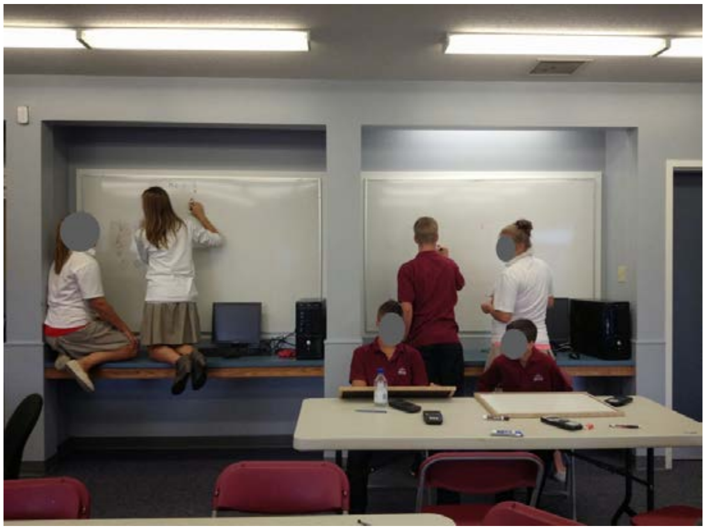


Figure 3.5 Students working in pairs using whiteboards

While working on problems, students usually started on a whiteboard for brainstorming (Figure 3.5). They often moved around and found tools and resources to help them solve their problems (Figure 3.6).

Figure 3.6 A few pairs join the center to form a super group. One pair splits to work individually.

While they were making sense of a problem, I would walk around the room and assess their learning and understanding. When a group was stuck, I would often step in and ask them a question or give them a suggestion that might help them out. Sometimes I would get them to talk to another group. If students asked me if their answer was right, I usually responded, “You tell me. Can you convince me that your solution is correct?” This would help me to assess not only their understanding of the problem, but also their ability to communicate a solution.

Depending on where the rest of the class was in the problem-solving process, I would either ask them questions to extend the task, or I would get them to pair up with another group. If another group was also finished, but had a different solution to the same problem, I would say, “You need to talk with this group. You both need to convince the other team that your solution is correct, and then decide at the end what solution you can all agree on.” This generally had the students arguing and improving their skills to communicate mathematics (Figure 3.7). Although I could generally follow their logic, another group was usually ready to jump on any mistakes or let them know explicitly that what they were saying did not make sense at all. Other times, I would try to extend the task. A few simple techniques I often used to extend the task included reversing the question, changing the initial conditions, or asking for a more algebraic or generalized solution.

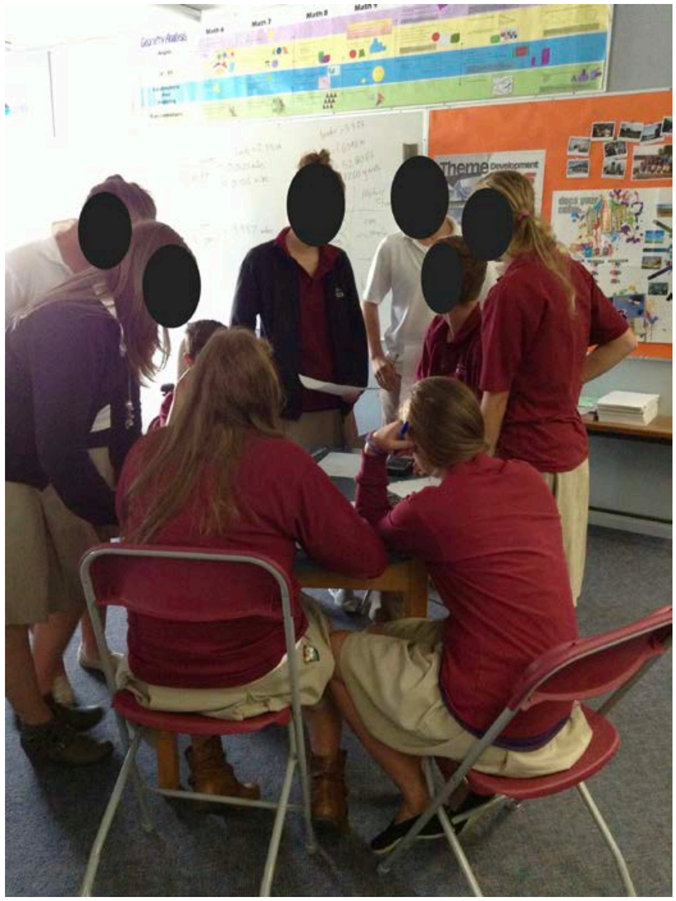


Figure 3.7 A group of students debates their solutions

As students became more familiar with the go-with-the-flow style lessons, and realized that the initial tasks I gave them were just a starting point, they started learning how to extend their own tasks or to just play with the mathematics of a task. I rarely would give the students an answer, only a strategy. And often, if I did teach a strategy to one group that was ready for it, then I did not teach that same strategy to the rest of the class at the same time. As students worked with each other in random groups every day, the various strategies would slowly move throughout the whole class as they explained them to each other.

3.5. Setting: Tasks

I was purposely not teaching the curriculum in a very specific order or in terms of 'units'. There were a few times throughout the year where I would choose tasks that focused on one concept, such as using the primary trigonometry ratios to calculate unknown angles and lengths in right triangles, for several days in a row. One reason I chose to do this is because I wanted to know how students would solve problems if they didn't expect the problem to relate to a certain concept. In previous years, I found that when I chose a task that fit in well with the unit we were working on, there was rarely a moment of surprise or "I didn't expect *this* to be related mathematically to *that*!" from students. Mathematical concepts and ideas that I found amazing came across to students as just another every day concept. The second reason I was avoiding teaching in units is because I was hoping that if students were working with the entire curriculum regularly – starting from the basics of each topic earlier in the year and building to more challenging concepts by the end of the year – that they would ultimately understand and remember everything much better by the end of the year. The final reason that I did not want to plan rigid units of teaching is because the purpose of this research was to determine what mathematical concepts students would work with when I let the learning emerge. By not forcing students to solve problems using a specific strategy or technique, I allowed the students to pull from all their prior mathematics knowledge to actively work towards a solution.

Although I worked at keeping a list of problem solving tasks that I felt would be great for students to work on, most lessons and tasks happened 'in the moment'. This is where my teaching felt like free-style teaching because I allowed myself to teach concepts in no particular order and often came up with problem solving tasks on the spot while students were in the room. Some problems which I had expected students to work through quickly, such as the goat problem, grew into multi-day tasks because the mathematical conversation in the classroom was really strong. Other days, I had a complex problem planned, but if students came in looking tired or unmotivated, I would switch the problem on the spot to try to better engage them. Whenever I was stuck for a task to get them started, I would either look at the list of outcomes the students need to learn for FPM10 and pick an outcome that had not been worked with recently, or look at my ever-changing list of interesting tasks. Whenever I came across an interesting task online or in my collection of problem solving books that I felt could be adapted for my students, I would add it to my written or mental list of interesting tasks. When I looked at the list of required curriculum outcomes I would pair it to one of tasks on my mental list that I thought could meet one of the outcomes or just make up a question on the spot to get students working with a prescribed learning outcome (Figure 3.1). Once students were working, I would assess how well they were taking to the task, and what sort of action to take to keep the problem-solving momentum going for the duration of the block.

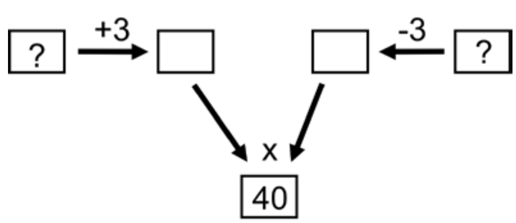


Figure 3.1 An example of a problem made on the spot. More about this problem in Chapter 4 in section Day 3 Part 1 Function Puzzles

3.6. Data: Journal Entries, photographs, and student work.

During the month of January 2015, I kept a record of the teaching and learning that was taking place in my classroom. Since the lessons were quite fluid, and I was very involved during the period of the lesson to keep the problem-solving process moving forward, I enlisted the help of my students to collect data in the form of photographs. The students took photographs of their whiteboard work to share with me as evidence of their learning for assessment purposes. At the end of the one-month period, the students needed to choose three of the tasks to formalize and hand in to me. To help them with this, they could take as many pictures of their whiteboard work as they wanted, using my iPad to refer back to later. For her assignment, Caroline submitted completed solutions to three of the tasks, which I have also included as data. I also went around the class regularly to take pictures of all student work, especially Caroline's.

In addition to photographs of student work, I kept a journal and took notes about the tasks I presented, Caroline's problem-solving process, how the task developed mathematically in her group, and the mathematical content that she was working with. These notes were based on what stood out to me and what I remembered about her work at the end of each class. Many days I was not able to record her whiteboard work with a photograph because she would erase work during the problem solving process. But I could easily take notes of the mathematical content that Caroline encountered because of my experience and background working with the mathematics curriculum. While a casual observer might not have realized the significance of certain activities or concepts, I was able to discern their connection to some of the bigger mathematical ideas. Although the tasks and problems did not always directly relate to the curriculum, the flexible nature of the classes allowed me to observe students brainstorming and to help the mathematical ideas that were loosely connected with the curriculum to grow in a very intentional way.

3.7. Analysis

To analyze the data and determine the curricular content that Caroline encountered through the tasks, mini lessons, and problem solving, I mapped all my data to the British Columbia Ministry of Education Prescribed Learning Outcomes in Mathematics from Grades 6 – 12. I grouped the data by day, and then isolated the mathematical concepts that Caroline encountered each day. These concepts were then categorized under a Prescribed Learning Outcome. I first looked for an appropriate PFM10 outcome to place a concept under; in the event that it did not quite fit the description, I put it under a Learning Outcome (LO) from any mathematics course between Grades 6 and 12. Concepts that stood out to me as unique or valuable that could not be linked to any learning outcome in mathematics from Grade 6 to 12, I marked with an asterisk.

Once all the mathematical concepts were categorized, I took a closer look at how a task was initiated during each lesson, and how Caroline worked through the problem and encountered these outcomes. To better understand the results of the data, and the extent of the curriculum that Caroline accessed, I organized the LOs that were encountered during the 18 mathematics classes by Grade level. The next chapter describes each of the initial tasks, the evolution of the problems, and any mathematical concepts that I was noticed Caroline encountered in her work or in observation.

Chapter 4. Results and Analysis

Students in a traditional setting typically encounter concepts, such as fraction, for an exact number of weeks and then move on to focus on a different topic. My students were in a process of encountering a broad spectrum of concepts on a regular basis each month throughout the entire school year. The understanding of the concepts was not as compartmentalized, and these results focus on the diversity of concepts encountered rather than the depth of understanding. In many of the tasks, Caroline's final solutions are not shown because I did not manage to photograph all of her whiteboard work. Her whiteboard work will be included in the task descriptions if I have a photograph, otherwise the description of her work is based off of my journal entries and any work she submitted on paper. I was also more interested in the mathematical concepts she encountered during the problem solving process.

In what follows, I first summarize the tasks and learning outcome that Caroline encountered in each of the 18 days for the duration of the study. Then I will describe the story of one lesson and how I linked the mathematics that Caroline encountered to the learning outcomes in the curriculum. Although my data focused on Caroline, I will include a brief glimpse into how the lesson panned out and what other students were doing during the lesson. Lastly I will discuss how the learning outcomes that Caroline encountered map to the overall mathematics curriculum to give a better view of the results of this study.

4.1. Description of Tasks

Day 1, Gears + Marching Band

If the gears shown in Figure 4.1 are turning, when will it spell TROY again? Caroline worked with a small group of students to determine a solution to this problem.

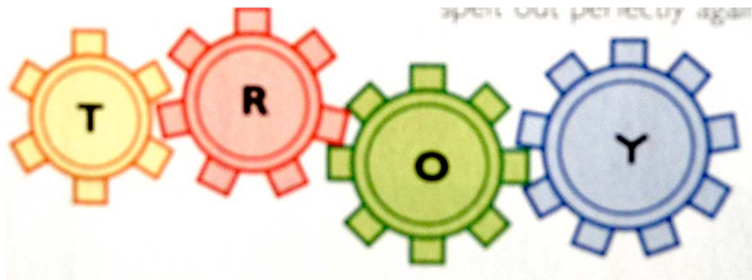


Figure 4.1 Arrangement of gears

They initially solved the problem assuming that the letters needed to make a full rotation to return to its original position. Their first strategy was to use percentages by calculating the ratio each gear turned to a complete turn if a specific gear made one complete turn. For example, if the T completely rotated, then the R would have rotated $6/7^{th}$ or 68% (8A3)[5]. After dropping this strategy, they reasoned that if the individual pegs of T turned a multiple of 6 times, it would be in the correct position again. They used this reasoning for each letter, and then calculated the least common multiple of these 4 numbers (6A3). After Caroline's group explained their process to me, I showed them that numbers could be written in terms of their prime factors and how that was useful in determining the least common multiple of a set of numbers (10B1). Then I challenged their assumption that all the letters needed to be rotated 360 degrees to spell out TROY. They quickly came to the realization that the O shown has infinite rotational symmetry, and the Y is written in such a way that it has order-3 rotational symmetry (9C5).

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[5] This code refers to the learning outcome encountered which can be found in Appendix A. The first number, 8 in this case, refers to the grade level.

This problem focused the students on properties of prime numbers, factors, and multiples, and gave me the opportunity to teach a quick mini lesson to each group about making factor trees to determine the prime factorization of a number. I followed it with posing the marching band question: *When I arrange the members of a marching band in rows of 2, 3, 4, 5, or 6, there is always one person short, but when I arrange them in rows of 7, it works perfectly. How many people are in the marching band?* Caroline and her group experimented with a few different strategies. One of the strategies they used was to test multiples of 7 and sketching that number of people in rows to check if it could be arranged into rows of 2,3,4,5, and 6 with only 1 remaining. Most of their strategies were cumbersome, although they were working on many lists and diagrams of multiples and factors (6A3). Eventually they discovered a single solution to the problem which they were excited to tell me. When I pushed their understanding with, “Is this the only solution? What about if there was one extra person instead of one short?” Caroline and her group became frustrated. This is likely because they did not have an efficient strategy and the problem extension felt like an overwhelming amount of work. I briefly discussed a possible technique using prime factorization to build a number and the idea of temporarily adding in one extra person. Then, the number of people is a multiple of 2, 3, 4, 5, and 6 but needs to be 1 more than a multiple of 7. Caroline used this idea and adapted this technique to work for the extension questions I posed earlier to their group (10B1).

Day 2, Making Groups

How many days can this class of 16 students be sorted into groups of size 2 (or 3 or 4), so that they don’t work with the same person twice and they work in new groups every day? The students were divided randomly into groups of four and each group was given a different size for the groups they had to make. The intention was that I could use the groups they made to quickly split the class into pairs, groups of 3, or groups of 4. Caroline's intuitive response was that this was almost an impossible task because there would be way too many ways to sort the class in groups of 4. With her group, she started organizing the students into lists, and using letter codes of generic people (A, B, C…) instead of names. Initially, they were listing all the possible arrangements of dividing the class into 4 groups (12C1). They dropped that strategy when they realized that no two students were allowed to work together twice, not just the one student they were focusing on.

For example, in the two groupings ABCD EFGH HIJK LMNO and EBCD AFGH HIJK LMNO, the person A is working with a different group, but all the other people are working with somebody they have already worked with. At some point during the lesson I eased up on the requirements and told them they could allow two people to work together on two different days as long as the rest of the group was different. Her group used numbers from 1-16 to represent the different students and her final (incorrect) solution is as follows:

Day 1:	1/2/3/4	5/6/7/8 9	/10/11/12	13/14/15/16
Day 2:	1/5/9/13	2/6/10/15	3/7/11/14	4/8/12/16
Day 3:	1/8/11/15	4/10/14/5	3/13/6/12	2/7/16/9
Day 4:	2/5/11/16	3/9/15/7	6/10/13/4	1/8/12/14
Day 5:	1/2/4/5	3/4/7/8	11/12/13/14	9/10/15/16
Day 6:	5/6/3/4	1/2/7/8	9/10/13/14	11/12/15/16

Although this problem did not bring up any of the prescribed learning outcomes in mathematics courses from Grade 6 through 10, Caroline found the task puzzling and mathematically challenging. The purpose of this task was to challenge the students to think critically and create a usable solution that I could use to put their class into groups.

Day 3 Part 1, Function Puzzles

Function diagrams were a series of problems that I made up on the spot and drew on the whiteboard for the students and presented as puzzles. *What numbers could you put in these two boxes with the question marks to make this work (Figure 4.2)?*

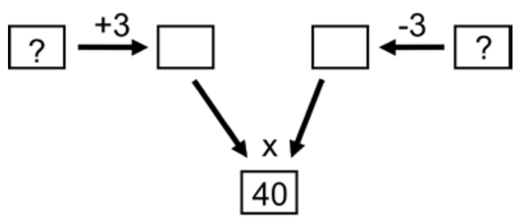


Figure 4.2 First function puzzle given to students.

Caroline and her group had little trouble working out a single solution using a guess and test strategy. *Is that the only combination that works?* This challenged them to look for a series of solutions. As they worked through the possible combinations, they also found all the factors of 40 (10B1). Each time they told me that they 'found all the answers', I challenged them with "*Are you sure you can't find any more?"* They asked me if negatives were allowed, to which I responded: *Are negative numbers not numbers? (8A7).* This task briefly introduces the vocabulary of subsets of the rational numbers that they are required to learn, including natural numbers and integers, as I continued to challenge them to think deeper. I was not well prepared on Day 3 with a problem solving task, so I made up the first problem as a way to get the students to start working on mathematics to give me a little extra time to choose a better learning task. Since the students became very engaged in the problem, I instead chose to keep extending the problem to see where it might go.

Caroline and her group initially started with a single solution, and then moved towards an understanding that there would be an infinite number of solutions. As soon as they recognized that the solution set is infinite, the students suddenly felt ready to throw their hands up in the air and gave up hope of ever finding them all. Caroline was initially satisfied with the one solution, but appeared frustrated and unhappy with her list of solutions when she realized that no matter how many solutions she would find, listing them all would be impossible. At that point, I stepped in again and modified the problem a little by merging the two boxes into a single box. That meant that the problem only took in a single input (Figure 4.3). *How many solutions are there now?*

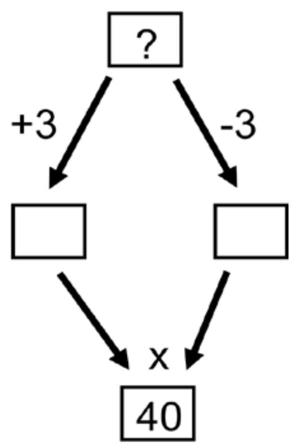


Figure 4.3 Function Puzzle Variation I

The focus was changed from the actual value of the solution value to the number of solutions possible. Using her work for the previous question, this task was quite simple, since she just needed to find out how many of the solutions had equivalent inputs. To make this into something even more challenging, I erased the 40 and asked her group: *What input would you choose to maximize the output in this box? Or minimize the output?* (Figure 4.4)

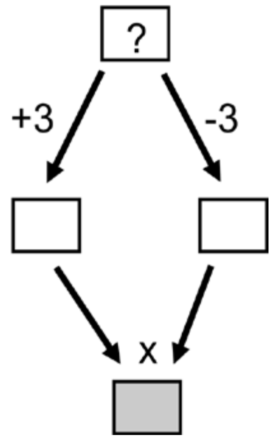


Figure 4.4 Function Puzzle Variation II

As they worked through the consequent changes to the function puzzle, Caroline became more comfortable with inserting variables to make equations (9B3) and graphing the relationships as a strategy to make sense of the problem instead of just using the guess-and-test and working-backwards strategies to determine a solution(*)[6]. She would insert a variable into the input boxes and then graph the resulting equation on Desmos. The main underlying concepts that she worked with throughout this task were optimization with quadratics (11C5), solving and graphing linear and quadratic functions (10C1, 10C4, 10C5), graphing with technology (*), and determining intersections on a graph to solve a problem (10C9). I was unable to collect samples of Caroline's actual solutions as I was focusing on her process and busy engaging the groups in dialogue to think critically, extend the tasks, and to make connections regarding the solutions, patterns, and graphs.

[6] This concept is marked in Appendix a with an asterisk because I decided it was a valuable and meets the goals of the curriculum even though it does not have a specific prescribed learning outcome. Throughout the rest of the document I will mark these situations with an asterisk.

Day 3, Part 2, Three Digit Sum

Use all the digits from 1-9 to make this equation true (Figure 4.5). Caroline started working on this task immediately using trial and error. Then she adjusted her strategy into

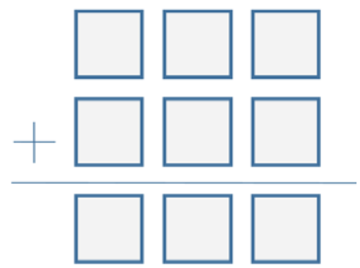


Figure 4.5 Three Digit Sum

determining all the possible combinations in a more organized order and used elimination to bring her closer to a solution (F11C2). Eventually, she came up with a solution that worked when the sum of the ones-column was greater than 10, and therefore allowed her to 'carry' a digit without the need to write it down in the tens-column.

I challenged her to determine a solution that did not use any carrying. After she struggled with this for a little while and was starting to feel that it was not possible, I had a discussion with Caroline and a few other students about odd and even numbers. *Is the sum of two odd numbers odd or even? What about the sum of two odds? Two evens?* Caroline thought about this and discussed it with a few other students. They tested it with a few sums to see if their rules were holding true for other numbers. *How can you apply this to the problem?* The students are not familiar with proofs, but they started puzzling with the task again, looking through the lens of odd numbers and even numbers.

I worked with Caroline and a few students on this concept, coaching them to think about adding even and odd numbers. They simplified this discussion into the following equations:

even + even = even
odd + odd = even
even + odd = odd

Then they tried building another combination in the diagram, using actual digits, but keeping this concept in mind. The real digits become placeholders for general even and odd numbers in this trial, and as they worked their way to the third column, they had an 'aha' moment. The realization that if a column has an odd number, it must have exactly two odd numbers for the addition to work. Given that there are five odd numbers from 1 to 0, they were able to justify that a solution was impossible without allowing an extra 1 to be 'carried' over from the previous column. I show them how they can use logic and reasoning to prove that it is impossible to find a solution (F11C1). Although Caroline's work on this task did not directly link to the mathematics curriculum in grades 6 through 12, she gained a deeper understanding of how numbers work through problem solving and articulating her thought process. She also learned that proving or demonstrating that no solutions exist is actually a solution to the problem.

Day 4, Goats

Charlie the goat is tied with a rope to the corner of a barn so that he can graze and eat grass. What is the area of grass that he can reach? The discussion that came up amongst the students as I presented this problem converted this simple warm-up problem into a more challenging task. The diagram I drew on the white board appeared quite distorted; attempting to poke fun at my bad sketch, one student questioned "How long is the bottom side of the barn?" Although I did not expect this, I responded as though it is perfectly normal to build a trapezoidal barn, "Oh, it's 10 m," and labeled the diagram as shown below in Figure 4.6. Students questioned and critiqued everything about the drawing and the situation from the length of Charlie the Goat's legs to the stretchiness of the rope.

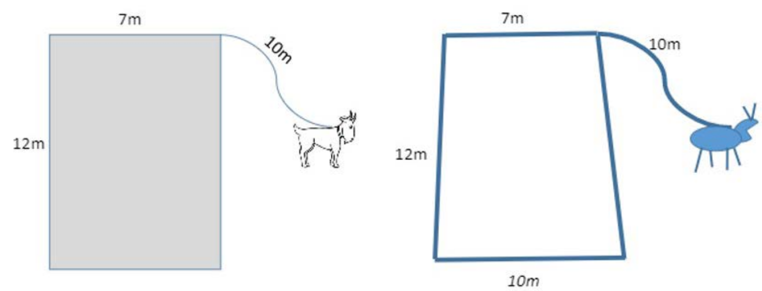


Figure 4.6 Left diagram is what I meant to draw although my sketch looked more like the version on the right.

Caroline worked in a random group of three students to start calculating out the area that the goat could reach when he was tied up. After drawing a diagram (Figure 4.7), she determined three quarter of the area of a full circle that a goat could walk in if it was tied to a 10m rope in an open field (7C2). Realizing that the rope would extend past the top-left corner of the barn, she worked out the area of a quarter circle with a 3m radius. The challenging part came when she needed to determine the area of the circle that was blocked by the barn since it was trapezoidal. She used trigonometry to determine the actual angle of the corner of the barn (10A4). Interestingly enough, she calculated both acute angles using trigonometry to check her answer.

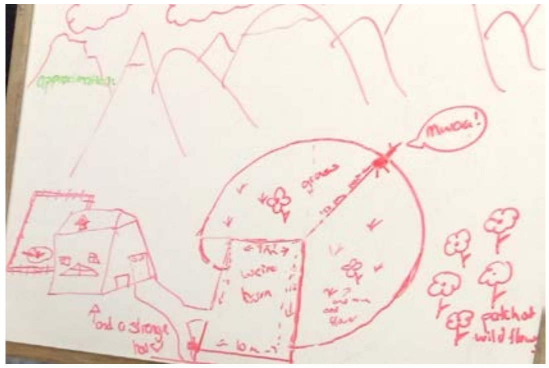


Figure 4.7 Caroline's sketch of the area the goat could access

She was satisfied when all three corners added up to 180°. As she explained this to me, I noticed that she used the sine ratio, opposite divided by hypotenuse, with the tangent function. She informed me that it doesn't make a difference whether she uses tangent ratio or the sine ratio because the answer will be the same! I took some time to turn her misconception into a learning opportunity as we investigated the fact that the value of the sine function approximates the tangent function for angles that are less than 10° (*). Finally, she determined the small wedge of area that she needed to subtract from her calculation due to the barn being trapezoidal. For this, she realized that the ratio of the small angle to a full 360 degrees would be the same proportion as the wedge area to the full circle (10A2). Of interest is the fact that she did not do this calculation for the small wedge on the opposite side of the barn. I am not sure if this is because she estimated it would be trivial, or if it went unnoticed by her.

Day 5, Goats Extension

The goat problem from the previous day went so well, that I decided to follow it up with some extensions. This time around, I gave each student group a different scenario of an animal being tied up outside of a triangular fenced area to eat grass. In some cases, the rope was long enough that it would meet or overlap on the opposite side of the pen, so students were challenged to think about what that overlap area would actually be. The enclosure in Caroline's task was a right triangle with one of the legs 7 m long and the hypotenuse 16m long. She needed to determine where she should tie the end of a 15m rope so that the pig would have the largest area of grass to eat.

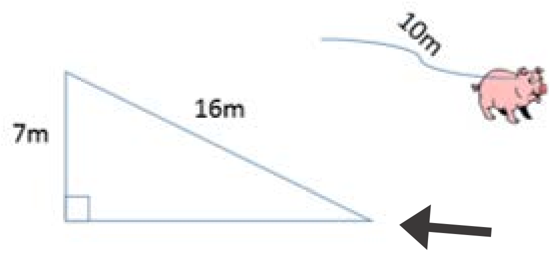


Figure 4.8 Diagram of goat extension task

Caroline used the Pythagorean theorem to calculate the length of the 3rd edge of the triangle (8C1). She then made the decision to tie the goat on the bottom right corner of the triangle as marked with the arrow in the diagram above (Figure 4.8). She then used trigonometry to calculate the angle of that corner (10A4) and subtracted the result from 360 to determine the exterior angle of that corner (7C1). Using area of the circle (7C2) and ratios (8A5) she calculated the total region that the pig could access when tied to that corner. Although she did not calculate any other regions, she justified why that specific corner would maximize the region because it was the smallest interior angle in the triangle. Her whiteboard calculations and extra-curricular artwork are shown in Figure 4.9.

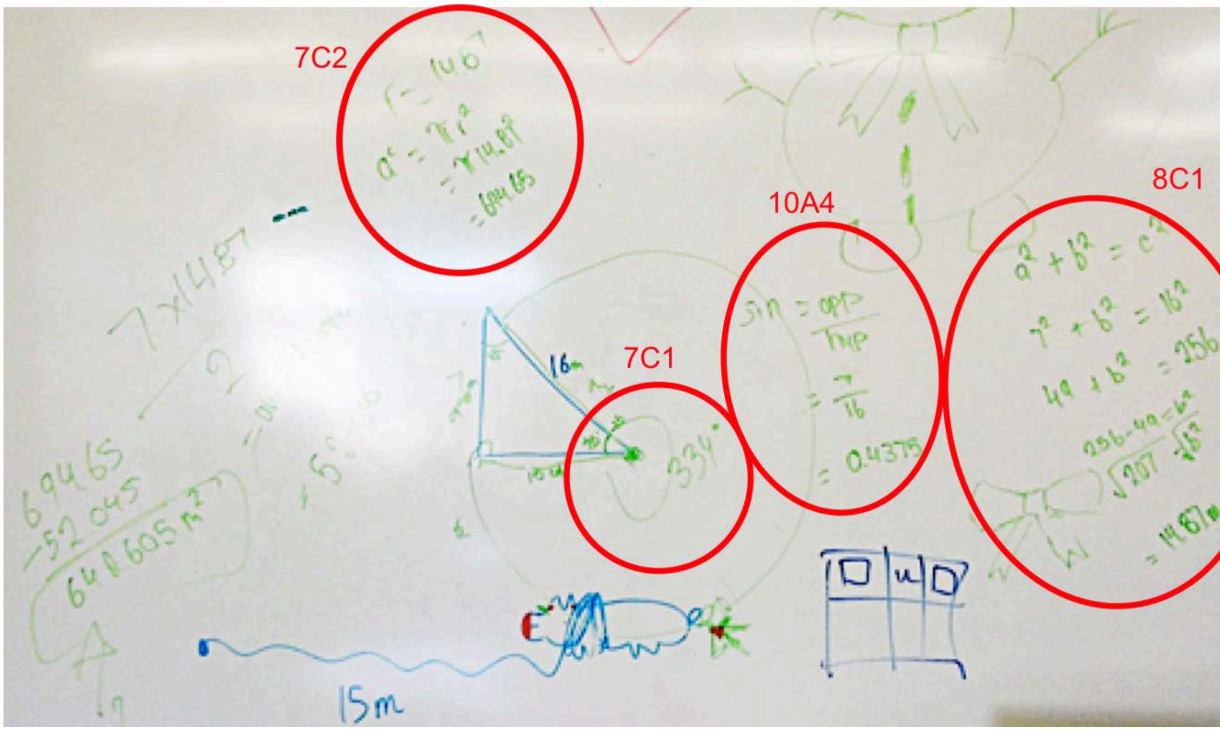


Figure 4.9 **Caroline's whiteboard work to determine the area accessible to a pig roped outside of a triangular enclosure. Calculations demonstrating encounters with the prescribed learning outcomes are circled and labeled.**

Day 6, Dutch Blitz

Dutch Blitz was a favourite card game amongst students at the school, so one of my colleagues and I decided to turn it into a giant card game that could be played in a large classroom with teams of students. Since we only had four sets of cards, the giant version was still too small for large class sizes. My class of pre-calculus students was designated to help with the construction of the expansion pack for this game earlier during the school year during one of their mathematics classes. They were familiar with this giant size game and involved in the construction, so I turned it into a mathematics task. I told that students that people have been asking if they can borrow that Giant Dutch Blitz game we have for their family reunions and other events. *What do you think? Should we let them? Would it be worth making more copies of this game to sell perhaps? Who knows, it might be a mega hit! What I want you to do is figure out the* *value of this handmade game. How much should someone need to pay us to buy the game? And, how much should we charge people to rent it out. Remember that people really can get into the game and the cards will start to show damage, so the game won't stay in perfect condition forever.* I gave them an email from the secretary that gave the prices of poster boards, whiteboard markers, the laminator, and the laminating rolls. Caroline and her group loved this task. They used their whiteboard to organize the given information, some of their calculations, and their final solution as shown in Figure 4.10. In their calculations, they used unit conversions (10A2) as shown in Figure 4.11, worked with rates to determine the expenses of labour and materials (8A5), and made reasonable estimates for any unknown values. They were the only group to incorporate a downgrade value to determine what the rental rate should be (Figure 4.12). Caroline explained that this was to determine the value of the product in relation to time and they used this to decide how long the game would last before it needed to be replaced. We discussed using a decreasing arithmetic sequence to model the downgrade (11C9), which is not shown on the whiteboard. The group miscalculated one value, resulting in their final game price being too low, but they fully understood the process. Caroline chose to submit the Dutch Blitz as one of the three tasks to submit for her assignment, so she was able to make the correction in their calculations. She organized all the information from the whiteboard and presented it on a poster as shown in Figure 4.13.

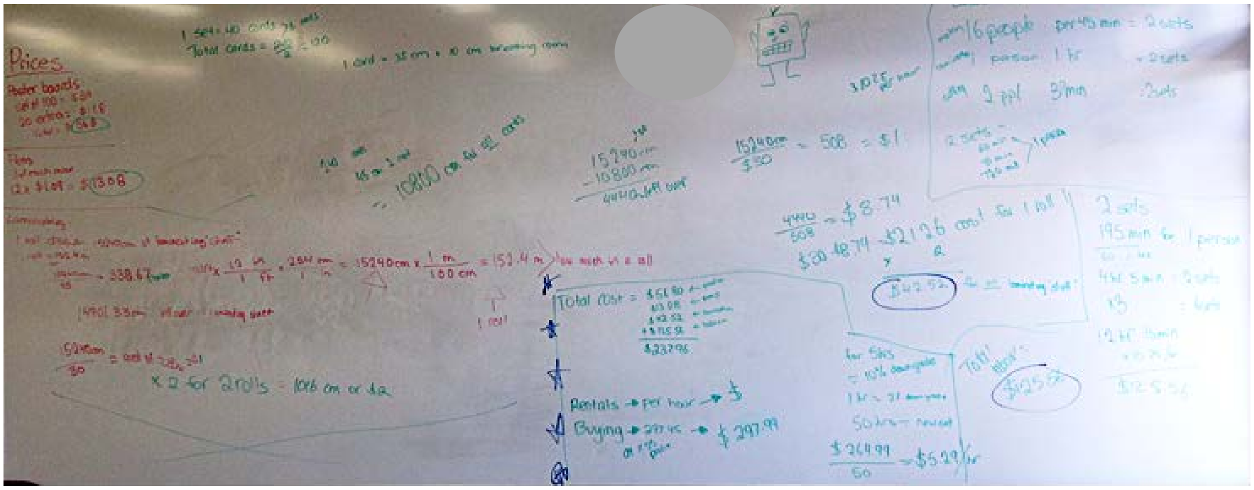


Figure 4.10 **Caroline's whiteboard with data, a few calculations, and their final answers.**

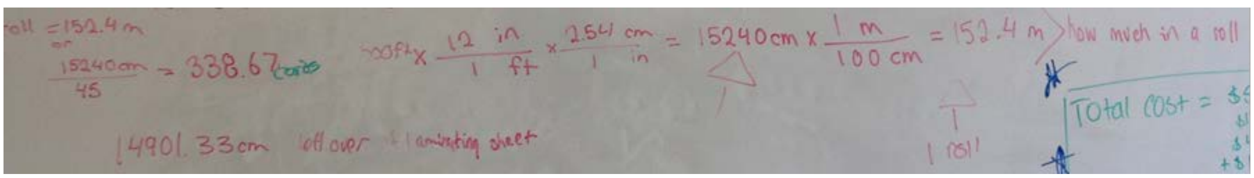


Figure 4.11 **Close up of whiteboard showing a unit conversion calculation**

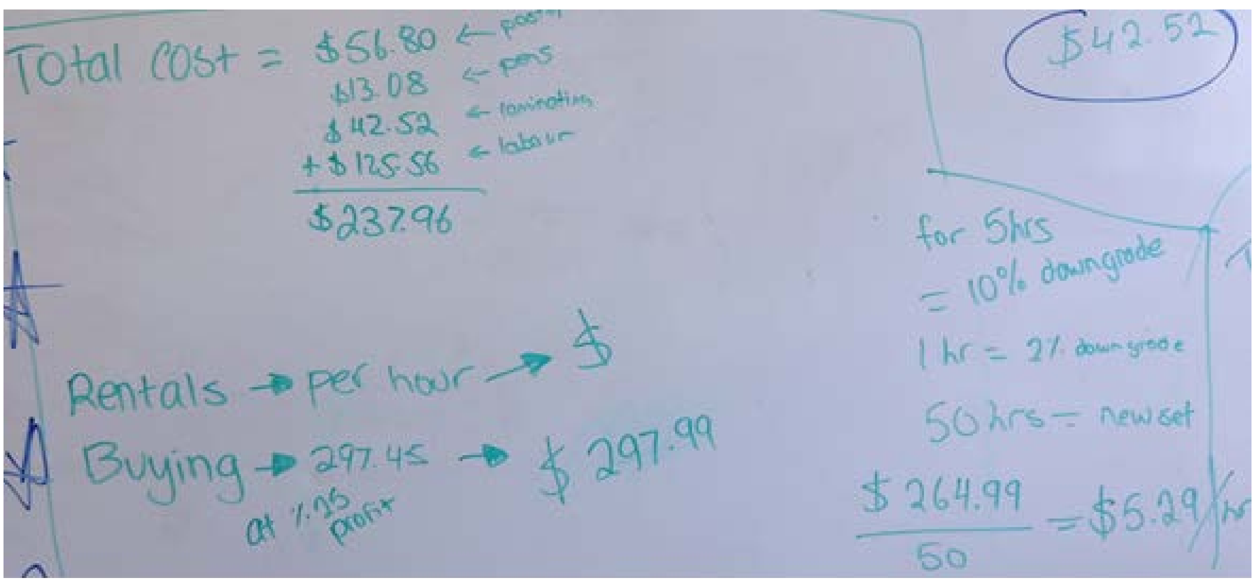


Figure 4.12 **Close up of whiteboard shows their total cost calculation with an added profit and their 'down grade' calculation.**

Figure 4.13 Caroline's poster summarizing her solution

Day 7, Hotel Snap

Hotel Snap is a task designed by Fawn Nguyen that has the students constructing a hotel using snap cubes (Nguyen, 2013) following the instructions in Figure 4.14.

<u>Hotel Snap</u>

Challenge

- As a team, build a hotel that yields the highest profit [score]

Rules and guidelines:

- Each cube represents a hotel room
- Exactly 50 cubes must be used.
- Hotel must stand freely on the face of cubes (not balancing on edges of the cubes)
- Entire hotel is one piece (blocks faces are linked)
- All rooms must have at least one window. A window is any exposed vertical side of cube.

Building costs and tax (daily rate)

- Land costs $400 per square unit
- Land refers to outline of tope view of building, including any enclosed regions.
- A roof costs $10 each. Roof is any exposed top side of cube.
- A window costs $5 each.
- Tax on height of building is calculated by multiplying the tax rate for the highest floor by the total land cost.
    - Floors 1-10 —> 50%
    - Floors 11-20 —> 1000%
    - Floors 21-30 —> 2000%
    - Floors 31-40 —> 3000%
    - Floors 41-50 —> 5000%

Income from each type of room (daily rate)

- The more windows, the more income.
    - 4 windows, 1 roof = $600
    - 4 windows, 0 roof = $500
    - 3 windows, 1 roof = $300
    - 3 windows, 0 roof = $250
    - 2 windows, 1 roof = $200
    - 2 windows, 0 roof = $175
    - 1 window, 1 roof = $150
    - 1 window, 0 roof = $125

Scoring

- Your net profit/loss will be checked for accuracy. A deduction of 50% of your error will be applied to the actual number. If your calculations are correct, then your team will be awarded an extra $1000.

Figure 4.14 Hotel Snap constraints and scoring.

The students need to create a hotel design that maximizes their daily hotel profit based on various rates for income, land taxes, building costs while also meeting a few constraints.

Caroline and her partner constructed a hotel and then began working through the calculations to determine the profit. They organized their information using tally charts (Figure 4.15). Once they filled in their first tally, Caroline realized their chart was not usable because the income also depended on whether or not the room had a roof, so they reorganized the information into a new tally chart (Figure 4.21). As they started filling in the information and making calculations, they started to better understand what features would make their hotel money, and what was costing them money. Although students are not supposed to redesign their hotel once they start making calculations, Caroline and her partner did anyway (Figure 4.20). Their second design (Figure 4.22) was definitely more optimal than their first construction, given the constraints.

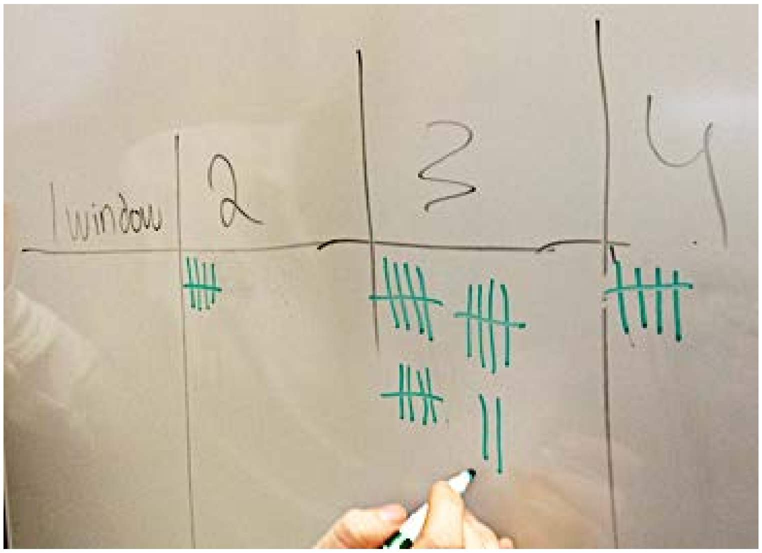


Figure 4.15 First tally

Figure 4.16 Building the hotel

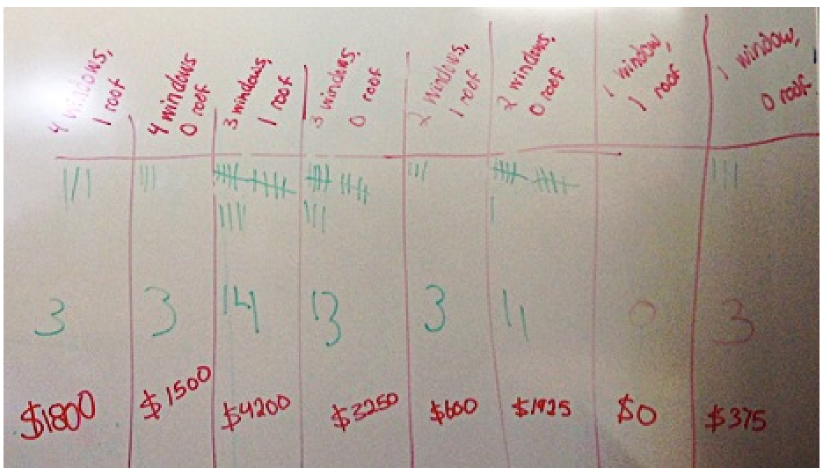


Figure 4.17 Second tally chart

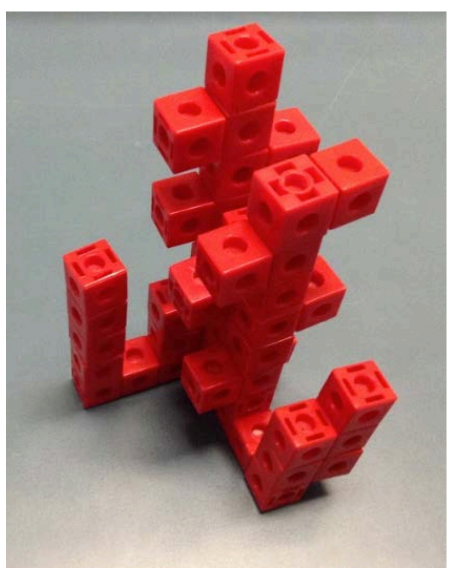


Figure 4.18 Final hotel design

Caroline did not work with any specific curricular mathematics outcomes during this task but was working with multiple variables and constraints, organizing data and making charts, and using strategy and problem solving.

Day 8, Pixel Pattern

A detailed description of this same lesson is also given in Section 4.1.1 where I also include a glimpse of how other groups experience the task and my role as a teacher in the lesson. Pixel Pattern is a 3 Act problem by Dan Meyer (2012) about an arrangement of coloured pixels that grows each second. For Act 1, a video shows the pattern growing as seen in Figure 4.19. After five seconds, the video zooms out to show that this pixel pattern is actually growing within a bounded region shown with a thin red line (Figure 4.20). The class discussed what they noticed about the pixel pattern and students started posing questions about what they wondered. I then wrote a question on the board to focus their attention in a particular direction: *When will the pixel pattern outgrow the box?* The students ask for the dimensions of the box and, in response, I gave them a printed copy of Figure 4.20 which included information that the box is 98 pixels vertically by 182 pixels horizontally.

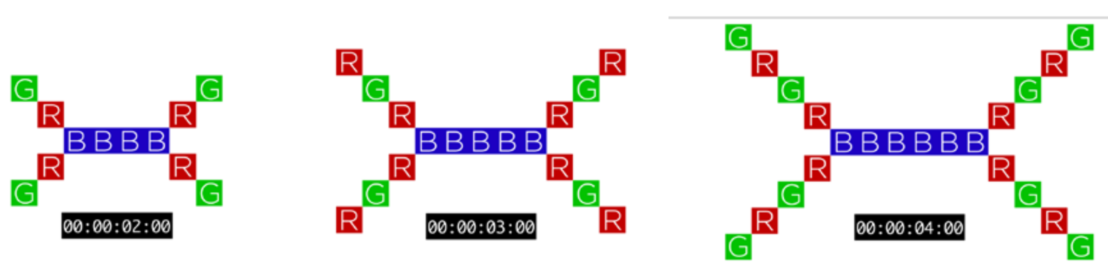


Figure 4.19 Growing pixel pattern

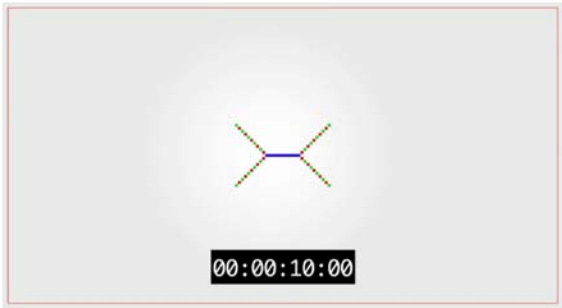


Figure 4.20 Pixel pattern shown in bounded box after 10 seconds. Red outline represents the boundary

Students were paired together randomly and moved to the white boards to make sense of the problem. Caroline and her partner immediately started making a table (Figure 4.21), and with her partner they analyzed how the pattern was growing to represent the relation using a table (10C4), interpreting and explaining the relationships between the data and situation (10C1). The pattern was relatively easy for them to calculate the values recursively, but there was good mathematical conversation about how the colours grow in relation to each other (10C2).

Figure 4.21 Caroline modelled the pattern in a table

They noticed that the growth is linear, but it only grows on alternating seconds, so they had difficulty determining a pattern rule for the number of red or green pixels after *t* seconds (10C5) since their pattern rule did not hold true for any second value, just alternating values. With a bit of creative thinking, Caroline came up with the idea of making one formula for when an odd number of seconds has passed, and another formula for when an even number of seconds has passed. The linear equations she determined to represent the relation (10C6) are shown in Figure 4.21 where she gives separate equation for odd and even inputs. Even though students were not familiar with formal piecewise notation, the general idea of using a piecewise function quickly moves through the classroom to the other students. With her partner, she combined the two separate functions into a single piecewise function of linear equations, which she later figured out how to graph on her own (Figure 4.22).

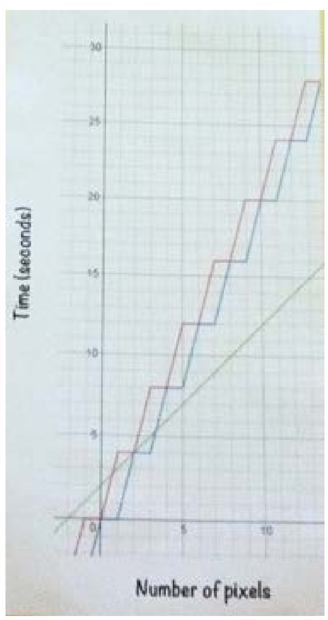


Figure 4.22 Piecewise function

Caroline went far beyond the original goal to determine how long it would take to outgrow the bounding box and demonstrated her understanding of representing linear patterns and determining equations of a linear relation (10C7) in a poster assignment that she later handed in. She was really interested in understanding and representing the pattern growth in a variety of ways even though the basic task was to determine when the pixel pattern outgrew the bounding box.

Day 9, Peg Debate

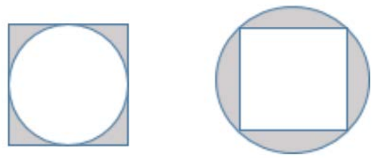


Figure 4.23 Diagram representing the peg scenarios

What is a better fit, a round peg in a square hole or a square peg in a round hole? (Figure 4.23) The class immediately started asking questions for more information: How big is the hole? What do you mean by a better fit? I kept my responses vague so that the students would think critically about the problem. *I don't know how big it is. You decide. Does the size even matter? You tell me what you think is a better fit.*

Caroline and her partner decided that the square hole and the square peg would each be 5cm by 5cm so that they had some values to work with. They used the Pythagorean theorem to calculate the radius of the circle in the case of the square peg (8C1). Their decision that the square peg was a better fit was based on the fact the there was less 'extra room'; they did not use ratios or percentages to make this decision. Figure 4.24 shows their calculations.

After each pair of students worked towards a solution they felt they could justify, the class was divided into two teams to prepare for a debate. Each team had one member of each original pair. Team A had to come up with a convincing argument that a round peg in a square hole is a better fit, and Team B had to argue that a square peg in a round hole was a better fit.

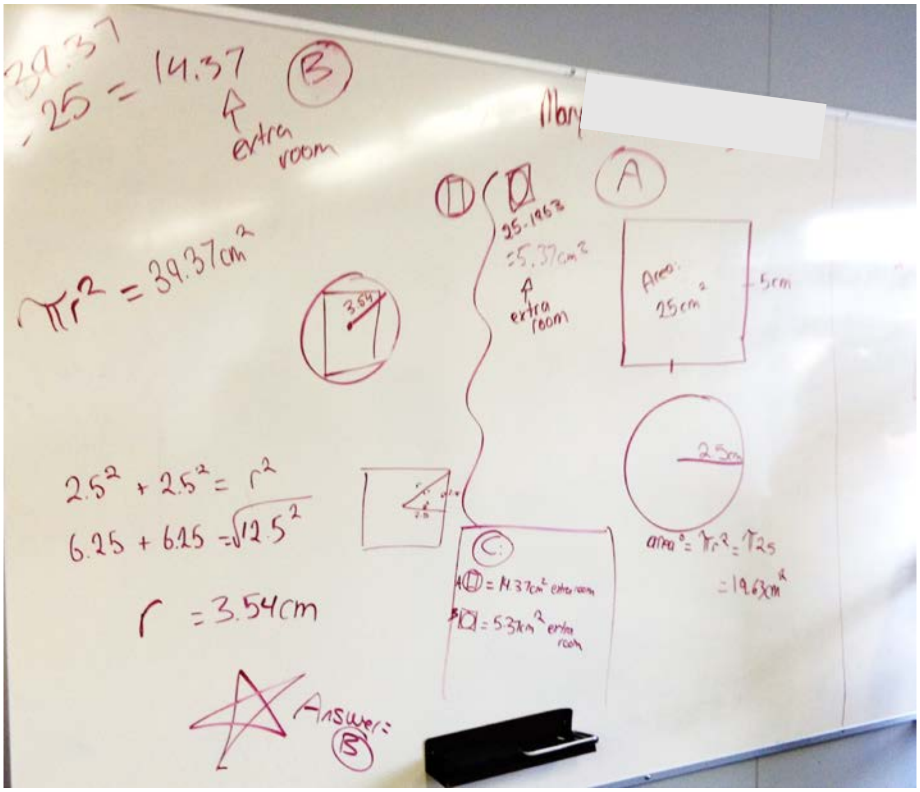


Figure 4.24 Calculations for peg debate

After 10 minutes of working through the problem in different ways with their team to come up with some arguments and counter arguments, the teams set up for a debate. Caroline was in Team B, which used a similar argument that she used in her original calculations. Her opponents challenged her that the hole sizes were not equal and therefore the argument was not valid. She was under the impression that using 5cm for both squares kept the two options equal, but by the end of the debate, she came to the conclusion that the areas of the holes in each scenario ought to be equal to justify her strategy. Once the debate concluded, I gave the students a mini lesson on how they could also use ratios and proportions can be used to justify their results (8A5). This showed the students that it would not matter how big the pegs were in each case, because they could compare the wasted space as a percentage of the entire hole, and that the percentage would be constant even if the peg and hole were scaled either up or down.

Day 10, Algebra Tiles I

One of the prescribed learning outcomes is that students need to concretely, pictorially, and symbolically demonstrate an understanding of the multiplication of monomials, binomials, and trinomials (10B4). Caroline and her peers did not remember ever using algebra tiles, I decided to teach a guided lesson about using the tiles to represent variables. Caroline worked with a partner to model trinomial expressions with algebra tiles. Then I demonstrated how to use algebra tiles to model trinomial addition and subtraction expressions, and challenged them to figure out the simplification of an expression that were a little more difficult than I had explained. For example, $(5x^2 – 3x + 4) – (4x^2 – 5x +7)$ was more challenging, because they needed to understand how to subtract 5 negative x tiles when there were only 3 negative x tiles present. As soon as I felt that the class understood how to use the algebra tiles to model polynomial addition and subtraction, I introduced multiplication expressions. I demonstrated how to use an area model to multiply binomials by an integer or a variable, and gave them exercises such as 3(2x+1) and 2x(x+3) to model. At the end of the lesson, the students were able to multiply two binomials, such as (2x+3)(x+12) using algebra tiles or by sketching what the model should look like if they did not have enough tiles. In summary, Caroline represented and simplified variable expressions with algebra tiles (9B5), added and subtracted trinomials with algebra tiles (9B6), and multiplied binomials with algebra tiles (9B7, 10B4) during this lesson.

Day 11, Algebra Tiles II

With algebra tiles still fresh in their minds, I put a series of problems on the board for the students to work through. *Here are some multiplication exercises. I would like you to work them out using algebra tiles, and then see if you can find any other strategies to multiply them without using the tiles.* Beyond showing them how they could multiply two binomials using an area model the previous day, this was a new concept for the students.

The students worked individually on these problems around tables, but discussed and debated their ideas with their peers. Caroline eagerly engaged in solving these problems as she felt that they were puzzle-like. By the time that she finished the first set (Figure 4.25), she had determined a strategy to solve them without the algebra tiles. She complained that the first questions in the second set (Figure 4.26) were 'too easy', so I added in an extra factor to a few of the problems (see questions 8 and 9).

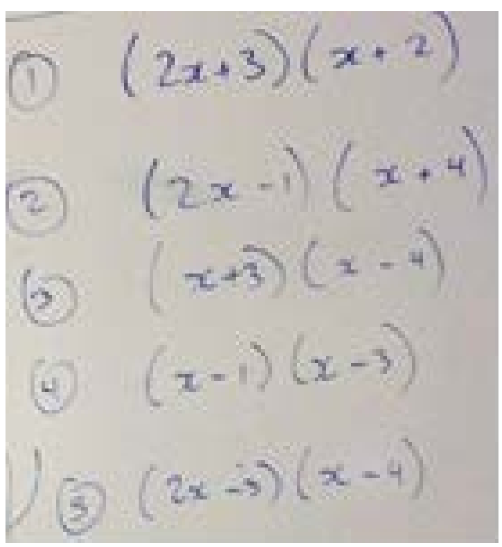


Figure 4.25 Set 1 of multiplying binomials exercises

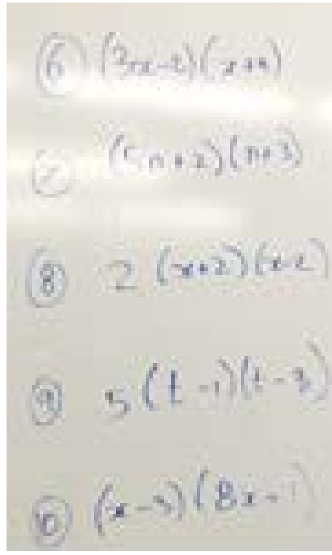


Figure 4.26 Set 2 of multiplying binomials exercises

I could sense that Caroline and the other students at her table were ready for an extra challenge while some of the other students in the class were still struggling. To keep them thinking, I reversed the questions so that they would start factoring. *What would you put in these brackets to make this equation true (Figure 4.27)*? She worked together with another student and the two of them were actively engaged in these exercises for the remainder of the class. Since the students had never heard of factoring or learned any strategies, the exercises in Figure 4.27 required were less like exercises and more like problems.

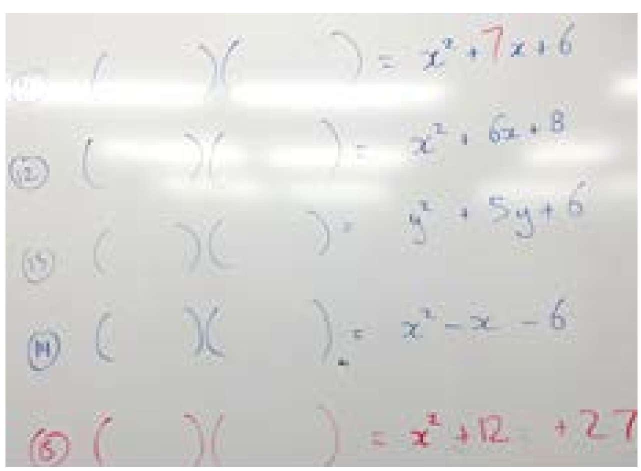


Figure 4.27 Set 3 of multiplying binomials exercises. These extend to factoring

Throughout the lesson, Caroline was multiplying binomials with algebra tiles (10B4) and working backwards to factor trinomials with algebra tiles (10B5).

Day 12, Algebra Tiles III

With the student engagement still going strong with the algebra tiles, I decided to keep the momentum going with some more problems for the students. To refresh their memory, I gave them a few warm-up problems that were similar to the problems from the previous day, and then increased the complexity of these tasks by adding in more variables. As demonstrated in Figure 4.28, the equivalent expressions in questions 4 and 5 are more challenging. Caroline was on a roll and quickly found a single solution, but struggled with the concept of multiple solutions when I asked her questions like: *Is that the only possible solution? How many different solutions can you find? Are there an infinite number of solutions or can you determine them all?* Although she was able to find multiple solutions, she was not confident whether or not she had found all of the solutions.

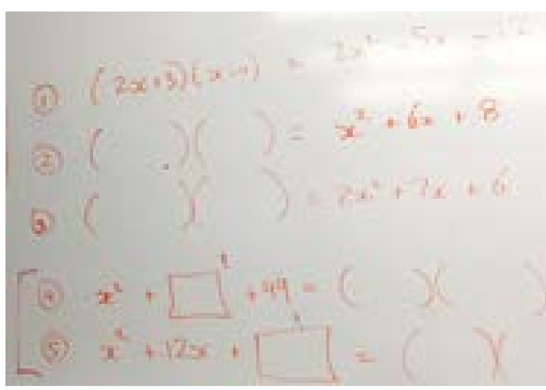


Figure 4.28 Multiplying and factoring exercises.

When she could only come up with two solutions, she made the assumption that she had them all, and when she started finding several solutions easily, she quickly assumed that there would be an unlimited number of solutions. Throughout these questions, Caroline multiplied binomials (10B4), factored trinomials (10B5), and determined a solution set for an unknown value in a quadratic equation (*).

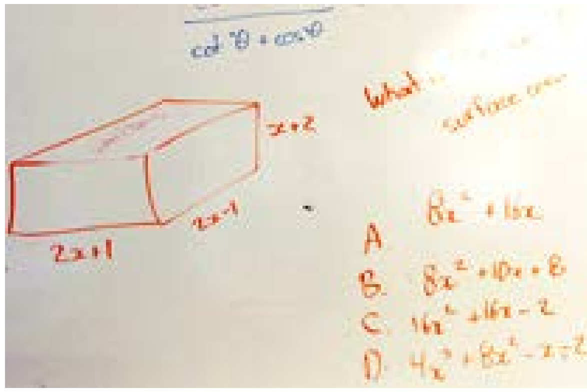
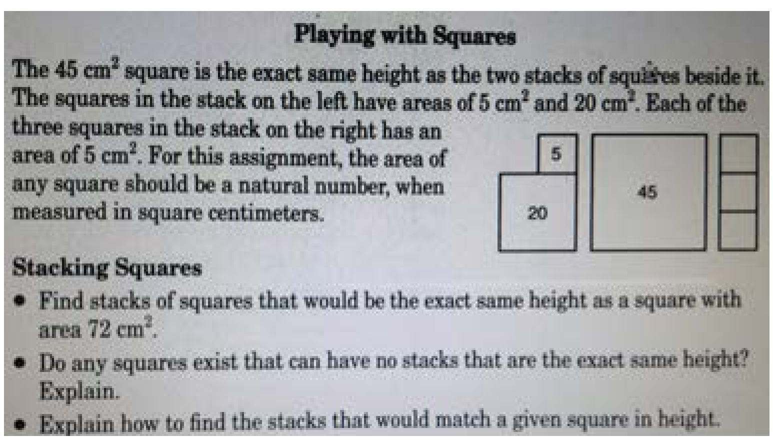


Figure 4.29 A multiple choice problem from a sample final exam.

These questions were followed up with a multiple-choice question from a sample final exam (Figure 4.29). In this task, Caroline calculated the surface area of a right prism (10A3) with variable side lengths. To do this she multiplied binomials to determine the area of the faces (10B4), and then added the expressions for each face to determine the total area.

Day 13, Stacking Squares



Playing with Squares

The 45 cm^2 square is the exact same height as the two stacks of squares beside it. The squares in the stack on the left have areas of 5 cm^2 and 20 cm^2. Each of the three squares in the stack on the right has an area of 5 cm^2. For this assignment, the area of any square should be a natural number, when measured in square centimeters.

Stacking Squares

- Find stacks of squares that would be the exact same height as a square with area 72 cm^2.
- Do any squares exist that can have no stacks that are the exact same height? Explain.
- Explain how to find the stacks that would match a given square in height.

Figure 4.30 Stacking Squares task instructions

The Stacking Squares task came from David Wagner (2003) and is quite challenging for students to get a handle on. I presented Figure 4.30 to the class on the projector and put the students into random pairs. Caroline approached this task with confidence and felt that it was easy, although her work and communication demonstrated that she did not understand how to work with squares and roots. She started with making a long list of calculations and converting square roots into decimals as she worked towards a solution as seen in Figure 4.31. I sat down at their table and worked with them to determine a solution that met the restrictions of the original challenge. We divided the square with area of 72 cm^2 into 4 equal squares of area 18 cm^2 to visually observe how a stack of two squares with area 18 cm^2 would have the same height as one square with an area of 72 cm^2. She realized that it was necessary to divide 72 by perfect square factors (9A5), such as 4 and 9. Throughout the discussion of the problem, we also explored how determining the prime factorization of 72 could help to determine how many solutions this problem would have (Figure 4.32).

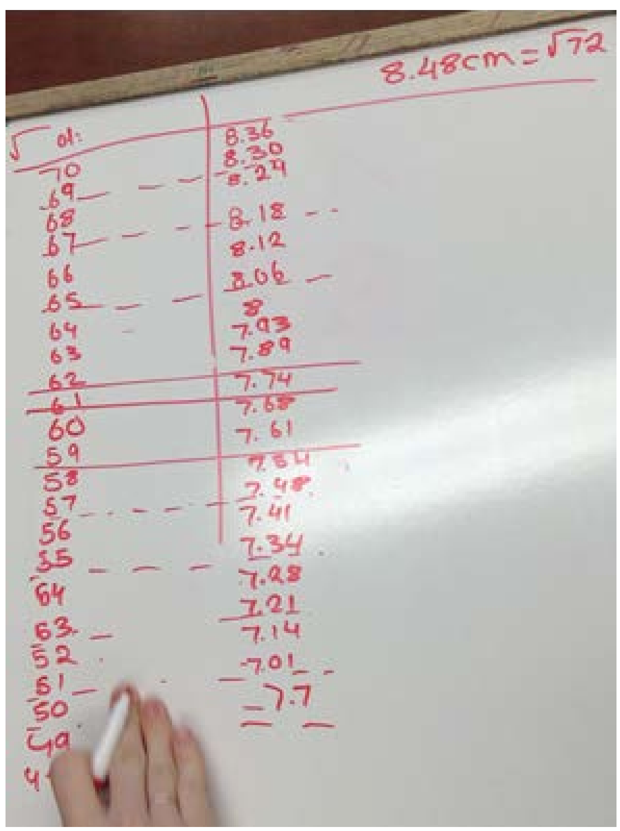


Figure 4.31 Caroline's first attempt to solve the Stacking Squares

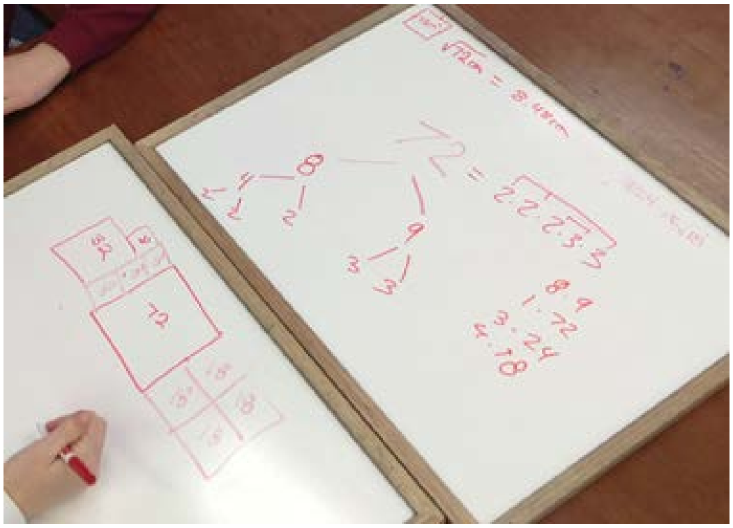


Figure 4.32 Caroline's whiteboard after I helped her

To calculate the prime factorization of 72, we drew a factor tree (10B1). Once Caroline and her partner started to understand the Stacking Squares task a little better, I grouped them together with another pair to continue working on the task to find more possible stacks (Figure 4.33). By the end of the class, she was well on her way to understanding the underlying concepts of simplifying mixed radicals and finding equivalent radicals (10B2).

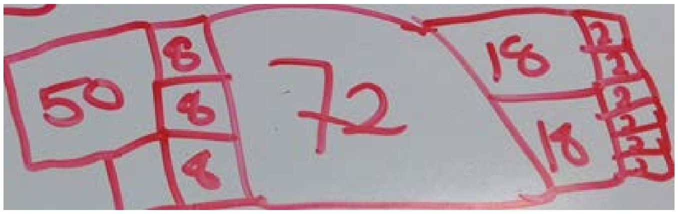


Figure 4.33 Caroline determined a few more stacks of squares

Day 14, Square Roots

To keeping the momentum going of working with squares in the Stacking Squares task the previous day, I decided to give a lesson about square roots (9A5). After diagramming what the square root means visually, I explained how $\sqrt{18}$ just refers to the length of the side of a square with an area of 18, even if the exact value cannot be calculated in decimal form. Most students were uncomfortable leaving an integer inside of a root, because they felt it was a mathematical operation acting on the number that needed to be calculated away! Using the concepts they explored with Stacking Squares, I explained the equivalence of $\sqrt{18} = 2\sqrt{4.5} = 3\sqrt{2}$ using a visual model similar to that in Figure 4.34 (10B2).

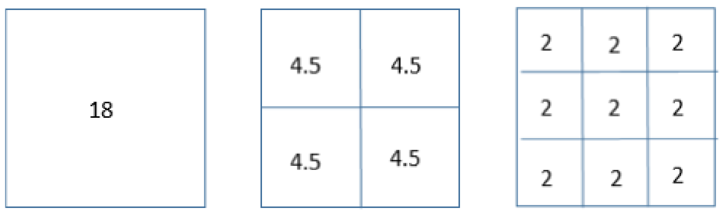


Figure 4.34 Visualization of equivalent mixed square roots

After this brief lesson, built on the Stacking Squares task, I gave the students a list of four mixed radicals to order from least to greatest (10B2). During the first round, the Caroline determined this visually with squares. Throughout the remainder of the class, I continued to make up lists of radicals for students to sort until they started finding strategies to determine the solution without drawing diagrams. Caroline was initially confused and continued to mix up side lengths with areas. She spent much of the time discussing the problems and working with a few peers making sense of simplifying radical expressions (10B2).

Day 15, Three Lines

Three Lines is a task that Dan Meyer posted on Twitter. *Determine the area enclosed by these three lines (Figure 4.35).* I decided this would be an interesting problem, as it requires the student to have a strategy to graph linear equations that are not in the familiar slope-intercept form, and to determine the area of a triangle when they are not directly given a height.

$$y = \frac{1}{2}x + 2 \qquad\qquad y + 2x = 7 \qquad\qquad y = 3(x+4)$$

Figure 4.35 Equations for Three Lines task

This problem has the potential to become easier if the students realize that two of the lines are perpendicular to each other and therefore the enclosed shape will make a right triangle. The students had not graphed relations given just an equation in quite a while, and were a little unsure how to start without guidance. With student giving suggestions, I made a small table of values for the first equation and talked about the shape of the graph as we plotted those points (Figure 4.36). I directed the class to graph the three equations and find the enclosed area, hinting at the fact that the shape of the other two graphs might not be linear. We also went on a tangent class discussion about names of other possible graph shapes, such as quadratic (Figure 4.37).

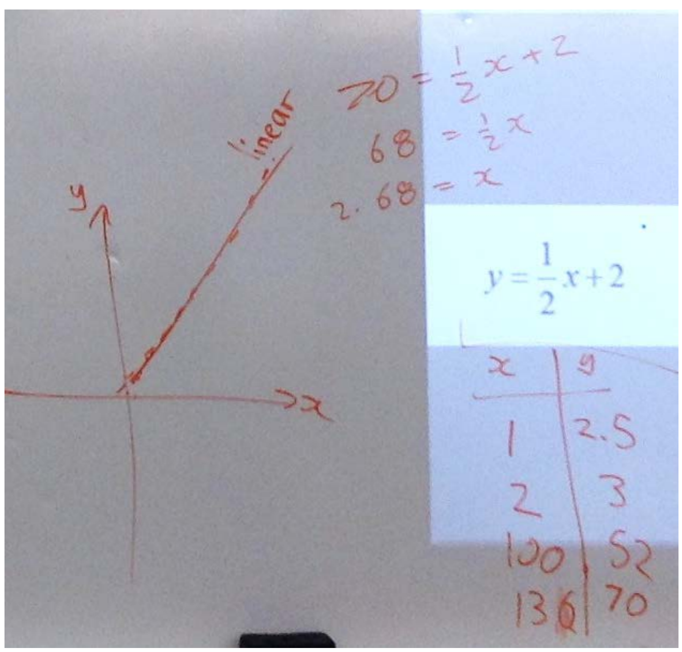


Figure 4.36 Graphing the first equation

Figure 4.37 Sketches from discussion about different types of functions and how the graphs of the three relations might enclose a surprising shape.

As I briefly described what sinusoidal graphs were and the different degrees of polynomial functions, one student (not Caroline) exclaimed, “So, the after cubic and quartic, would the next ones be quantic and septic?!” While I continued in conversation about different polynomial functions with that student individually, Caroline made a table of values for each of the functions as shown in Figure 4.38.

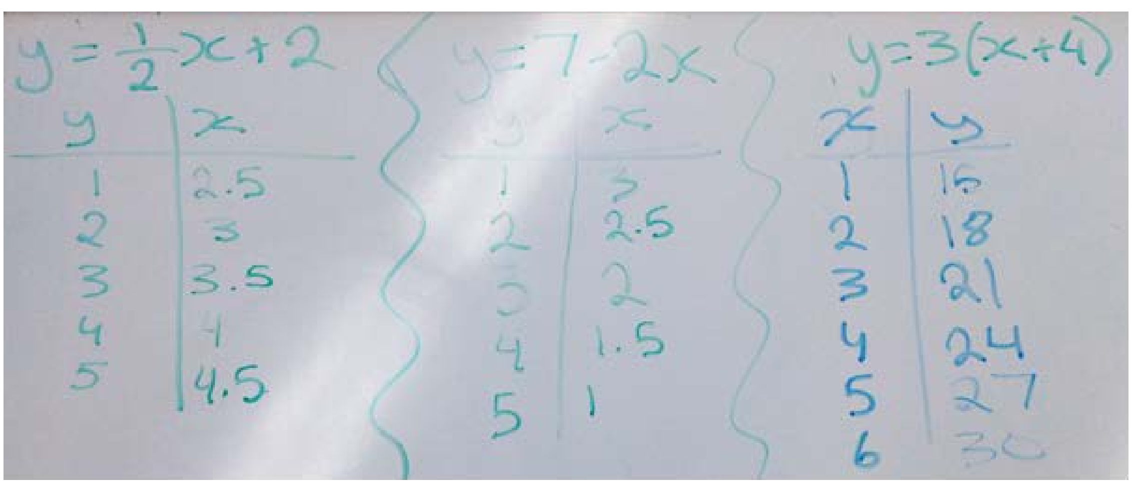


Figure 4.38 Caroline’s table of values each of the equations

On paper she carefully drew a set of axes and plotted the points. Although she graphed it correctly, she did not use the same scale on both the horizontal and vertical axes, making it more challenging to accurately calculate the area of the triangle. She re-graphed the equations on a new set of axes so that her triangle would be to scale (Figure 4.39). Caroline created a table from the equations and then used the table to graph the three lines (10C4). She recognized that two of the lines had perpendicular slopes (10C3) and was able to identify characteristics of the graphs, such as intercepts and intersections (10C5). After graphing the linear equations, she solved three systems of linear equations graphically (10C9). To calculate the area of the enclosed triangle (7C2), Caroline calculated the distance between each of the intersecting corners of the triangle using the Pythagorean Theorem (8C1).

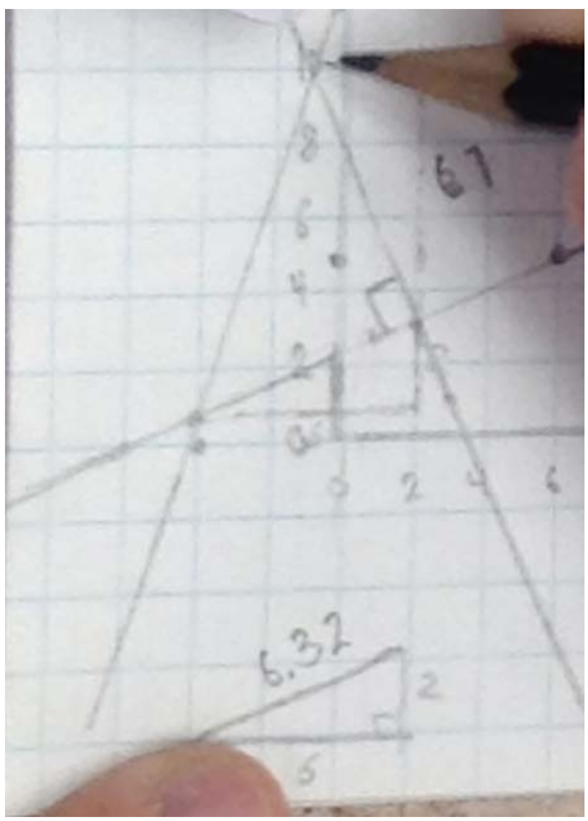
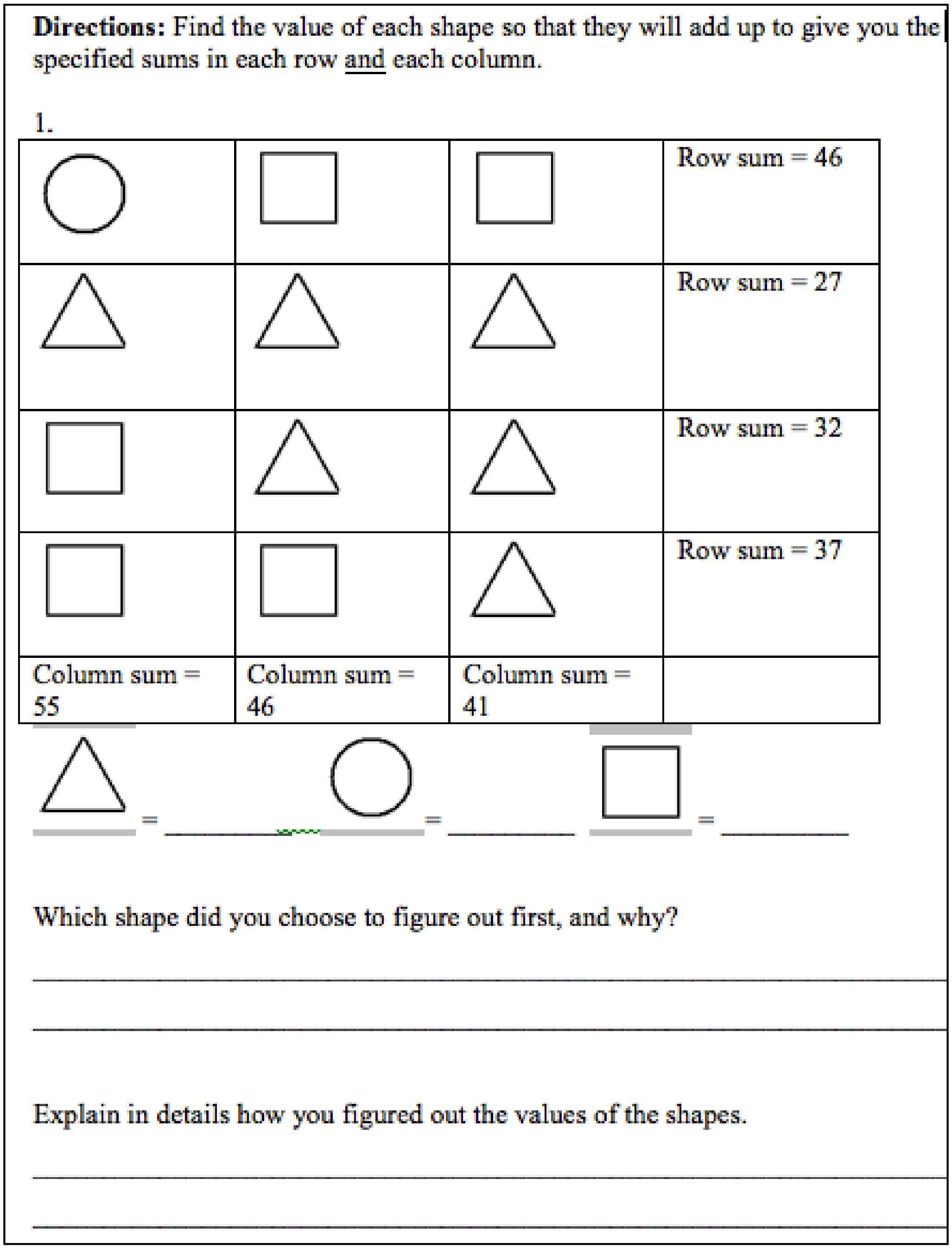


Figure 4.39 Caroline's graph of the three linear equations

Day 16, Shape Equations

For this task, I gave the students a guided worksheet package I found on an online blog (Mimi, 2010). The students started out with solving systems of linear equations with three variables pictorially as puzzles. One of these puzzles is shown in Figure 4.40. Each puzzle is followed by a question or two asking students to explain some element of their problem-solving process in words. In part two of this task, students are shown how equations using the variables x, y, and z can be redrawn as shape puzzles. The level of difficulty really increases as the puzzles move from shapes towards a more algebraic approach of solving the systems of equations. Throughout the worksheets, students are required to really think about what they are doing and make sense of the algebraic form of linear systems. If she concluded in one row that a square must be equal to 2 triangles, she would carry out the substitution by circling shapes or groups of shapes and drawing in the substitution. Caroline worked through these guided tasks with a problem-solving mindset and was able to solve systems of linear equations pictorially, by substitution, and by the addition and subtraction method to eliminate variables (10C9).



Directions: Find the value of each shape so that they will add up to give you the specified sums in each row and each column.

1.

○	□	□	Row sum = 46
△	△	△	Row sum = 27
□	△	△	Row sum = 32
□	□	△	Row sum = 37
Column sum = 55	Column sum = 46	Column sum = 41	

△ = ________ ○ = ________ □ = ________

Which shape did you choose to figure out first, and why?

____

____

Explain in details how you figured out the values of the shapes.

____

____

Figure 4.40 Example shape equation puzzle

Day 17, Number Puzzles

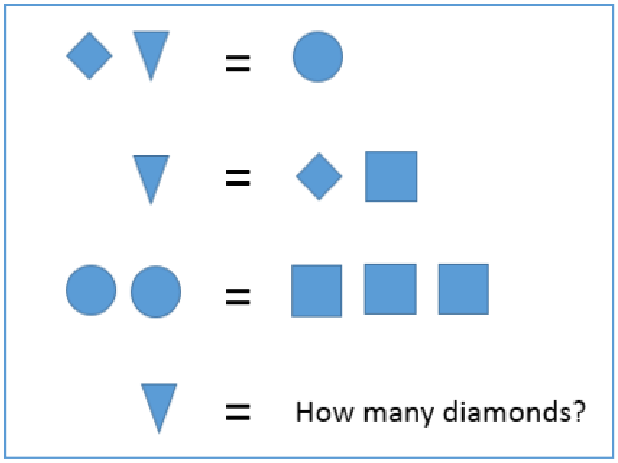


Figure 4.41 Warm up problem solving activity involving systems of equations

Given that these three equalities are true, how many diamonds would be equal to 1 one triangle? Caroline and her partner solved the system of equations shown in Figure 4.41 pictorially using substitution (10C9). Their work can be seen in Figure 4.42. This short task was followed up by two number puzzles.

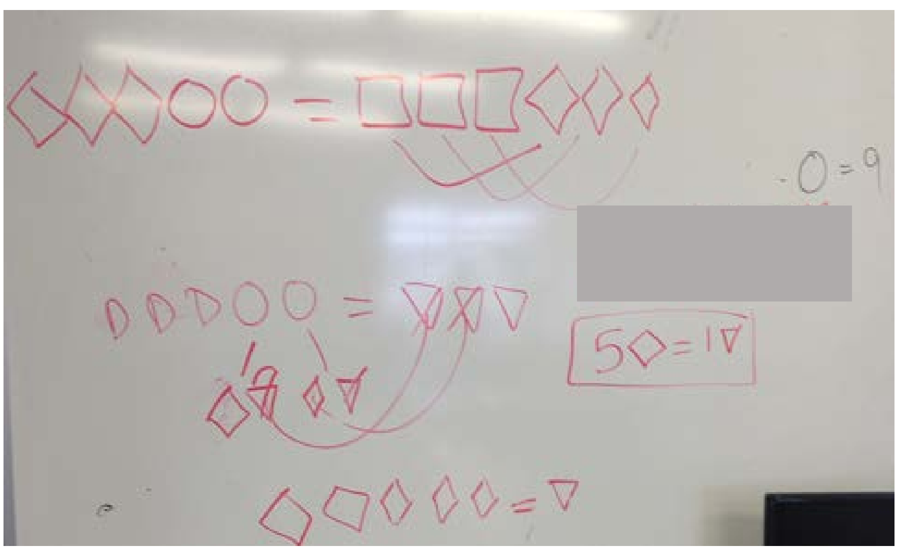


Figure 4.42 Solving a system of equations pictorially

In the second puzzle, students needed to place the numbers from 1-8 in the boxes shown in Figure 4.43 such that no two consecutive numbers share an edge or corner. In the 3rd puzzle, students needed to place each of the natural numbers from 1-

17 in a section of the Olympic rings shown in Figure 4.43, so that every section contains a number and the sum of the numbers in each ring is the same.

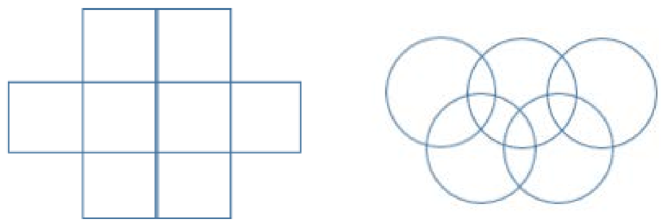


Figure 4.43 Diagrams for the two number puzzles

Caroline used strategic guess and test strategies for these last two tasks but did not engage in any specific curricular content from PFM10.

Day 18, Road Lines

As I was driving to school this morning I noticed the dashed road lines coming towards me on the highway. Have you ever wondered how long they are? I really wanted to know, so I stopped at the top of the overpass and got out of my car to take a few pictures (Figure 4.44). What do you think? Let's write down some guesses on the whiteboard.

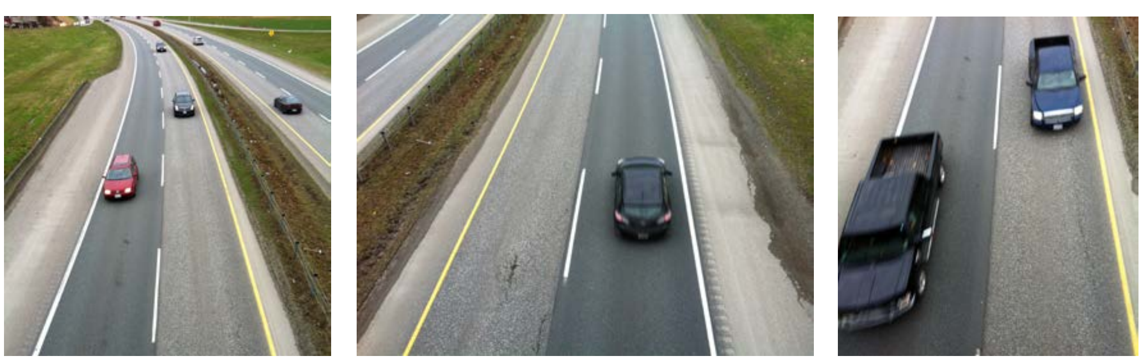


Figure 4.44 Photos I took of the highway to show a comparison

The students were surprised that the lines were even longer than some vehicles. After the class discussion about road lines, the students were put into random groups to design their own mathematics question. *Explore the mathematics of a question that comes to mind when you think of road lines. For example, how should the speed limit effect the length of a road line? Or maybe figure something out about the line painting machines.*

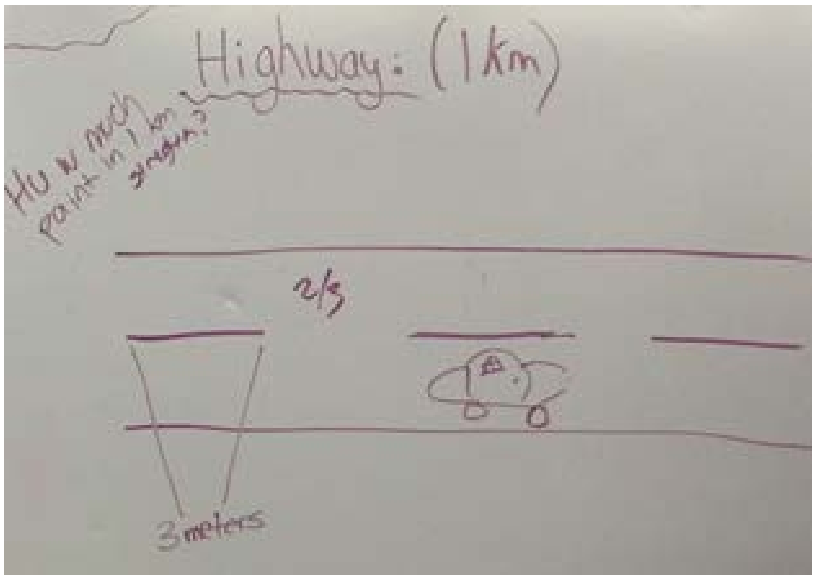


Figure 4.45 Diagram of the highway road-lines

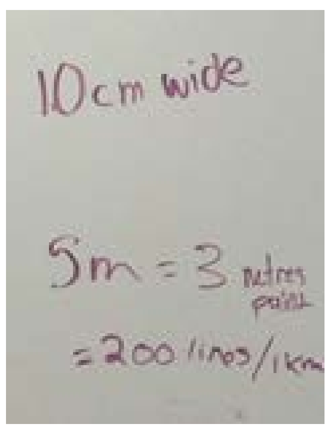


Figure 4.46 Estimations about length and dimensions of the roadlines

Caroline and her group decided to calculate how much paint would be required to paint a one-km stretch of broken line on this highway. They began with a diagram (Figure 4.45) and made some estimations to work with (Figure 4.46).

Figure 4.47 Area calculation of a single dashed line

They decided that each painted line was about three meters long and that the gaps in between were about two meters. From this they calculated that for every five meters of distance, they will need to paint a three-meter line, for a total of 200 lines in one kilometer. They also estimated that each line would be about 10 cm wide. From this they calculated the area of each painted line (Figure 4.47). Caroline's group was not sure what to do about the fact that paint is measured in litres, which is a unit of volume, but their calculations resulted in an area. They were not sure how to solve this problem and resorted to a few online searches about paint. They found a website that sold paint by the gallon which they converted into litres (Figure 4.48).

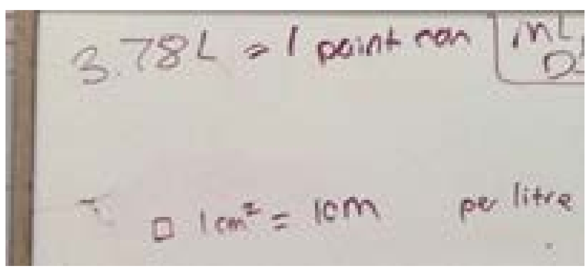


Figure 4.48 Volume of paint can converted into litres

I do not have data on the rest of Caroline's problem solving because I was working closely with another group, nor a final solution, since they did not write down all their calculations. During the course of this task, Caroline was thinking mathematically, discussing and debating strategies with her peers, problem solving, estimating measurements and converting between SI and Imperial units of measurement (10A1), calculating the area and volume of the road lines while discussing the difference between the two for thin surfaces (10A3), and finding unknown data using available resources (*).

In what follows, I will give a more detailed analysis of one of the lessons and describe how I linked the data about Caroline's problem solving to the learning outcomes in the curriculum. This will be followed by discussion of how the FPM10 curriculum was encountered almost in its entirety during the entire 18 days.

4.2. Analysis of One Lesson

4.2.1. Story of the Lesson

When the students came into class on Day 8 of my data collection period, I presented them with a Three Act Problem called Pixel Pattern from Dan Meyer. In a 3 Act Problem, the students are first presented with an incomplete video that promotes discussion. The students were encouraged to call out anything they noticed in the video or some questions they wonder so I can write it out on the white board. Generally with a Three Act Problem, we would vote on a favourite question, but for this particular problem I proposed the question I wanted them to solve: When will the pixel pattern outgrow the bounding box? For Act 2, the students may ask for any extra information they think will be useful to solve the problem, and I give them anything that they ask for if I know the information. In this problem, a few students wanted to know the dimensions of the bounding box and a few wanted to see what the pattern looked like. I already had some papers printed out with this information on it that I handed out. These diagrams are shown on Page x where I explain this task in a little more detail. Once the students had the information they think they need, I divided the class into random groups of 2 students each and they all move to find a whiteboard to work at. Most students start making a diagram of the pattern almost immediately and several groups start making a table.

One group of two girls started by drawing out several more stages of the pattern and then put the totals number of pixels of each colour in a table. They first made a note that the pattern rule for the blue pixels is always two more than the number of seconds passed. Then they came up with a separate pattern rule for the number of odds and evens of each second. One of their equations looked something like this:

For odd numbered seconds: time (in seconds) + Blue pixels = Red pixels

Both of their equations were dependent on knowing how many pixels were the other colour, so we had a discussion about how to calculate the number of red pixels with only 1 variable, time. We discussed how when the functions depend on each other, we get little information. I used the analogy that it is much like asking their mom for something and she says, go ask your dad. But when you go ask your dad, he says, go ask your mom. So you run back and forth indefinitely and get no information after a lot of effort. These two girls eventually came up with two different relations for width/height of the pattern to determine when it out grew its boundary.

While these two girls were working, Caroline worked with another partner and really focused on the problem that the red and green pixels grow during alternating seconds. By the time I moved to where they were working, the idea of creating two separate functions, one for odd numbered seconds and one for even numbered seconds, had already travelled around the room. Caroline and her partner had no problem determining separate formulas for the number of red and green pixels based on the number of seconds passed, and created a piecewise function for odd and even seconds. They calculated the width of the pattern to determine when it outgrew the bounding box horizontally, but did not initially realize that it might outgrow its boundary vertically first. Instead of finding a formula or relation of the growth of the width, they took the maximum width allowed and divided that by 3 to get an approximation of the time it took. Their rationale was that each section, the diagonal on the left, the blue strip in the middle, and the diagonals on the right each grew by one pixel every second and therefore colour did not matter. Since they solved the original question fairly quickly using this strategy, they moved on to creating tables, and relations, and functions for each of the pixel colours. These two girls paid very little attention to the rest of the students in the class as they were fully engaged in the mathematical concepts behind the task. Even determining a solution to the question posed did not stop them from further investigating the problem. They were determined to figure out how to make a graph on Desmos that would show growth one second, and would pause the next second.

A third pair of students in the class also made a table to tract the number of each pixel colour in one of the diagonal sections. These two did not realize initially that they were missing three other diagonals until they compared their results with the work of their peers working beside them on the whiteboard. They decided to just correct the results later by just multiplying the table results by 4 instead of recounting all their table values.

Caroline and her partner explained to me how they created their table of values for each colour and how they extended the table without drawing diagrams for each situation. We also discussed the unique situation that the output of their function (number of red pixels) would stay fixed while the time (seconds) changed and that this would not a cause a problem in making a function since at any moment of time there would still only be one possible value for the number of pixels. Caroline was able to describe and represent the pattern as a linear function on alternating seconds using words, a table of values, a graph, and an equation. She also noticed certain characteristics about the domain of the function, and created separate linear functions in slope-intercept form for even numbered inputs and odd numbered inputs.

The most interesting component in this lesson for me is how the concept of creating a piecewise function moved through the classroom quickly. Most students stopped once they created the functions, but Caroline and her partner went on to figure out how to graph the relation as well. They were determined to figure out how to graph a piecewise function on the Desmos graphing application. Although the domain on their Desmos graph is not entirely correct, they exceeded the expectations of FPM10 by going beyond graphing a simple linear equation or only using a table to determine a solution to the problem.

4.2.2. Analysis of the Data

For the Pixel Pattern lesson, I collected a variety of data about the mathematics that Caroline encountered to determine which learning outcomes could be linked to the task. During the lesson, I took a photograph of the whiteboard work from her group as shown in Figure 4.49. After the lesson, I recorded all the mathematical concepts that I recalled either discussing with Caroline during my meetings with her group, that I observed her working on, or noticed her communicating with others about. Caroline chose to use this task for one of her assignments and designed a poster outside of class time summarizing her work on this task which she submitted a few weeks later (Figure 4.50). She included a paragraph of how she solved the problem on her poster as shown in Figure 4.51, but I chose to not tag any of this data as extra encounters with specific learning outcomes, since it is not very clear. The same applies to most of the information on her poster assignment, or it could be considered redundant since.

To code the learning outcomes that Caroline encountered, I began with my written record of observations. The whiteboard photographs generally support my anecdotal record, but only represent a single moment during the lesson since the students often erase or rework their solution as they are problem solving. As Caroline and her group explained to me how they created their table for each colour and how they extended the table, they demonstrated an ability to interpret and explain the relationships among data, graphs, and situations. I tagged this as an encounter with the learning outcome 10C1 (Interpret and explain relationships among data, graphs, and situations) from the FPM10 curriculum.

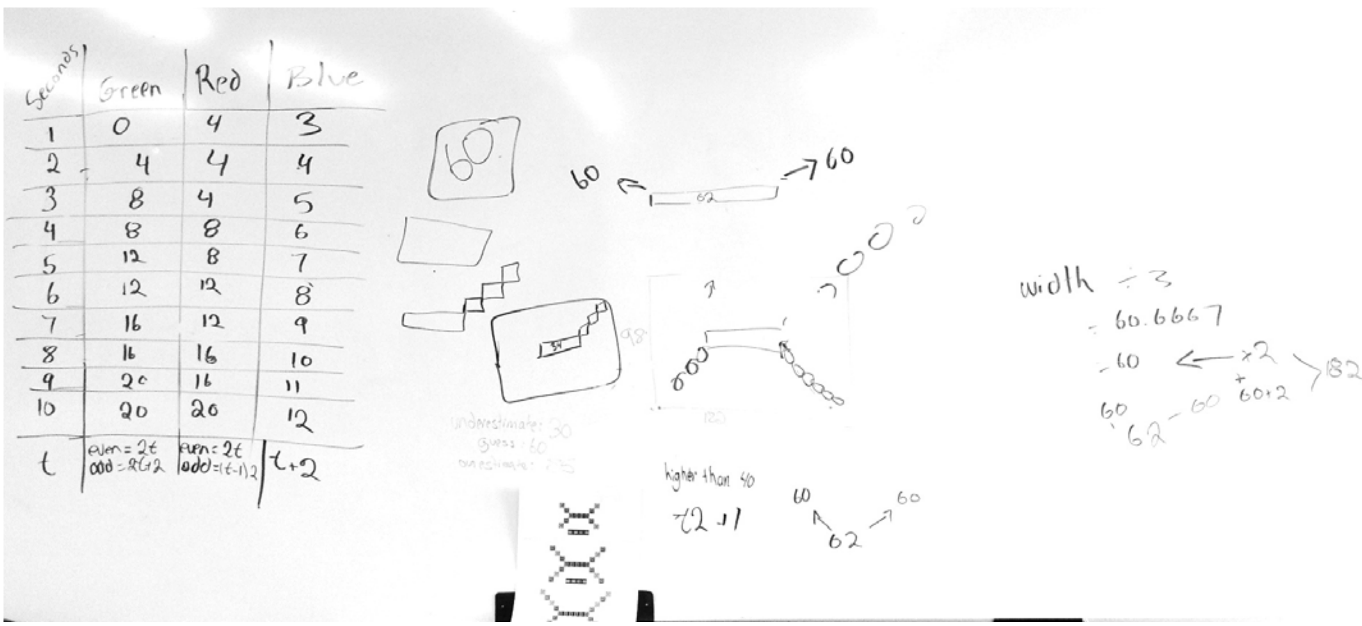


Figure 4.49 Photo of whiteboard work from Caroline's group during Pixel Pattern Task.

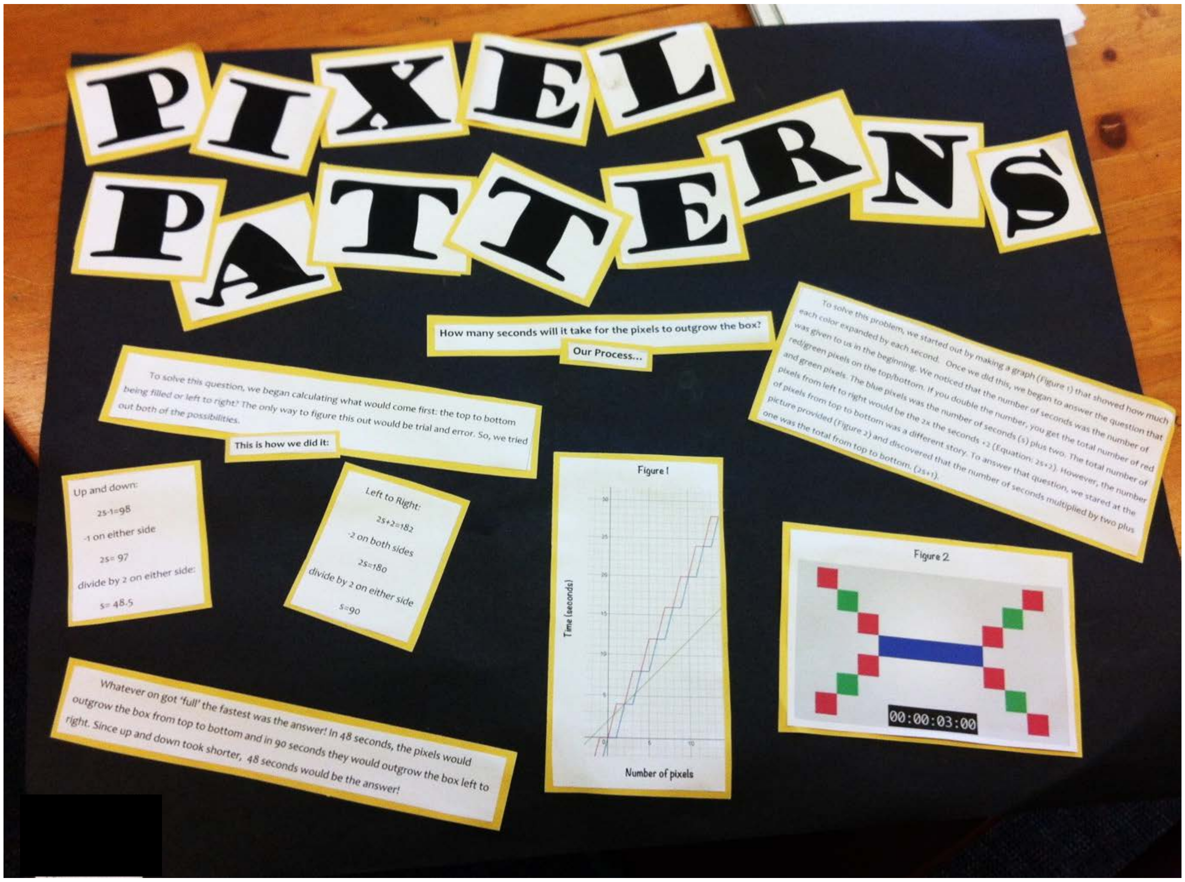


Figure 4.50 Poster that Caroline submitted for Pixel Pattern

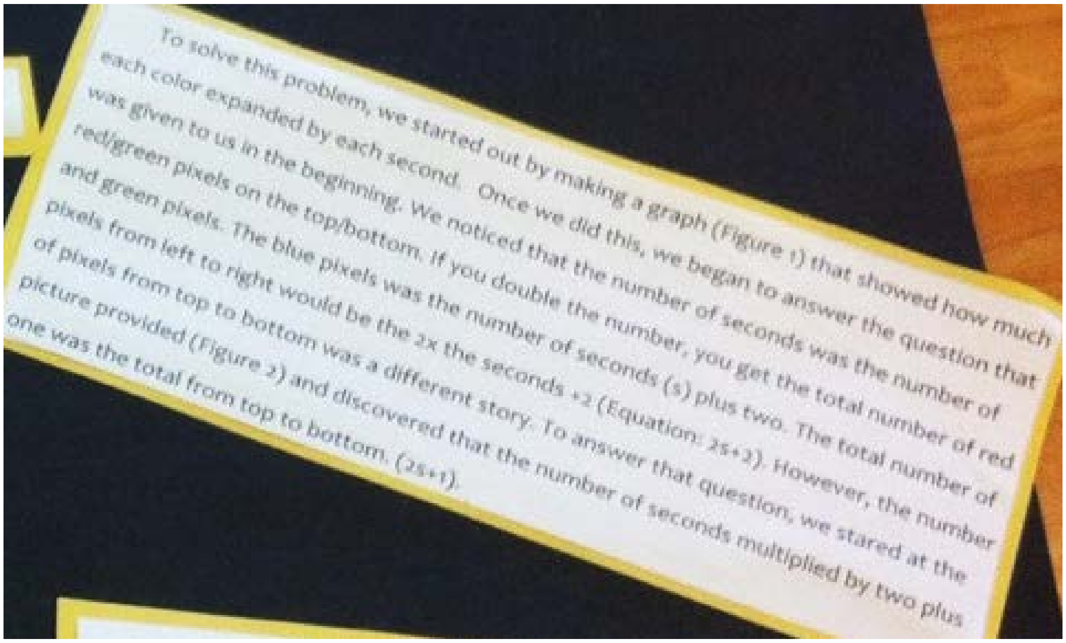


Figure 4.51 Close up of Caroline's summary of what she did in this task from poster.

I marked our conversation about the relation having only one possible value for the number of pixels at any point in time, as an encounter with the concept of relations and functions, which is learning outcome 10C2 (Demonstrate an understanding of relations and functions). In my notes, and also supported by the table of values she drew on the whiteboard (Figure 4.49) and in the work on her poster (Figure 4.50) Caroline demonstrated she was able to describe and represent the pattern as a linear function on alternating seconds using words, a table of values, a graph, and an equation. I marked this as an encounter with 10C4 (Describe and represent linear relations using words, ordered pairs, tables of values, graphs, and equations) from the curriculum. In my notes, I wrote that she noticed certain characteristics about the domain of the function, so this is a brief encounter with 10C5 (Determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range).

Caroline initially graphed the growth of each colour as a linear function ignoring the fact that during alternating seconds the value did not grow continuously. These three functions that she created and graphed were marked as encounters with both 10C6 (Relate linear relations expressed in slope-intercept form, general form, and slope-point form to their graphs) and 10C7 (Determine the equation of a linear relation, given a graph, point and slope, two points, or point and parallel or perpendicular line, to solve problems) since she made connections between graphs and their equations in slope-intercept form, and used a table (which is more than two points) to create an equation. Once she made those equations and graphs, I casually gave her a challenge to figure out how to make a graph in Desmos that only grows on alternating seconds to match her original piece-wise functions shown in the bottom of the table shown in her whiteboard work. I left her to work on puzzling with only a suggestion to use a step function. She managed to figure this out somewhat as shown on the graph on her poster, but I am not certain exactly how she accomplished this since her graph does not include the equations she inserted into Desmos. I marked this as an encounter with piecewise functions which is not a required learning outcome in the curriculum. Throughout the 18 lessons outlined in the following section, I marked instances like these with an asterisk because I felt they were worth noting even though I could not flag them with a learning outcome.

Caroline encountered multiple learning outcomes during this particular lesson that are all closely related. There were some days that she did not encounter any of the required learning outcomes for the course, and some days that she encountered an interesting variety of learning outcomes. In what follows, I will discuss the extent of the mathematics curriculum that she encountered during the 18 lessons of this study.

4.3. Discussion

After mapping the data to the curriculum, I noticed that Caroline encountered an extensive amount of mathematics during the 18 lessons. During this time, she explored a wide variety of content and skills from the curriculum through problem solving. I first discuss the content and curriculum she encountered in individual lessons. Then, because this was a FPM10 course, I discuss the extent and redundancy of the required curriculum outcomes that Caroline encountered during the entire period.

4.3.1. Learning Outcomes Encountered by Day

As she worked through the different tasks, Caroline was in a process of sense-making and grappled with up to 14 different learning outcomes that are in the mathematics curricula from Grades 6 to 12 each day. Some days, such as Day 7, working on the Hotel Snap task, she did not directly encountered any prescribed learning outcomes in the Grade 6 – 12 curricula; but the problem-solving skills she developed during those tasks were valuable. On Day 3, her problem solving and investigations dipped into content from Math 8, Math 9, FPM10, and both the Foundations and Pre-Calculus Math 11 courses. This shows that there was time within a single 50-minute lesson for both a review component and an extension component in addition to working with the required Grade 10 curriculum.

Other days, Caroline's work did not extend much beyond the FPM10 curriculum, but stayed more focused on a series of related curriculum outcomes. An example of this is on Day 8 with the Pixel Pattern activity from Dan Meyer. She worked almost entirely with prescribed learning outcomes in the section of the FPM10 curriculum which only focuses on relations and linear functions. There still was time for a bit of an extension that day as she developed a piece-wise function to represent the patterns and used problem solving strategies that can be found in the Foundations 11 Math curriculum.

A summary of the data can be seen in Appendix B organized chronologically by day. The Mathematical Concept column in the table briefly identifies the topic for each prescribed learning outcome. Further clarification about the codes for the prescribed learning outcomes can be found in Appendix A. It is evident in the table, that Caroline encountered many concepts. There were moments of review and extension that popped up regularly throughout the entire period as well. These can be seen by the codes for the prescribed learning outcomes that are listed in the columns to the left (review) or right (extension) of the PFM10 column in the table. You will notice that most of the concepts fall under the PFM10 column, which demonstrates that Caroline was mostly engaged with the required learning outcomes for the course. Even though I was not explicitly telling the students how to solve the problems, or directly teaching the concepts, Caroline engaged with the curriculum on a regular basis through the process of problem solving. In addition to working with the mandated learning outcomes for the course, I noticed that Caroline was making sense of the mathematics in respect to the tasks she was working on, and not just imitating the ideas that I had shown her.

4.3.2. Overall Curriculum Encountered

In order to fully understand the extent of the curriculum that Caroline encountered during the 18 lessons, I rearranged the prescribed learning outcomes by Grade and concept rather than by activity. This revealed a better picture of the content encountered. Not only was the majority of the content encountered from PFM10, except for 2 outcomes, 10B3 and 10C8, Caroline worked with every single outcome from the entire curriculum at least once. 10C8 requires students to represent a linear function using function notation, so it is not at all surprising that Caroline did not work with this LO while problem solving, as it requires students to know about a mathematical convention. Some prescribed outcomes she even encountered multiple times over different tasks, such as 10B1[7], which Caroline engaged with eight times during four different tasks. Caroline also engaged with seven LOs from the Math 9 curriculum, four LOs from Math 8, two LOs from Math 7, and one LO from Math 6. These represent the time she was engaged in reviewing mathematics from prior years. On the other end of the spectrum, she also engaged with two LOs from PC11, two LOs from F11, and one from PC12, which demonstrates that she took opportunities to extend her engagement with mathematics beyond the expectation for PFM10. In addition to working with prescribed learning outcomes, Caroline also engaged with mathematics beyond the scope of the curriculum in many of the tasks. In 16 instances, I noted these mathematical concepts with an asterisk, since I felt as a teacher that the tasks had value worth recognizing. In several cases, such as in the Hotel Snap task, Caroline did not encounter any curricular mathematics, but engaged in problem solving and strategizing in a meaningful way. Table 4.1 below summarizes the data as organized by grade and LOs.

[7] Grade 10, section B, outcome 1: Demonstrate an understanding of factors of whole numbers by determining the: prime factors, greatest common factor, least common multiple, square root, and cube root.

Table 4.1 Data Sorted by Course and Learning Outcome

Day / Task	**Mathematical Concept**	*Math 6*	*Math 7*	*Math 8*	*Math 9*	*PFM10*	*PC11*	*F11*	*PC12*	*Extra*
Day 1, Gears + Marching Band	Calculating lists of multiples	6A3								
Day 1, Gears + Marching Band	Common factors	6A3								
Day 1, Gears + Marching Band	Prime Numbers	6A3								
Day 5, Goats Extension	Degrees in a circle		7C1							
Day 4, Goats	Area of circle		7C2							
Day 9, Peg	Area of circle		7C2							
Day 15, Three Lines	Area of triangle		7C2							
Day 1, Gears + Marching Band	Percentages			8A3						
Day 4, Goats	Solve problems involving rates and ratios			8A5						
Day 5, Goats Extension	Solve problems involving rates and ratios			8A5						
Day 6, Dutch Blitz	Solve problems involving rates and ratios			8A5						
Day 9, Peg	Solve problems involving rates and ratios			8A5						
Day 3, Function Puzzles + 3digit Sums	Multiplying negatives			8A7						
Day 5, Goats Extension	Pythagorean Theorem			8C1						
Day 9, Peg	Pythagorean Theorem			8C1						
Day 15, Three Lines	Calculating distances			8C1						
Day 13, Stacking Squares	Square roots				9A5					
Day 14, Simplifying + Ordering square roots	Square roots				9A5					
Day 3, Function Puzzles + 3digit Sums	Linear and Quadratic equations				9B2					
Day 3, Function Puzzles + 3digit Sums	Algebra				9B3					
Day 10, Algebra Tiles (1)	Represent variable expressions with algebra tiles				9B5					
Day 10, Algebra Tiles (1)	Simplifying expressions with algebra tiles				9B5					
Day 10, Algebra Tiles (1)	Adding/Subtracting trinomials with algebra tiles				9B6					
Day 10, Algebra Tiles (1)	Multiplying binomials with algebra tiles with degree 2 or less				9B7					

Day 2, Making Groups	Organizing data				9D3					
Day 18, Road Lines	Estimating SI/Imperial measurement					10A1				
Day 4, Goats	Use proportional reasoning and convert units					10A2				
Day 5, Goats Extension	Use proportional reasoning and convert units					10A2				
Day 6, Dutch Blitz	Unit Conversions					10A2				
Day 12, Algebra Tiles (3)	Surface area of box with variable side lengths					10A3				
Day 18, Road Lines	Volume vs surface area for thin surfaces					10A3				
Day 4, Goats	Trig ratios					10A4				
Day 4, Goats	Inverse trig ratios					10A4				
Day 5, Goats Extension	Trig ratios / inverse trig ratios					10A4				
Day 1, Gears + Marching Band	Prime factorization					10B1				
Day 1, Gears + Marching Band	Least Common Multiples					10B1				
Day 1, Gears + Marching Band	Greatest common factors					10B1				
Day 1, Gears + Marching Band	Prime Numbers					10B1				
Day 2, Making Groups	Multiples					10B1				
Day 3, Function Puzzles + 3digit Sums	Factors					10B1				
Day 13, Stacking Squares	Factor Trees					10B1				
Day 13, Stacking Squares	Prime factorization					10B1				
Day 3, Function Puzzles + 3digit Sums	Subsets of Rational numbers					10B2				
Day 13, Stacking Squares	Simplifying radicals					10B2				
Day 13, Stacking Squares	Equivalent radicals					10B2				
Day 14, Simplifying + Ordering square roots	Simplifying square roots					10B2				
Day 14, Simplifying + Ordering square roots	Ordering square roots					10B2				
Day 14, Simplifying + Ordering square roots	Equivalent mixed radials					10B2				
Day 10, Algebra Tiles (1)	Multiplying monomials, binomials, and trinomials with algebra tiles					10B4				
Day 11, Algebra Tiles (2)	Multiplying monomials, binomials, and trinomials with algebra tiles					10B4				

Day 12, Algebra Tiles (3)	Multiplying monomials, binomials, and trinomials with algebra tiles					10B4				
Day 11, Algebra Tiles (2)	Factor trinomials with algebra tiles					10B5				
Day 12, Algebra Tiles (3)	Factoring trinomials					10B5				
Day 3, Function Puzzles + 3digit Sums	Interpret and explain relations					10C1				
Day 6, Dutch Blitz	Interpret and explain relations					10C1				
Day 8, Pixel Pattern	Interpret and explain relations					10C1				
Day 8, Pixel Pattern	Relations and functions					10C2				
Day 15, Three Lines	Slope					10C3				
Day 3, Function Puzzles + 3digit Sums	Describe and represent linear relations					10C4				
Day 8, Pixel Pattern	Describe and represent linear relations					10C4				
Day 15, Three Lines	Describe and represent linear relations					10C4				
Day 3, Function Puzzles + 3digit Sums	Linear relation graph characteristics					10C5				
Day 8, Pixel Pattern	Linear relation graph characteristics					10C5				
Day 15, Three Lines	Linear relation graph characteristics					10C5				
Day 8, Pixel Pattern	Linear equations and graphs					10C6				
Day 8, Pixel Pattern	Create linear relations					10C7				
Day 3, Function Puzzles + 3digit Sums	Finding intersections					10C9				
Day 15, Three Lines	Solving system of equations graphically					10C9				
Day 15, Three Lines	Finding intersections					10C9				
Day 16, Shape Equations	Solving systems of linear equations pictorially					10C9				
Day 16, Shape Equations	Solving by substitution					10C9				
Day 16, Shape Equations	Solving by adding/subtracting rows					10C9				
Day 17, Number Puzzles	Solving systems of linear equations pictorially					10C9				
Day 17, Number Puzzles	Substitution					10C9				
Day 3, Function Puzzles + 3digit Sums	Minimization/Maximization						11C5			
Day 6, Dutch Blitz	Decreasing arithmetic sequence						11C9			
Day 3, Function Puzzles + 3digit Sums	Odd and Even Number concepts for proofs							F11C1		

Day 17, Number Puzzles	Problem solving							F11C1		
Day 3, Function Puzzles + 3digit Sums	Problem Solving / working backwards							F11C2		
Day 3, Function Puzzles + 3digit Sums	Problem Solving: Trial/Error, Elimination							F11C2		
Day 7, Hotel Snap	Problem Solving							F11C2		
Day 8, Pixel Pattern	Problem Solving							F11C2		
Day 2, Making Groups	Organizing data								12C1	
Day 2, Making Groups	Combinations								12C1	
Day 6, Dutch Blitz	Problem Solving									*
Day 3, Function Puzzles + 3digit Sums	Graphing with technology,									*
Day 4, Goats	Tangent ~ sine for angles <10degrees									*
Day 5, Goats Extension	Making decisions									*
Day 6, Dutch Blitz	Estimation									*
Day 7, Hotel Snap	Working with multiple variables									*
Day 7, Hotel Snap	Working with constraints									*
Day 7, Hotel Snap	Organizing data / Charts									*
Day 7, Hotel Snap	Strategizing									*
Day 8, Pixel Pattern	Piecewise functions									*
Day 9, Peg	Communicating mathematics									*
Day 9, Peg	Area of square									*
Day 12, Algebra Tiles (3)	Solution set for open ended questions.									*
Day 18, Road Lines	Making/Solving own questions									*
Day 18, Road Lines	Finding data									*
Day 18, Road Lines	Estimating									*

A description of each of the prescribed learning outcomes from the Ministry of Education in British Columbia that are discussed in throughout this document can be found in Appendix A. This includes all the PFM10 outcomes, both the ones that Caroline encountered and the 2 she did not encounter during the 18-lesson period. From Math 6-9, Pre-Calculus11, Foundations of Math 11, and Pre-Calculus 12, only the outcomes that Caroline actually encountered and are referred to in this document are included in Appendix A.

Chapter 5. Conclusion

I experimented with using activities, games, applications of mathematics, and problem-solving tasks to improve my students' understanding, enjoyment, and appreciation of mathematics since I first starting teaching. It always bothered me that so many students had such a strong preconceived notion that they were bad at mathematics because it was both boring and difficult; or on the flip side, that they liked mathematics because they were good at it or they liked how it was so black and white, the answer was either right or wrong. I initially tried to solve this by putting my energy into entertaining students with mathematical tidbits and trying to teach the mathematical procedures and concepts as clear and simple as possible in a lecture style lesson. That energy would quickly drain, students would easily lose interest, and lessons fell flat on a regular basis. After building a Thinking Classroom and shifting to a problem-based approach, lessons felt much more energizing as students actively engaged in mathematics. But even if the mathematics lessons were more enjoyable than they had been, it was important for me to ensure that students were actively engaging in the required curricular content as well. Keeping track of the mathematical concepts that Caroline encountered for one month gave me better insight into the feasibility of teaching a high school mathematics course through problem solving, given the time constraints.

In all my attempts to change that opinion, making the dramatic shift in my perspective on teaching mathematics, is the closest I have gotten to making any sort of significant impact on the way that students perceive mathematics. My students appeared to be understanding that there was a bigger purpose to mathematics than just knowing facts and strategies. As they wrestled through problem solving tasks that were not directly linked to a specific strategy or learning unit (e.g. similar triangles), they were making connections to any strategy in their knowledge base. When I taught them about a concept, such as using the primary trigonometric ratios to solve a right triangle, there was often an energy and excitement amongst the students to have more efficient strategies to use! "Why didn't you teach us this sooner?" became a common theme throughout the year as students were relieved that new strategies generally made previously difficult tasks much easier. For me, learning to teach through problem solving was an exciting process, because I could feel that my students were engaged in learning and were asking for better mathematical strategies instead of me putting energy into convincing them that mathematics is useful and enjoyable!

5.1. Answering the Research Question

To what extent can a curriculum be encountered through problem-based learning in a thinking classroom?

In this study, I isolated the mathematical concepts that Caroline worked with and categorized them under the corresponding learning outcome required by the Ministry of Education for Pre-Calculus and Foundations of Mathematics 10. During the one-month period of 18 class periods, the students engaged in 15 different tasks, excluding the extension problems that took a few days. Organizing these data, it was evident that Caroline encountered nearly the entire curriculum for FPM10. During the period of data collection, there were only two of 18 curriculum outcomes that she did not encounter in the data I collected. Each of the LOs that Caroline encountered are listed in Appendix A. In addition to encountering almost an entire mathematics curriculum during the period of one month, Caroline also encountered a review component and an extension component. This is also typical in a traditional classroom setting in that when students start learning a new concept, the teacher has them review some curricular content from earlier mathematics courses, and when they are nearly finished a unit, they will have the opportunity to extend their learning. Having said that, it is definitely not the case in a traditional classroom that every student has the opportunity to extend his or her learning on a daily basis.

I also made two other observations in my data regarding the content that Caroline encountered. The first is that she encountered more than one LO each day while problem solving. This can be seen in Table 6.1 in Appendix B, where the LOs that Caroline encountered are organized by day. During a few of the tasks, Caroline did not work directly with the FPM10 curriculum, but did encounter a few lower level LOs (review), higher level LO (extension), and worked on non-curricular competencies such as communication, problem solving, and using technology. The second observation is that she encountered many of the LOs on two or more days. This can be seen in Table 4.2 of Chapter 4, where the LOs that Caroline encountered are sorted by Grade level and by outcome. Many of the LOs were encountered on several different days, or multiple times on the same day.

Research in psychology shows that student learning is greatly increased if they are actively engaged with the knowledge or skills (Halpern & Hakel, 2004; Donovan, Bransford, & Pellegrino, 1999). Students engaging with knowledge and skills repeatedly and in different contexts, have an increased likelihood of storing this information in their long-term memory. When students work with new concepts in a variety of tasks and activities, they also improve the ability to create a more generalized understanding and have an increased ability to apply the skill in new situations or problems (Halpern & Hakel, 2004; Donovan, Bransford, & Pellegrino, 1999). During this small window of time, Caroline consistently worked with the entire mathematics curriculum on a regular basis.

5.2. Limitations of This Research

By giving students autonomy in how they work through tasks and solve problems in my classroom, Caroline's experiences represent only that of a single student in a class of 16 individuals and could be very different from her peers. Each group of students often worked through the mathematical tasks differently and, therefore, over the course of a month, each student would have had a unique experience of encountering the curriculum outcomes. As teacher, I was also responsible for the education of all 16 students while collecting data on Caroline's experience. While I worked with Caroline's peers during lessons, I would not have noticed every encounter that she made with the curriculum unless she took the time to write it down in such a way that I could notice it later or record it with a photograph. It is likely she encountered more mathematical concepts than I could keep track of. I chose Caroline for this case study as she was generally engaged in the mathematical activities and because she was rarely absent from my class. It is possible that this may have resulted in her encountering more of the curriculum during the given time frame than many of the other students in the class.

Since I was conducting this small case study on a student of my own choosing in a classroom environment that I constructed myself as teacher, there may be some concern of bias in the data. To avoid bias, I deferred making conclusions about the depth of the student's understanding and only took note of encountered curriculum that I either heard and took note of during lessons, and concepts that Caroline demonstrated in her written problem-solving equations and diagrams. When I do make reference to the students being stuck, surprised, confused, amused or any other similar responses that arose during the lessons and problem-solving tasks, then it would have been obvious student reactions that I felt could have been interpreted as such. As a teacher, it is generally easy to realize a student is experiencing extreme confusion or amusement. Caroline has a very open personality, and never hesitated to tell me that she had no idea what was going on, to exclaim her frustration, or laugh loudly at any situation that amused her.

The depth of understanding was not at all taken into account when I drew a connection between Caroline's problem-solving process and the curriculum. The lens, through which I viewed the concepts which Caroline encountered while engaging in these problem-solving tasks, was more that of an observer than an educator. I made the connections between the mathematical concepts that Caroline worked with and the ministry prescribed learning outcomes with no regard to how well she understood the concept. Through my observations, assessments, and discussions with Caroline, I could tell that while she fully understood some learning outcomes, sometimes she barely touched the surface of an outcome. In a few cases, although she was fully engaging with a specific learning outcome, it was clear to me as her teacher that her understanding was full of misconceptions. Because I was not concerned about mastery of content during the period of this study, I often let this go with the assumption that we would come back to each of the learning targets again throughout the remainder of the course through more problem solving tasks and mini lessons. I made the decision as an educator that any attempt Caroline made to use a mathematical concept had value. Similarly, in a traditional classroom, when a teacher teaches a certain concept, he or she also considers the lesson a valuable encounter with the curriculum even if the students did not fully engage with or master the concept.

5.3. Possibilities for Further Study

Given that this case study has a narrow focus, of encounters experienced by one student during the period of one month, there are many unanswered questions. It would be interesting to know the depth of understanding for each encounter, as well as to what extent Caroline encountered the curriculum during the course of the school year from September until June. Since each month was very similar, with students working on problem-solving tasks for the course, it is likely that a student could encounter each learning outcome many times during the course of the school year. How might this approach of cycling through the curriculum, with little evidence of order, affect their depth of understanding? In this classroom, the cyclic approach to teaching mathematics is not only dramatically different than in a traditional classroom, where students usually focus on a few learning outcomes at a time before being assessed and moving on to different outcomes, but also different from most problem-based learning environments. I see a lot of value in studying the impact of students learning the mathematics curriculum at random or cyclically, rather than in a linear fashion. Another study could be to look at the mathematics that an entire class encounters while working on a single open-ended mathematics task. This could shed some light on how personalized the curriculum becomes for each student or how uniquely each student encounters the curriculum with a single task.

5.4. What have I learned as a teacher?

Letting students encounter mathematics in a way that felt authentic has been a rewarding experience. Teaching mathematics through problem solving has allowed students to begin to experience the nature of mathematics that I tried so hard to force them to see in my first years of teaching. By letting go and standing back, I am no longer standing in their way of enjoying mathematics. Often I find it difficult to give my students the room to problem solve because I want to rescue them before they have even struggled with a mathematical concept or problem. When I step in and give answers before students have played with a concept, sweated over it, and struggled with it, I steal the opportunity for them to appreciate the beauty of the mathematics.

I have learned that students are genuinely interested in discovering and learning about mathematics once they start experiencing the joy of problem solving. Usually, when I give them structured problem-solving tasks with little room for autonomy in determining a solution, the students quickly complain or give up.

My greatest challenge is still assessment in the learning environment that I have built for my mathematics courses. When students learn through problem solving, I find that assessment and feedback is as important as it is dangerous. The students need feedback to improve their understanding and mathematical competencies, but formal feedback in the form of percentages and letter grades at the end of each term can have a dramatic impact on their attitudes during the remainder of the year. I value students' problem solving even if their solutions are full of mistakes, because mistakes are opportunities for learning and understanding. Incorrect mathematics combined with problem solving has room for growth and making deeper connections, while students that just memorize procedures correctly and don't really solve problems authentically generally have challenges later in mathematics because they either forget the procedures or are unable to generalize concepts.

Afterward

At the start of the course, I made the decision that in order to successfully create an environment where students could learn through problem solving, I would need to take the focus off the final exam. Near the end of the course, the students did spend some time becoming familiar with exam style questions and vocabulary. Since they looked at every question as a problem solving task and did not always know the quickest method or algorithm to find a solution, it was very difficult for them to complete the exam in the given amount of time. For this reason, there was a discrepancy on their final exam compared to their in class grade. By the end of the course, Caroline demonstrated that she exceeded expectations in the prescribed learning outcomes in class work with an A+ grade, but received a B grade on her provincial exam. On average, students in this course received a 22% higher score in class than on their final exam. I believe that the final exam scores did not accurately reflect the depth of their knowledge of the content, but their ability to recall the correct solution algorithms.

While I believe that learning through problem solving is well worth the challenge of working with a more unorganized learning environment, structured theoretical mathematics lessons and practice still have their place in the current curriculum. Since it is still expected that students master certain procedures and algorithms to solve specific types of problems, it is important that students do practice applying those procedures with some repetition to be successful on a standardized final exam. Although rote practice should not be the norm, neither the starting place for problem solving, it will help students develop a more automatic recall of certain procedures while problem solving in the future. During the course of the year of this research, the students learned to solve almost any problem-solving task that I put before them and worked with mathematical concepts far beyond what was expected of the course. Yet, they did not always choose efficient strategies to solve problems, struggled with the procedural questions, and did not learn to use the mathematical notation and vocabulary that came up on the provincial final exam.

References

Barrows, H. S. (1996). Problem- based learning in medicine and beyond: A brief overview. *New directions for teaching and learning*, *1996*(68), 3-12.

Ben-Hur, M. (2006). Concept-rich mathematics instruction building a strong foundation for reasoning and problem solving. Alexandria, VA: Association for Supervision and Curriculum Development.

Boaler, J. (2013, March). Ability and Mathematics: The mindset revolution that is reshaping education. FORUM, 55(1), 143-152. Symposium Journals. Retrieved July 10, 2015, from http://www.ncpdf.org/pdf/steering/2013-09-06/12.0 Boaler_FORUM_55_1_web.pd

Boaler, J. (2002). Experiencing school mathematics: Traditional and reform approaches to teaching and their impact on student learning (Rev. and expanded ed.). Mahwah, N.J.: L. Erlbaum.

Boaler, J. (2008). What's math got to do with it?: How parents and teachers can help children learn to love their least favorite subject. New York: Penguin Books.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. Journal for research in mathematics education, 41-62. Retrieved July 9, 2015 from http://www.jstor.org/stable/749717?seq=1&cid=pdf-reference#references_tab_contents

Brown, S. I., & Walter, M. I. (2005). The art of problem posing (3rd ed.). Mahwah, N.J.: Lawrence Erlbaum.

Halmos, P. R. (1994). What is teaching?. *The American Mathematical Monthly*, *101*(9), 848-854.

Halmos, P. R. (1982). A Hilbert space problem book (2nd ed.). New York: Springer-Verlag

Henderson, K. B., & Pingry, R. E. (1953). Problem solving in mathematics. *The learning of mathematics: Its theory and practice*, 228-270.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., ... & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. Educational researcher, 25(4), 12-21.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. Second handbook of research on mathematics teaching and learning, 1, 371-404.

Hilbert, D. (1900). Mathematical problems.

Hung, W., Jonassen, D. H., & Liu, R. (2008). Problem-based learning. *Handbook of research on educational communications and technology*, *3*, 485-506.

Halpern, D. F., & Hakel, M. D. (2003). Applying the Science of Learning to the University and Beyond: Teaching for Long-Term Retention and Transfer. *Change*, 35(4), 36-41.

Hmelo-Silver, C. E. (2004). Problem-based learning: What and how do students learn? *Educational psychology review*, *16*(3), 235-266.

Johnson, L. (2005). Teaching outside the box: How to grab your students by their brains. San Francisco: Jossey-Bass.

Donovan, M. S., Bransford, J. D., & Pellegrino, J. W. (1999). How people learn: Bridging research and practice

Learn. (n.d.). Retrieved October 21, 2017, from https://www.merriam-webster.com/dictionary/learn

Levy, S. (1996). Starting from scratch: One classroom builds its own curriculum. Portsmouth, NH: Heinemann.

Lockhart, P. (2009). A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form. *New York, NY: Bellevue Literary Review*

Mason, J. (1993). Assessing what sense pupils make of mathematics. In M. Selinger (Ed.), Teaching mathematics, 153-166, London: Routledge.

Mason, J., Burton, L., & Stacey, K. (2010). Thinking Mathematically (2d ed.). Harlow: Pearson Education UK.

Mathematics. (n.d.). Retrieved July 21, 2017, from https://www.merriam-webster.com/dictionary/mathematics

Mimi. (2010, July 28). My Take on Using Puzzles to Teach Substitution [Web blog]. Retrieved from http://untilnextstop.blogspot.ca/2010/07/my-take-on-using-puzzles-to-teach.html

Meyer, D (2011). The Three Acts Of A Mathematical Story [Web blog]. Retrieved from http://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/

Meyer, D (2012). Pixel Pattern. Retrieved from http://threeacts.mrmeyer.com/

National Research Council. (1989). Everybody counts: A report to the nation on the future of mathematics education. Washington, DC: National Academy Press.

Nguyen, F. (2013, December 11). Hotel Snap [Web log]. Retrieved from fawnnguyen.com/hotel-snap/

Norman, G. T., & Schmidt, H. G. (1992). The psychological basis of problem-based learning: a review of the evidence. *Academic medicine*, 67(9), 557-565.

Polya, G. (1957). How to solve it; a new aspect of mathematical method. (2d ed.). Princeton University Press.

Savery, J. R., & Duffy, T. M. (1995). Problem based learning: An instructional model and its constructivist framework. *Educational technology*, *35*(5), 31-38.

Schoenfeld, A. (2009). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. *Colección Digital Eudoxus*, (7).

Schoenfeld, A. H. (2016). Mathematical thinking and problem solving. Routledge.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive science*, *12*(2), 257-285.

Tans, R. W., Schmidt, H. G., Schade-Hoogeveen, B. E., & Gijselaers, W. H. (1986). Sturing van het onderwijsleerproces door middel van problemen: een veldexperiment [Directing the Learning Process by Means of Problems: A Field Experiment]. *Tijdschrift voor Onderwijsresearch*

Tobias, S. (1978). Overcoming math anxiety. New York: Norton.

Wagner, David (December 2003). “We have a Problem Here: 5 + 20 =45?” *Mathematics Teacher 96: 612 - 616.*

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. *Research ideas for the classroom: High school mathematics*, 57-78.

Ministry of Education, Province of British Columbia. (2008). The Common Curriculum Framework for Grades 10-12 Mathematics: Western and Northern Canadian Protocol. Retrieved from https://www2.gov.bc.ca/assets/gov/education/kindergarten-to-grade-12/teach/pdfs/curriculum/mathematics/2008math1012wncp_ccf.pdf

Appendix A:
Ministry of Education Curriculum with Codes

Required Curriculum for this Pre-Calculus and Foundations of Mathematics 10		
10A1	Solve problems that involve linear measurement using:	
		SI and imperial units of measure
		Estimation strategies
		Measurement strategies
10A2	Apply proportional reasoning to problems that involve conversions between SI and imperial units of measure.	
10A3	Solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including:	
		right cones
		right cylinders
		right prisms
		right pyramids
		spheres
10A4	Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles.	
10B1	Demonstrate an understanding of factors of whole numbers by determining the:	
		prime factors
		greatest common factor
		least common multiple
		square root
		cube root
10B2	Demonstrate an understanding of irrational numbers by:	
		representing, identifying and simplifying irrational numbers
		ordering irrational numbers.
10B3*	Demonstrate an understanding of powers with integral and rational exponents	
10B4	Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials)	
		concretely
		pictorially
		symbolically
10B5	Demonstrate an understanding of common factors and trinomial factoring concretely, pictorially, and symbolically.	

10C1	Interpret and explain the relationships among data, graphs, and situations.	
10C2	Demonstrate an understanding of relations and functions.	
10C3	Demonstrate an understanding of slope with respect to	
		rise and run
		line segments and lines
		rate of change
		parallel lines
		perpendicular lines
10C4	Describe and represent linear relations using:	
		words
		ordered pairs
		tables of values
		graphs
		equations
10C5	Determine the characteristics of the graphs of linear relations, including the:	
		intercepts
		slope
		domain
		range
10C6	Relate linear relations expressed in: …to their graphs	
		slope-intercept form (y=mx+b)
		general form (Ax + By + C = 0)
		slope-point form (y-y1=m(x-x1))
10C7	Determine the equation of a linear relation, given: …to solve problems	
		a graph
		a point and the slope
		two points
		a point and the equation of a parallel or perpendicular line
10C8*	Represent a linear function, using function notation	
10C9	Solve problems that involve systems of linear equations in two variables, graphically and algebraically.	

*Required course outcomes that were not at all encountered during the one-month research period.

Review Component: Mathematics from Grade 6 - 9	
Mathematics 6	
6A3	Demonstrate an understanding of factors and multiples by determining multiples and factors of numbers less than 100, identifying prime and composite numbers, and solving problems involving multiples.
Mathematics 7	
7C1	Demonstrate an understanding of circles by describing the relationships among radius, diameter, and circumference of circles, relating circumference to pi, determining the sum of the central angles, constructing circles with a given radius or diameter, and solving problems involving the radii, diameters, and circumference of circles.
7C2	Develop and apply a formula for determining the area of triangles, parallelograms, and circles.
Mathematics 8	
8A3	Demonstrate an understanding of percentages greater than or equal to 0%
8A5	Solve problems that involve rates, ratios, and proportional reasoning
8A7	Demonstrate an understanding of multiplication and division of integers concretely, pictorially, and symbolically
8C1	Develop and apply the Pythagorean theorem to solve problems
Mathematics 9	
9A5	Determine the square root of positive rational numbers that are perfect squares
9B2	Graph linear relations, analyze the graph, and interpolate or extrapolate to solve problems
9B3	Model and solve problems using linear equations of the form: $ax = b$, $\frac{x}{a} = b, a \neq 0$, $ax + b = c$, $\frac{x}{a} + b = c, a \neq 0$, $ax = b + cx$, $a(x + b) = c$, $ax + b = cx + d$, $a(bx + c) = d(ex + f)$, $\frac{a}{x} = b, x \neq 0$, where a, b, c, d, e, and f are rational numbers.
9B5	Demonstrate an understanding of polynomials (limited to degree less than or equal to 2)
9B6	Model, record, and explain the operations of addition and subtraction of polynomial expressions, concretely, pictorially, and symbolically (limited to degree less than or equal to 2)
9B7	Model, record, and explain the operations of multiplication and division of polynomials expressions (degree less than or equal to 2) by monomials, concretely, pictorially and symbolically
9D3	Develop and implement a project plan for the collection, display, and analysis of data by formulating a question for investigation, choosing a data collection method that includes social considerations, selecting a population or sample, collecting the data, displaying the collected data in an appropriate manner, and drawing conclusions to answer the question

Extension Component: Mathematics for Grade 11 - 12	
Pre-Calculus Mathematics 11	
11C5	Solve problems that involve quadratic equations
11C9	Analyze arithmetic sequences and series to solve problems
Foundations of Mathematics 11	
F11C1	Analyze and prove conjectures, using inductive and deductive reasoning, to solve problems
F11C2	Analyze puzzles and games that involve spatial reasoning, using problem-solving strategies
Pre-Calculus Mathematics 12	
12C1	Apply the fundamental theory of counting to solve problems

Appendix B:
Data Sorted by Day

Table 6.1: Data Sorted by Day

Day / Task	Mathematical Concept	*Math 6*	*Math 7*	*Math 8*	*Math 9*	*PFM10*	*PC11*	*F11*	*PC12*	*Extra*
Day 1, Gears + Marching Band	Calculating lists of multiples	6A3								
Day 1, Gears + Marching Band	Common factors	6A3								
Day 1, Gears + Marching Band	Prime Numbers	6A3								
Day 1, Gears + Marching Band	Percentages			8A3						
Day 1, Gears + Marching Band	Prime factorization					10B1				
Day 1, Gears + Marching Band	Least Common Multiples					10B1				
Day 1, Gears + Marching Band	Greatest common factors					10B1				
Day 1, Gears + Marching Band	Prime Numbers					10B1				
Day 2, Making Groups	Organizing data				9D3					
Day 2, Making Groups	Multiples					10B1				
Day 2, Making Groups	Organizing data								12C1	
Day 2, Making Groups	Combinations								12C1	
Day 3, Function Puzzles + 3digit Sums	Multiplying negatives			8A7						
Day 3, Function Puzzles + 3digit Sums	Finding intersections					10C9				
Day 3, Function Puzzles + 3digit Sums	Graph and analyze linear equations				9B2					
Day 3, Function Puzzles + 3digit Sums	Algebra, Model and solve linear equations				9B3					
Day 3, Function Puzzles + 3digit Sums	Interpret and explain relations					10C1				
Day 3, Function Puzzles + 3digit Sums	Describe and represent linear relations					10C4				
Day 3, Function Puzzles + 3digit Sums	Linear and Quadratic equations					10C5				
Day 3, Function Puzzles + 3digit Sums	Minimization/Maximization						11C5			
Day 3, Function Puzzles + 3digit Sums	Problem Solving / working backwards							F11C2		
Day 3, Function Puzzles + 3digit Sums	Graphing with technology,									*

Day 3, Function Puzzles + 3digit Sums	Problem Solving: Trial/Error, Elimination							F11C2		
Day 3, Function Puzzles + 3digit Sums	Odd and Even Number concepts for proofs							F11C1		
Day 3, Function Puzzles + 3digit Sums	Factors					10B1				
Day 3, Function Puzzles + 3digit Sums	Subsets of Rational numbers					10B2				
Day 4, Goats	Area of Circle		7C2							
Day 4, Goats	Solve problems involving rates and ratios			8A5						
Day 4, Goats	Use proportional reasoning and convert					10A2				
Day 4, Goats	Trig ratios					10A4				
Day 4, Goats	Inverse trig ratios					10A4				
Day 4, Goats	Tangent ~ sine for angles <10 degrees									*
Day 5, Goats Extension	Degrees in a circle		7C1							
Day 5, Goats Extension	Solve problems involving rates and ratios			8A5						
Day 5, Goats Extension	Pythagorean Theorem			8C1						
Day 5, Goats Extension	Use proportional reasoning and convert					10A2				
Day 5, Goats Extension	Trig ratios / inverse trig ratios					10A4				
Day 5, Goats Extension	Making decisions									*
Day 6, Dutch Blitz	Solve problems involving rates and ratios			8A5						
Day 6, Dutch Blitz	Unit Conversions					10A2				
Day 6, Dutch Blitz	Interpret and explain relations					10C1				
Day 6, Dutch Blitz	Decreasing arithmetic sequence						11C9			
Day 6, Dutch Blitz	Problem Solving									*
Day 6, Dutch Blitz	Estimation									*
Day 7, Hotel Snap	Problem Solving							F11C2		
Day 7, Hotel Snap	Working with multiple variables									*
Day 7, Hotel Snap	Working with constraints									*
Day 7, Hotel Snap	Organizing data / Charts									*
Day 7, Hotel Snap	Strategizing									*
Day 8, Pixel Pattern	Interpret and explain relations					10C1				
Day 8, Pixel Pattern	Relations and functions					10C2				

Day 8, Pixel Pattern	Describe and represent linear relations					10C4				
Day 8, Pixel Pattern	Linear relation graph characteristics					10C5				
Day 8, Pixel Pattern	Linear equations and graphs					10C6				
Day 8, Pixel Pattern	Create linear relations					10C7				
Day 8, Pixel Pattern	Piecewise functions									*
Day 8, Pixel Pattern	Problem Solving							F11C2		
Day 9, Peg	Area of circle		7C2							
Day 9, Peg	Solve problems involving rates and ratios			8A5						
Day 9, Peg	Pythagorean Theorem			8C1						
Day 9, Peg	Communicating mathematics									*
Day 9, Peg	Area of square									*
Day 10, Algebra Tiles (1)	Represent variable expressions with				9B5					
Day 10, Algebra Tiles (1)	Simplifying expressions with algebra tiles				9B5					
Day 10, Algebra Tiles (1)	Adding/Subtracting trinomials with algebra				9B6					
Day 10, Algebra Tiles (1)	Multiplying binomials with algebra tiles with degree 2 or less.				9B7					
Day 10, Algebra Tiles (1)	Multiplying monomials, binomials, and trinomials with algebra tiles					10B4				
Day 11, Algebra Tiles (2)	Multiplying monomials, binomials, and trinomials with algebra tiles					10B4				
Day 11, Algebra Tiles (2)	Factor trinomials with algebra tiles					10B5				
Day 12, Algebra Tiles (3)	Surface area of box with variable side					10A3				
Day 12, Algebra Tiles (3)	Multiplying binomials					10B4				
Day 12, Algebra Tiles (3)	Factoring trinomials					10B5				
Day 12, Algebra Tiles (3)	Surface area of box with variable side					10B5				
Day 12, Algebra Tiles (3)	Solution set for open ended questions.									*
Day 13, Stacking Squares	Square roots				9A5					
Day 13, Stacking Squares	Factor Trees					10B1				
Day 13, Stacking Squares	Prime factorization					10B1				
Day 13, Stacking Squares	Simplifying radicals					10B2				

Day 13, Stacking Squares	Equivalent radicals					10B2				
Day 14, Simplifying + Ordering square	Square roots				9A5					
Day 14, Simplifying + Ordering square	Simplifying square roots					10B2				
Day 14, Simplifying + Ordering square	Ordering square roots					10B2				
Day 14, Simplifying + Ordering square	Equivalent mixed radials					10B2				
Day 15, Three Lines	Area of triangle		7C2							
Day 15, Three Lines	Calculating distances			8C1						
Day 15, Three Lines	Slope					10C3				
Day 15, Three Lines	Describe and represent linear relations					10C4				
Day 15, Three Lines	Linear relation graph characteristics					10C5				
Day 15, Three Lines	Solving system of equations graphically					10C9				
Day 15, Three Lines	Finding intersections					10C9				
Day 16, Shape Equations	Solving systems of linear equations					10C9				
Day 16, Shape Equations	Solving by substitution					10C9				
Day 16, Shape Equations	Solving by adding/subtracting rows					10C9				
Day 17, Number Puzzles	Solving systems of linear equations					10C9				
Day 17, Number Puzzles	Substitution					10C9				
Day 17, Number Puzzles	Problem solving							F11C1		
Day 18, Road Lines	Estimating SI/Imperial measurement					10A1				
Day 18, Road Lines	Volume vs surface area for thin surfaces					10A3				
Day 18, Road Lines	Making/Solving own questions									*
Day 18, Road Lines	Finding data									*
Day 18, Road Lines	Estimating									*