Complexity and Collaborative Problem Solving

by

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Abstract

Through research conducted in a classroom, this dissertation explores problem solving in choice-affluent environments where students have abundant access to resources beyond their own and their group's knowledge and experience. Contrary to the conventional notion of problem solving as an isolated activity reliant on individual resources, such as knowledge, experience, or sudden insights, this dissertation highlights the collaborative nature of problem solving. Being similar to problem solving in society and among mathematicians, problem solving amongst students in mathematics classrooms should involve accessing external resources like the work of their peers, technology, the internet, and social connections. Using classroom video, I conducted an analysis of students engaged in problem-solving activities. Combining Schoenfeld’s theory on resources with Koichu’s shifts and choices model for problem solving in choice-affluent environments and Mason’s work on shifts of attention, I investigated how groups collaborate in their own group and between other groups to make progress in solving a problem. The findings suggest that collaborative problem solving is not a deterministic process – the stages in the process do not follow a sequence. The processes that students follow when solving problems are non-linear and unpredictable due in part to the complex nature of the learning environment. Additionally, the research showcases a relatively new methodological tool, gaze-dialogue transcripts, to document the dialogue, gestures and gazes during collaborative problem solving from a video source.

Keywords: collaboration; problem solving; mathematics; complexity; resources; Thinking Classrooms

For my Mom.

You instilled in me the value of education from such an early age.

"You can be a garbage collector Michael, but you need to get your degree first."

For my Dad.

You taught me how to solve problems with persistence and a sense of humour.

So much of who I am comes directly from you.

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Each person mentioned here has played a crucial role in this journey, contributing their support, patience, and inspiration. This work stands as a reflection of their collective impact on my academic and personal growth.

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Chapter One - Introduction

I have been teaching using a Thinking Classroom framework (Liljedahl, 2019) since 2013, and over this time, I have noticed many benefits. I will describe the Thinking Classroom framework in detail later in this thesis, but for the unaware reader, I will say that Thinking Classrooms are classrooms where students work collaboratively on carefully chosen mathematical tasks in the public and vertical space of the classroom whiteboard. Over the years, I have continued to use this framework for many reasons, but primarily, I value this framework for the high level of student engagement and continuous opportunities for students to solve problems. In this educational model, teaching typically occurs after students have been working on a task – not before. And because of this quality, most tasks that students engage in are tasks that have not yet been taught and tasks that students are not yet familiar with; therefore, students are frequently positioned in the space of solving problems.

Thinking Classrooms, as a learning space, look and sound much different than classrooms where I used to teach. In these spaces, I have noticed student behaviour that I have rarely seen in my 18 years of teaching prior to 2013. Some of this behaviour is just plain obvious: students laughing in their groups, celebrating by giving others fist-pumps or high-fives, running across the room to speak to another group, and even students solving problems well past the bell; and some of this behaviour is more subtle: students switching from on-task to off-task and back to on-task, asking other students for help to understand, and even students switching groups in order to get a better explanation of a mathematical topic. I have noticed these changes in behaviour, but I have also noticed other changes.

Over the years, I have become increasingly aware of knowledge and ideas developing in different areas of the room. Sometimes, I have seen the operator in this knowledge movement and other times, it is a complete mystery as to how the ideas have moved through the classroom. On some occasions, I can see a solution or idea form on one board in the room, and I can see a student in another group point to the idea and add the outside idea to their own group's work. But, other times, I have witnessed an idea move through the room with no apparent student conversation or even student gazing.

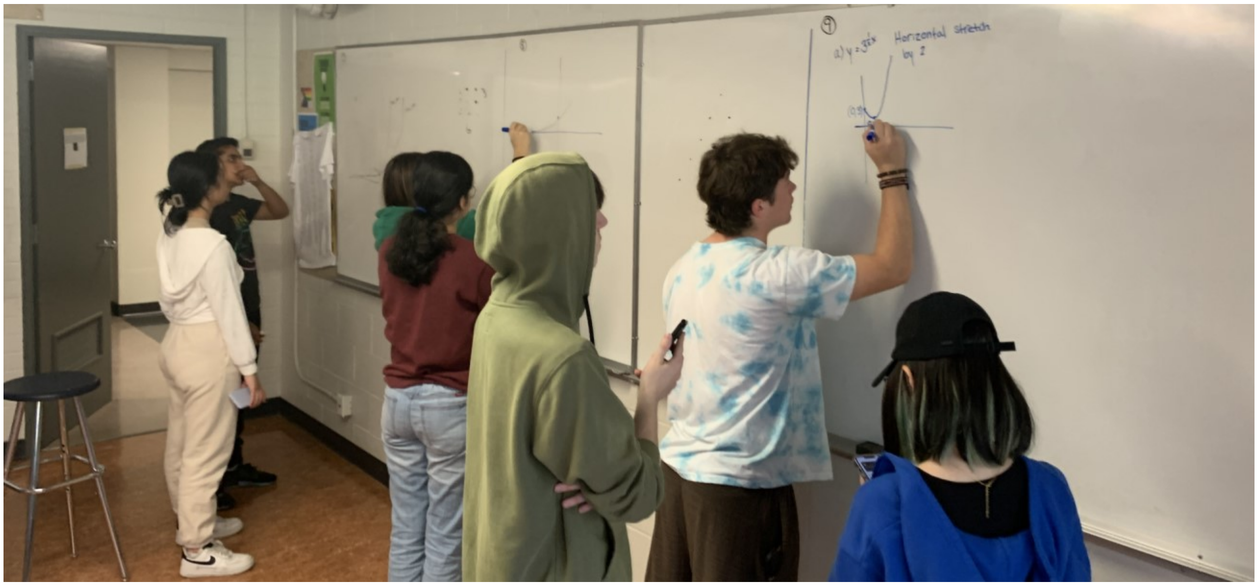

Figure 1 A group of students working at the whiteboard.

> *Imagine you work in a cycling shop building unicycles, bicycles, and tricycles for customers. One day, you receive a shipment of 10 wheels. Presuming that each cycle uses the same type and size of wheel, what are the combinations of cycles you can make using all 10 wheels?*

Figure 2 A Thinking Task.

I gave this task (fig. 2) to a mathematics class of grade 9 students on the first day of our fall semester. As my students were working in groups of three at whiteboards around the perimeter of my classroom, I circulated through the room watching them collaborate and listening to their conversations. I always appreciate listening to these early conversations and seeing how they begin to represent their mathematical thinking. Some groups were telling the story of building these cycles and taking care in writing out sentences describing each build. I was a little surprised by their lack of symbols and the care taken to write out each word (fig. 3).

3 bicycles, 1 tricycle, 1 unicycle

5 unicycles, 1 tricycle, 1 bicycle

FIGURE 3 A GROUP'S REPRESENTATION.

Similar groups would use letters or diagrams to represent each cycle (fig. 4).

3 b 1 t 1 u

FIGURE 4 A SYMBOLIC REPRESENTATION

And some groups would move to a table to organize their thinking (fig. 5).

Unicycles	*Bicycles*	*Tricycles*
10	*0*	*0*
1	*3*	*1*

FIGURE 5 AN ORGANIZED TABLE

As I was making my way throughout the room and visiting different groups, I often asked, "How do you know when you are done?" I asked this to get students thinking about organizing their thinking or to order their lists. On this day last Fall, I noticed something different on the other side of my room. I saw one group generating a list on their board that looked subtly different from the others (fig. 6).

3, 3, 3, 1

3, 3, 2, 2

3, 3, 2, 1, 1

3, 3, 1, 1, 1, 1

FIGURE 6 AN EFFICIENT REPRESENTATION.

I made my way over to this group and asked them to describe what they were thinking. One student in the group responded by saying that the 3's represented the tricycles, the 2's were the bicycles and the 1's were the unicycles. Having given this task to other classes many times before, I was not surprised with this representation showing itself – I had seen this before. I am always impressed with this mathematical representation. Not only were these students using a very efficient representation to solve the problem, but they were also generating their list in a carefully designed order. This solution method was in effect changing the problem from building cycles with ten wheels to a problem of making 10 with 1's, 2's and 3's. As I already stated, I was not surprised to see this solution representation emerge in a class of 30 students, but I was surprised by what happened next.

After my conversation with these students, I stepped back towards the center of my room to survey my class and see what group I needed to attend to next. As I looked around the room, I noticed this unique solution representation was on the two neighbouring boards to the original group. Did these other groups come up with this unique representation on their own? Or is it possible that they overheard my conversation with the originating group?

On another day, I began my lesson by giving my students the following diagram (fig. 7):

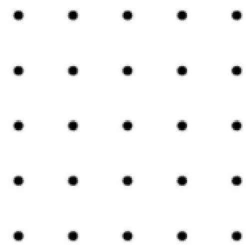

FIGURE 7 ANOTHER THINKING TASK.

I said that I can make squares by connecting four dots with lines. I then asked, “How many different squares can I make?” After posing this question, I asked my students to go to their whiteboards again, in their groups of three, and work towards a solution. I moved to the center of the room again to begin my observations.

The groups began working on their solutions with an energetic engagement. There is something about these kinds of problems that have an easy entry, that really generates conversation and activity amongst all students. This is why I like beginning many of my classes with these tasks – they provide an early opportunity for mathematical conversation and thinking for all students, and they provide situations that can improve student confidence in mathematics. As I was watching my students working on their whiteboards, I noticed a lot of typical solutions around the room. Some students would trace out different squares and keep tallies of their counting (fig. 8), some groups would complete their counting, organize their counting in a table and notice patterns in the table (fig. 9) and some groups would just count squares in a haphazard way, with no particular plan or order (fig. 10).

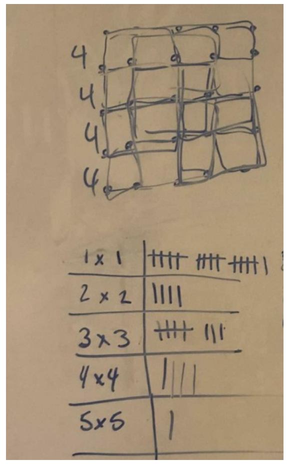

FIGURE 8 TALLIES OF SQUARES.

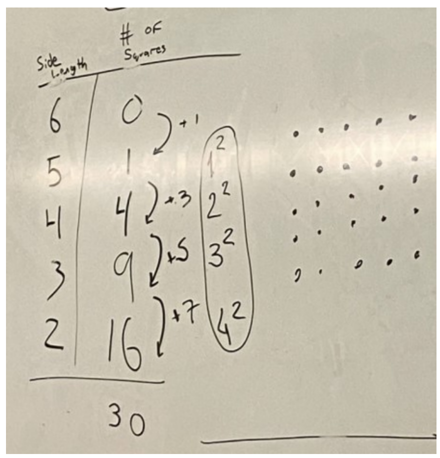

FIGURE 9 ORGANIZED COUNTING IN A TABLE.

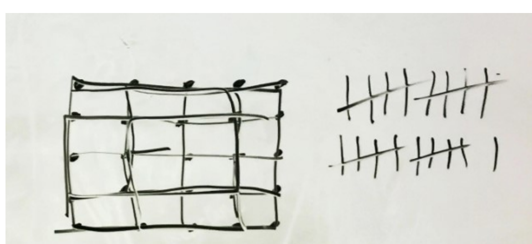

FIGURE 10 COUNTING SQUARES WITH NO PLAN.

After ten or fifteen minutes, activity and engagement began to diminish, and many groups around the room seemed to settle around a solution of 30 squares. When groups begin to feel that they are done, or when they get stuck, this is when classroom management can become more of a problem, especially with a room full of grade 9's. At this time, I found myself trying to keep groups on task by encouraging them to keep looking and saying "there may be more." I noticed a group in the far corner of the room that was still very actively engaged in their discussion, and on their whiteboard, I noticed a slightly different diagram (fig. 11).

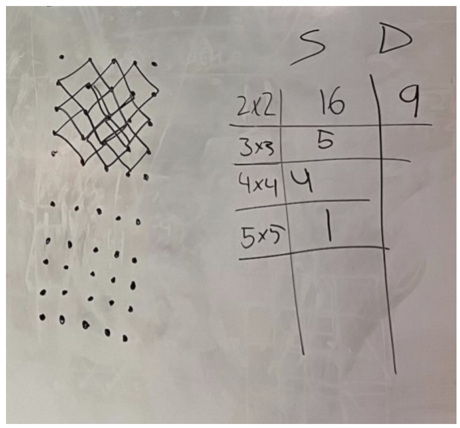

FIGURE 11 A DIFFERENT SQUARE TO COUNT.

This group had noticed that their squares did not have to have horizontal and vertical sides; rather, their squares could possibly have tilted sides. I moved towards them to give encouragement and maybe ask some questions, and as I did so, I noticed a change in my classroom. The distraction and off-task behaviour was changing back to the sound of engagement. It's hard to describe this change, but I can tell you that a room full of distracted and off-task grade nines sounds different than a room full of actively engaged problem solvers. As I was making my way over to this group, the whole class was becoming actively engaged again. I looked around the room, and almost every board had tilted squares on them (fig. 12).

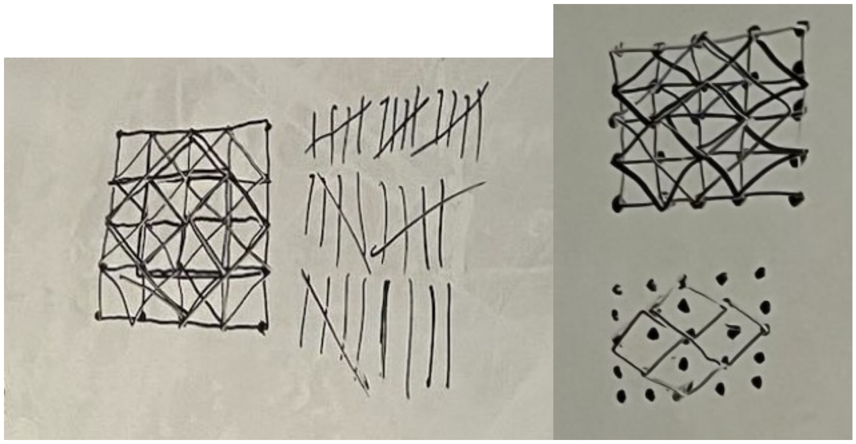

FIGURE 12 MORE BOARDS WITH TILTED SQUARES.

How did this happen? Did they see the tilted squares on their own and then use it? Did they notice me moving to that first group in the corner? Did these other groups notice the tilted square possibility in isolation, or did they somehow gather this idea from the originating group?

Problem solving is what one does when one does not know what to do (Resnick & Glasser, 1976). Problem solving permeates all content at all levels in a mathematics classroom. It is a stated goal in most curriculum documents (Introduction to Mathematics | Building Student Success - B.C. Curriculum, n.d.), and most educators would describe problem solving on a shortlist of desired learning outcomes for their students. For much of the last 80 years, problem solving has been studied at the individual level (e.g., Dewey, 1933; Pólya, 1961; Schoenfeld, 1985; Carlson & Bloom, 2005; Mason, 1989). In the last two decades, with the acceptance that mathematics and problem solving is often highly collaborative, there has been a shift in interest towards studying problem solving in groups (Ambrus & Barczi-Veres, 2016; Nelson, 1999). This is a step in the right direction, as it comes closer to resembling actual problem solving, but it is still mostly confined to problem solving within a group – as far as I have found, there has not been much investigation looking at problem solving within the whole classroom and how students work individually, within groups and between groups, and problem solving in groups where groups are able to see and share ideas simultaneously with other groups is nonexistent.

Problem solving in mathematics classrooms has been a focus for mathematics educators, curriculum designers, and education leaders since at least Pólya (1949). There is little argument around the desire to have students engage in solving problems as mathematicians do in order to develop understanding, improve motivation and engagement and to encourage connections in the subject. The focus on problem solving in mathematics education research can be sorted into three categories (Stanic & Kilpatrick, 1989): as a cognitive enterprise, as something to be taught, and as something to teach through. The study of problem solving in collaborative spaces may be related to each of these three, but in many ways, it is also unique.

In the example of students solving the bicycle problem (p. 7), I had noticed that a particularly successful problem representation originated in one group; and then after a short time, the same representation was being used by three other groups in the class. In the square problem, I noticed that including tilted squares in the count began in one group and was very quickly adopted by other groups. These particular representations or solutions could be considered as a *breakthrough moment* for these groups, as they provided a shift in the interpretation of the problems and allowed a more direct pathway to a final solution. Given that the shift was likely a moment of inspiration for one of the groups, was it developed in the same way by the other groups? Or, perhaps more likely, did the other groups acquire these ideas by overhearing or observing the ideas?

These are just two examples of this phenomena. I have been teaching using Thinking Classrooms for 10 years, and I have been becoming increasingly aware of this phenomenon. It is quite different from my experiences in my earlier classrooms. I remember walking through and observing student work in my earlier teaching days, and most progress that was made was due to my direct involvement with each student. I would rarely see a unique insight or a creative approach emerge from more than one or two students. In fact, when I did see something from one student, my only method for sharing with the rest of the class would be to stop the class, place the student's work under a document camera, and ask for the rest of the class to notice and hopefully understand the method. Contrast this with today's Thinking Classroom and I am almost expecting whole class progress once I see something emerge from one collaborative group.

Over the past 10 years of using the Thinking Classroom framework, I have been noticing a different mode of knowledge transfer. I have become more aware of the lessening need for me, the teacher, to orchestrate when, how and with whom the ideas need to be shared. I have become more aware of ideas emerging from multiple groups, sometimes almost simultaneously, and then before too long, the ideas become common to the whole class. In fact, instead of me having to share an idea with each of my thirty students, I am seeing the ideas move through the room, and I may have to spend more directed energy with only one or a few students.

Ideas and strategies are being shared in this Thinking Classroom, but I am not aware of exactly how this is happening. Somehow, progress and learning is moving from the student, to the group and eventually to the whole class, and I am not a necessary component in this chain of events. If I am not the one to point to a new idea or learning moment, then how can the whole class still make this progress? Not only am I not the most instrumental actor in this arrangement, but I am also seeing that progress is more efficient without me. Somehow, students and groups and the whole class are making progress in tasks without my direct input and more efficiently than in my earlier teaching days.

Thinking Classrooms are spaces where students appear to solve problems with more success than when they work individually. There is something that is happening in these highly collaborative environments that is acting like a catalyst in the reaction of solving problems. In this dissertation, through a detailed analysis of classroom video, I plan on uncovering the details of this catalyst and to understand the interactions that are at play during collaborative problem solving and the interactions that support knowledge and ideas to move through these environments.

Chapter Two - Literature Review

In an online interview (https://www.youcubed.org/resources/the-nature-of-21st-century-mathematics), Jo Boaler, a Stanford University mathematics educator, was speaking with Keith Devlin, a Stanford University mathematician. The interview was generally about the current state of mathematics education from a mathematician's vantage point. This commentary was drawing attention to how many students were experiencing mathematics through their high school education, and how this current state might be seen as far different than the mathematics that mathematicians engaged in virtually for the whole of history; and perhaps more importantly, the mathematics that is used in society today is completely different than the mathematics that is taught in schools. There were two points that Devlin made that resonated with me and my work on problem solving in high school classrooms. The first was a metaphor that Devlin used to compare the work of mathematicians with conducting an orchestra, and the second point was how Devlin described the tools and resources used by mathematicians in the 21st century.

Devlin described the work of a mathematician today as being similar in ways to a conductor of an orchestra, where a mathematician from history (pre-1980's) was more like being a musician in the orchestra itself. Before the 1980's, mathematicians had to learn and be experts in a large variety of different mathematics: number, algebra, geometry, calculus, and more. Like an orchestra, in order to make good music, mathematicians needed to be skilled and proficient in all of the *instruments* of mathematics. After the 1980's, computer software and applications were able to do all of the procedural mathematics faster and more accurately than humans, and Devlin suggests that the mathematician's role became more like that of the conductor of the orchestra. As a conductor, a mathematician needs to be familiar with the quality and purpose of all of the different instruments (areas in mathematics), but one no longer needs to be an expert on each instrument. To solve problems, mathematicians need to know what types of mathematics is necessary, and how different areas might need to work together, but they do not need to be experts in each area. This is a different message than that being communicated in most school mathematics today. In schools today, students are expected to become highly proficient in number, in geometry, in algebra, and they are engaged in these topics much like mathematicians have engaged in them for the past hundreds of years. This makes me wonder if schools today should be teaching students the ways that mathematicians do mathematics today?

The second point that Devlin made in this interview was that mathematicians today solve problems much differently than how they did in the past. In the past, many mathematicians would work on problems as a solitary exercise, using pencil and paper or chalk and a chalk board. If not rare, collaboration was slow, often only occurring through letters exchanged with colleagues or through discussions at annual conferences. Today, the resources (or tools) available and used by mathematicians are far different, and in order to use these tools, mathematicians need to learn a new skillset. Devlin shared a slide (fig. 13) showing the tools that he now uses as a mathematician to solve problems:

FIGURE 13 SLIDE FROM DEVLIN'S PRESENTATION

Google, Wikipedia, email, YouTube, WolframAlpha, spreadsheets, MathWorks, mathoverflow, Wolfram Mathematica, and graphing calculators are all extremely powerful tools for solving problems, and Devlin suggests that these are the tools that he uses as a mathematician to solve problems. Today, I would add Desmos, Geogebra and Sketchpad to this list. This made me realize that mathematics in our world has changed dramatically since the 1980’s, so shouldn’t the mathematics that we teach in our schools reflect this change?

The ways that mathematicians solve problems today has changed. They do not need to be experts in every area of mathematics and mathematicians today have access to resources that are very effective and efficient, so perhaps the ways in which we educate our students to be capable and confident problem solvers should also change. In this review, I will look at the history of problem solving in mathematics education.

Problem solving has long been a valued component in mathematics education (NCTM, 2000; Burnard & White, 2008; Craft, 2000; Sawyer, 2011; Sternberg, 1988; Szabo et al., 2020), but the ways in which we teach our students to problem solve and what researchers know about problem solving has changed. In this literature review, I will share how our knowledge and understanding of problem solving from the early days of Dewey and Pólya has evolved, and how it now better represents the problem solving needed in citizens of our 21st century societies. Much like Devlin drew attention to how mathematics today is far different from the mathematics of yesterday, I will also draw attention to how problem solving in the literature does not fully meet the demands of problem solving in society today.

Problem solving – An individual activity

Let us begin by looking at some different ways of defining a problem and the process of problem solving. From a psychology perspective, problem solving is the simple act of trying to achieve a goal where the pathway to the goal is blocked (Kilpatrick, 1985). John Dewey wrote more generally about problem solving when he described how we overcome a difficulty through some sort of action and reflection (Dewey, 1910). Brownell (1942) put a more mathematical tone to problem solving when he wrote:

> Problem solving refers (a) only to perceptual and conceptual tasks, (b) the nature of which the subject by reason of original nature, of previous learning, or of organization of the task, is able to understand, but (c) for which at the time he knows no direct means of satisfaction. (d) The subject experiences perplexity in the problem situation, but *he* does not experience utter confusion.... Problem solving becomes the process by which the subject extricates himself from his problem .... Defined thus, problems may be thought of as occupying intermediate territory in a continuum which stretches from the 'puzzle' at one extreme to the completely familiar and understandable situation at the other. (p. 416)

In this definition, we understand problem solving to be unique to the individual, meaning that something that is a problem to one is not necessarily a problem to another nor a problem in the future. Liljedahl et al. (2016) describes problems as “tasks that cannot be solved by direct effort and will require some creative insight to solve” (p. 6), and problem solving as the process by which one extricates oneself from the problem and achieves satisfaction.

John Dewey (1859-1952) wrote about problem solving in 1933 when he wrote about overcoming difficulty through intelligent action and reflection. Dewey suggests that in order to overcome a difficulty, one requires both ideas and facts. “Mere facts or data are dead, as far as the mind is concerned, unless they are used to suggest and test some idea, some way out of a difficulty. Ideas on the other hand are mere ideas, idle speculations, fantasies, dreams, unless they are used to guide new observations of, and reflections upon, actual situations, past present, or future” (Dewey, p. 199). For Dewey, ideas guide the problem-solving process, but they are difficult to explain or predict. Having ideas are a critical component to solving a problem with intentionality, but it is “not so much something we do, as it is something that happens to us” (Dewey, p. 145). In Dewey’s description of problem solving, ideas need to be tested with facts, and then there needs to be a reflection phase to judge the process. If the process does not solve the problem, then the idea needs to be adjusted or a new idea needs to be found. Dewey draws comparison to the scientific method for solving problems, in that the idea leads to a conjecture that then must be tested and evaluated (or reflected upon). Dewey does not provide much detail on the conception of ideas. Do ideas come from the environment, or are they developed through an inner mental process? Is it past experiences that generate ideas, do ideas just magically occur or do ideas come from outside sources? These are all questions that would take years to develop responses to, and in a little more than ten years, George Pólya comes on the scene with some newer and more refined *ideas* on problem solving.

George Pólya (1887-1985), a Hungarian mathematician, published a ground-breaking book on Problem solving titled *How to Solve it* (Pólya, 1945). The purpose of the book was to teach students and professionals a systematic approach to solving problems. In addition to providing a structured approach to problem solving, Pólya also emphasized the importance of creativity, intuition, and persistence in the problem-solving process. He believed that problem solving was a skill that could be learned and improved upon, and his book provided practical strategies and examples to help readers develop their problem-solving skills. In this seminal book, Pólya breaks problem solving into four distinct stages:

Stage 1 - Understanding the problem

Stage 2 - Devising a plan

Stage 3 - Carrying out the plan

Stage 4 - Looking back

It is this second stage, devising a plan, where Pólya makes some suggestions as to where ideas may come from by providing a list of heuristics:

Stage 2 - Devise a Plan

- Can you find an analogous problem that you can solve or have already solved?
- Can you find a more general problem that you can solve or have already solved?
- Can you find a more specialized version of the problem that you can solve or have already solved?
- Can you find a related problem that you can solve or have already solved?
- Can you change the problem into a problem that you can solve or have already solved?
- Can you find a subproblem that you can solve or have already solved?
- Can you add some new element to the problem to create a problem that you can solve or have already solved?
- Can you decompose the problem and recombine it into a problem that you can solve or have already solved?
- Can you work backwards?
- Can you draw a picture?

In this list of heuristics, the first eight reference personal experience as a source for ideas, and the last two reference specific heuristics that are developed later in the book. Mathematicians and experienced problem solvers approved of these stages and the list of heuristics because it aligned with their own personal experiences. A difficulty arises, however, if you are a student with no prior experiences to draw from for ideas. For example, how do you teach students to find an analogous problem that they have already solved if they have solved few or no analogous problems? In this way, problem solving is like being a member of an elite club – a club whose membership is restricted to people who can already solve problems. If prior knowledge and prior experience is all that a problem solver can draw on to gain an idea and make progress in a problem, then this model does little to support beginning problem solvers, and yet, this model for teaching problem solving gained so much momentum in the 1950's and 60's that there are still traces of Pólya's 4 stages to solving problems in most school textbooks to this day.

While Pólya's focus was on how to solve problems, Schoenfeld's book (Schoenfeld, 1985) entitled Mathematical Problem Solving is an empirical critique of Pólya and is focused on how to teach problem solving. Where Pólya was interested in how mathematicians solve problems, Schoenfeld was more interested in how students solve problems. Schoenfeld emphasizes the importance of metacognition, or the ability to reflect on and monitor one's own thinking processes. He argues that successful problem solvers are able to engage in metacognitive strategies such as monitoring their progress, evaluating their strategies, and revising their approaches when necessary. His book is framed by four categories of knowledge and behaviour that are used to explain human problem-solving behaviour: Resources, Heuristics, Control and Belief Systems. Resources are the "foundations of basic mathematical knowledge" (Schoenfeld, p. 12) that students bring to the problem. Heuristics are the "techniques used by problem-solvers when they run into difficulty" (Schoenfeld, p. 74). Control describes the executive decisions made when students are solving problems: "making plans, selecting goals and sub-goals, monitoring and assessing solutions as they evolve, and revising or abandoning plans when the assessments indicate that such actions should be taken" (Schoenfeld, p. 27).

Schoenfeld's description of control begins to consider a collaborative component. Referencing an experiment by Mugny and Doise (1978), Schoenfeld points out that more progress is made when students with different cognitive strategies work together when compared to children with similar cognitive strategies. Not only is Schoenfeld describing value in the number of students when students solve problems, but he is also describing value in the variety of students. Problem-solving approaches “evolve during the solution as a result of interactions between the two students. Once it emerges, it can then become part of the individual students’ repertoire” (Schoenfeld, p. 142), but this is the extent to which collaboration is addressed.

Schoenfeld also states that some of Pólya's heuristics for problem solving could not be directly taught. Schoenfeld notes that some of the heuristics, such as "look for a pattern," "solve a simpler problem," and "make a drawing or diagram," were easy to teach and could be explicitly taught to students; however, he argues that other heuristics, such as "use your intuition" and "be creative," were more difficult to teach directly. Schoenfeld describes these heuristics as more like habits of mind that develop over time through practice and experience. He suggests that the development of these habits of mind requires students to engage in authentic problem-solving activities, rather than just learning a set of problem-solving strategies in isolation.

My focus in this discussion on problem solving is on the foundations of mathematical knowledge that students bring to the problem-solving experience – resources. Resources are the facts, procedures and skills that a problem solver brings to a problem – in short, the mathematical knowledge. Schoenfeld was interested in the nature of the knowledge that students have at their disposal and in how this knowledge is organized and accessed.

Regarding the nature of knowledge, Schoenfeld quotes research (de Groot, 1965, 1966; Simon, 1980) that describes this knowledge as “stored chunks in memory.” Simon’s research shows that chess masters have approximately 50000 chunks stored in memory just for routine play. They use the word ‘chunk’ to describe not only a unit of memory but also an action or response associated with it. Schoenfeld compares this to how a literate person reacts when reading a stop sign. A literate person will instantly access the associated chunk that contains a meaning, and an appropriate action may follow. Schoenfeld suggests that this may be similar to how mathematicians access their resources when solving problems, and that these chunks are learned through years of experience in problem solving (Schoenfeld, 1985, p. 50). Schoenfeld moves from this itemized description of resources to another that is more situational, suggesting that problem solvers are able to identify a problem type and recall the associated technique or procedure for a solution. This type of problem schemata approach to resources is still problematic for beginning (or novice) problem solvers. Expert problem solvers have the experience to be able to draw on resources such as chunks or problem types, because of their experience and large repertoire of problem-solving performance; however, novice problem solvers have a smaller resource bank, and worse, many of their resources may be applied incorrectly (Schoenfeld, 1985, p. 51).

Schoenfeld summarizes that the resources that problem solvers access are like automatic responses, stereotypical responses to stereotypical situations. On their own, a resource drawn on and applied to answer a problem does not necessarily fit with our understanding of problem solving. If a task can be completed by simply implementing a resource recalled, then this would be like answering a routine problem. It is with respect to non-routine problems that we are interested. With non-routine problems, students will still access their resources of scripts, schemata or frames accumulated from previous experiences, but these resources will be drawn on in order to make steps towards solving a problem. This explains why there is more to Schoenfeld's model than just resources; students also need to make use of heuristics, control and belief systems to complete the problem-solving picture. In problem solving, a variety of factors shape behaviour. These include the problem solver's intuition or informal knowledge about the problem; knowledge of facts and definitions; the ability to correctly complete algorithmic processes; ability to use routine procedures; and knowledge about the rules of discourse in the problem domain (Schoenfeld, 1985). With beginning problem solvers, their resource inventory may be small, but worse, some of their inventory may also be incorrect. This could be catastrophic if the solver is working alone, but it might be mitigated in collaborative environments.

Schoenfeld notes that one of the challenges in teaching problem solving is helping students develop a rich repertoire of resources that they can draw upon when needed. He suggests that this can be achieved by providing students with a range of problem-solving experiences and by encouraging them to reflect on their own problem-solving processes. Good problem-solvers need to have a lot of resources to draw upon, and the best way to accumulate these resources is to have lots of success in solving problems. This is something we have seen before and leaves us with the same question, how do novice problem-solvers solve problems if they do not have the past experiences to draw upon?

Mason, Burton and Stacey (1982) combine both Schoenfeld's framework for successful problem solving and Pólya's four stages for solving problems in their book titled Thinking Mathematically. In this book, Mason employs a series of tasks to help describe the phases of problem-solving: Entry, Attack and Review. These three stages resemble Pólya's stages by combining the middle two: Pólya's Make a plan and Carry out a plan are combined by Mason into the Attack phase. Mason pays particular attention to the part in problem-solving when the solver is stuck, and he describes processes that specifically deal with this state. He refers to these processes as specializing and generalizing.

After the entry phase where students work to understand the problem, Mason suggests entering the attack phase by Specializing. Specializing is examining the problem by looking at specific examples. It is at this point where the solver may get *stuck* and then go back to the *entry* phase to better understand the problem before attacking again, or the solver experiences an AHA moment and progresses to the review stage by generalizing their solution. Mason describes this position of being Stuck in a very positive light, as it is a necessary precondition to the problem-solving process: "I hope that you will get happily stuck and learn from it!" (Masonet al., p. 45). He first describes the process of Specializing as a means to get unstuck, and then introduces the process called "Mulling" as a means to get unstuck. Mulling is the process where the subconscious plays a role. "Attack now turns into a waiting game, waiting for a fresh idea or insight… If you feel desperate to do something, then I recommend fresh air and exercise" (Mason et al., p. 99). Creativity is often demonstrated when moving from stuck to unstuck, so this provides a nice model for the analysis of creativity in the classroom (Chamberlin et al., 2022). It is the moment where solvers move from the entry phase (understanding the problem) into the attack phase (or making a plan). Unlike Pólya's heuristics, Mason is suggesting the solver might need to rely on extra-logical processes such as illumination, insight and creativity to go beyond their resources that their repertoire of past experience might afford them.

Although Mason seems to have solved the Pólya/Schoenfeld conundrum for novice problem-solvers, he has introduced a new problem. How does one teach novice problem-solvers creativity, insight and illumination? How do these ideas manifest into resources for the student problem-solvers? This idea of mathematicians, or problem-solvers in general, relying on extra-logical processes to make progress in a problem was not new. In fact, there is generally two different categories for models of problem-solving processes (Rott, 2021).

Much of the literature on problem solving in mathematics education describes problem solving from a non-empirical process (Rott, 2021). Models of the problem-solving process were developed through author's observations of their own problem-solving processes or processes of people they are familiar with; and as a result, models for problem-solving processes are not based on empirical observations or data. Classic models of problem-solving processes can be divided into two categories: Intuitive or Creative models and Logical models.

The concept of intuitive or creative problem-solving models can be traced back to Poincaré's (1908) self-reflection on his own problem-solving processes. Hadamard (1945), a mathematician, and Wallas (1926), a psychologist, further developed Poincaré's ideas by emphasizing the role of subconscious activities in problem-solving processes. They proposed a four-phase model, which includes (i) preparation, where the problem solver works on a difficult problem but cannot find a solution, (ii) incubation, where the problem solver thinks about the problem but in a non-focused way, (iii) illumination, where suddenly a brilliant idea appears after some time, often hours or even weeks, and (iv) verification, where the idea is checked for its accuracy. This model is commonly used to summarize the ideas of Hadamard and Wallas.

Dewey (1933) introduced the idea of logical models for problem-solving processes, which consist of five phases: (i) encountering a problem and generating suggestions, (ii) specifying the problem and intellectualizing it, (iii) approaching possible solutions and developing a guiding idea or hypothesis, (iv) reasoning in a narrow sense by developing logical consequences of the approach, and (v) testing the hypothesis by action and accepting or rejecting the idea based on experiments. Unlike Wallas' model, Dewey's model does not include any subconscious activities. Pólya's (1961) famous four-phase model – (i) understanding the problem, (ii) devising a plan, (iii) carrying out the plan, and (iv) looking back – draws upon Dewey's work, as noted by Neuhaus (2002).

Most of the research on problem solving has been around the logical model because it is very linear and organized and easier to describe (Rott, 2021). However, most of this research acknowledges an extra-logical process as part of the genuine problem-solving experience, but it does not address it directly due to the difficult nature in describing events like an a-ha moment or a creative insight.

By reading Pólya, through Schoenfeld and into Mason et al., one sees a progression in the research on problem solving. It is interesting to think about Pólya's four-stages in solving problems from a collaborative perspective. Although not directly referenced, it is implied through much of this book that students are working individually. In fact, most of the literature on problem solving contained an individual perspective until later in the 20th century. Mason et al. (1982), while using a model very similar to that of Pólya's, focused on the value of being stuck in the problem-solving process. Schoenfeld (1985, 2016) describes resources, heuristics, control and belief systems as contributing factors to a person's success in problem solving. In order to be good at problem solving, a person would need to have lots of resources, but in order to have lots of resources, a person needed to have lots of success in problem solving (Pruner & Liljedahl, 2021). This circular problem is similar to that which came up in Pólya's work requiring prior experiences to draw from in stage 2. How can a person develop a repertoire of resources/experiences when they have not yet been successful in problem solving? This highlights the problem with focusing only on the individual in problem solving.

In the previous two decades of the 20th century, mathematicians were becoming more accustomed to working with others with a higher frequency of conferencing due to the ease of world travel and with the beginning of the internet age where communication and collaboration could happen with the click of a button (Schoenfeld, 2016). So, the time was ripe for mathematics researchers to shift their attention from the individual to the collective in problem-solving research.

Problem solving – A collective activity

By the 1980's, researchers were becoming more attuned to the social make-up and benefits provided in classroom settings. The math classroom is a dynamic social environment created collaboratively by the participants, where the teacher and students mutually interpret each other's actions and intentions based on their individual perspectives and objectives (Mehan, 1978). Researchers were increasingly using group problem solving in classrooms as data for their research (Dees, 1983; Lesh & Akerstrom, 1982; Noddings, 1982; Schoenfeld, 1982; Granberg, 2016), and they were also suggesting that collaborative problem solving should be considered for classroom instruction.

Through the 1990's, there was a trend towards an understanding of problem solving in terms of situated cognition (Resnick, 1991; Lave & Wenger, 1991; Artzt & Femia, 1999; Bjuland, 2002). Situated cognition suggests that the way people think and reason is not separate from their physical and social surroundings, but is instead grounded in those surroundings. This means that people's cognitive processes are influenced by the tools, technologies, and social practices they use and the tasks they are trying to accomplish. The research focus in the 1990's was on mathematical reasoning in solving problems and reasoning processes of students working in collaborative groups. Resnick, Levine and Teasley (1991) pushed the conception of cognition from the individual to the collective, arguing "that the social context in which cognitive activity takes place is an integral part of that activity" (p. 4). "Groups are especially preferred when several kinds of knowledge and expertise are required" (p. 14).

Although the problem-solving framework developed by Pólya, Schoenfeld and Mason in the earlier years has remained relatively intact, the research was shifting its focus from problem solving as an individual endeavour to problem solving in a social context. The benefits of this shift were manifold. Problem solving in a social environment more accurately reflected problem solving in the real world where resources were not limited to what was individually remembered or individually experienced, heuristics could emerge from the collaboration of a group that may not have belonged to any single member and, above all, collaborative problem solving was much more in line with how mathematicians solved problems.

The trend towards valuing the collaborative was also happening within the larger community. Likely due to the introduction of computers and a more technologized society, the community was beginning to value a skillset more oriented toward group thinking and collaboration. In Boaler's book, Mathematical Mindsets (2015), she shares two tables (fig. 14) showing the Fortune 500 "most valued" skills in 1970 and in 1999 (Boaler, p. 28). Problem solving and Teamwork move from the bottom of the list in 1970 to the top of the list in 1999. This clearly shows a trend within the broader community of placing value on problem solving and collaboration.

TABLE 3.1 Fortune 500 "most valued" skills in 1970

1	Writing
2	Computational Skills
3	Reading Skills
4	Oral Communications
5	Listening Skills
6	Personal Career Development
7	Creative Thinking
8	Leadership
9	Goal Setting/Motivation
10	Teamwork
11	Organizational Effectiveness
12	Problem Solving
13	Interpersonal Skills

TABLE 3.2 Fortune 500 "most valued" skills in 1999

1	Teamwork
2	Problem Solving
3	Interpersonal Skills
4	Oral Communications
5	Listening Skills
6	Personal Career Development
7	Creative Thinking
8	Leadership
9	Goal Setting/Motivation
10	Writing
11	Organizational Effectiveness
12	Computational Skills
13	Reading Skills

FIGURE 14: MOST VALUED SKILLS FROM FORTUNE 500

In the year 2000, the National Council of Teachers of Mathematics (NCTM, 2000) published its Principles and Standards for School Mathematics. In this document, not only was problem solving the first of five process standards, but collaboration and communication were also highlighted as important mathematical processes to support student learning. "Solving problems is not only a goal of learning mathematics but also a major means of doing so. It is an integral part of mathematics, not an isolated piece of the mathematics program … Mathematical communication is a way of sharing ideas and clarifying understanding. Through communication, ideas become objects of reflection, refinement, discussion, and amendment" (NCTM, 2000). By the end of the 20th century, problem solving has established itself as the central focus in the teaching and learning of mathematics, and the mathematics education research field has shifted its focus from problem solving as an individual activity to problem solving as a collaborative activity. This trend to the collaborative not only recognizes the shift in values for the general community, but it also shows a recognition in how mathematicians have changed how they work on mathematics.

Collaborative problem solving in this time was limited to studying students solving problems within small groups (Schoenfeld, 1982; Noddings, 1982; Dees, 1983; Bjuland, 2002; Granberg, 2016; Carbonneaue et al., 2020; Roberta & Marta, 2021) where resources were shared and heuristics emerge from the small group. I have not found many studies that looked at large group, group to group, or even whole class collaboration for problem solving. Furthermore, researchers still struggle to explain how some problems were solved by illumination, a-ha moments or other extra-logical processes.

Collaborative problem solving has become more common in research in recent years as educators are being made more aware of the multiple benefits that arise from using these models (Angawi, 2014; Nebesniak, 2007). Students tend to enjoy the process of solving problems more when they are collaborating with their peers (Camacho-Morles et al., 2019). This enjoyment leads naturally to more positive feelings about the subject and improved self-confidence with respect to mathematical ability (Nebesniak, 2007). The mathematical output is better than in the case of the more traditional individual problem-solving model (Dillenbourg, 1999; Johnson & Johnson, 1999). There is a by-product of improved social and communication skills from collaborative problem solving that is also highly valued in today's society (Fiore et al., 2010). Lastly, while working within a group, students are exposed to a wider range of problem-solving strategies, alternative solutions and in general, resources (Gillies, 2000).

Collaborative problem solving in classroom environments can take many forms. Students can be put into small groups to solve problems within their group, whole classes can participate in a problem-solving exercise through discussion like the Harkness method (Soutter & Clark, 2021), or students could be left to their own direction and seek out collaboration from peers of their choice. Of course, this list is not complete, but it is intended to provide some scope as to the variety of contexts in which collaborative problem solving can occur. Collaboration is a means to improve problem solving in the classroom, but how does it improve problem solving and what does current research say about developing problem-solving skills in our students?

Contemporary Research on Collaborative Problem -Solving

With the emergence of problem solving as a collaborative classroom activity came a focus on research into how best to do this. According to Lester and Cai (2016), there have been significant improvements in our understanding of the various factors that influence problem solving in mathematics education, including emotional, cognitive, and meta-cognitive aspects (Lesh & Zawojewski, 2007; Lester & Kehle, 2003; Schoenfeld, 1985, 1992, 2013; Silver, 1985). A great deal of research has also been conducted on teaching problem solving in math classrooms (Kroll & Miller, 1993; Lesh & Zawojewski, 2007; Wilson, Fernandez, & Hadaway, 1993) as well as teaching math through problem solving (Lester & Charles, 2003; Schoen & Charles, 2003). However, there are still many more questions than answers when it comes to this complex form of activity (Cai, 2003; Lesh & Zawojewski, 2007; Lester, 1994, 2013; Lester & Kehle, 2003; Schoenfeld, 1992, 2013; Silver, 1985). There is general agreement within the field of mathematics education that developing students' problem-solving skills should be a main focus of classroom instruction (National Council of Teachers of Mathematics, 1989, 2000), and there has been much progress made in understanding how to improve students' problem-solving skills through classroom instruction, and research-based insights are now available (Cai, 2010). Lester & Cai (2016) describe six features with respect to problem solving that current research has been able to address:

1. Should problem solving be taught as a separate topic or through problem-solving activities?

    Teaching problem solving is usually done through isolating strategies or heuristics and demonstrating which types of problems are best used for each heuristic. Over the past several decades, research has shown that teaching problem solving does not improve student's problem-solving skills (Begle, 1973; Charles & Silver, 1988; Lester, 1994; Schoenfeld, 1985). Teaching through problem-solving activities is the teacher selecting a series of tasks for the students to work through as problems, and then highlighting the learning goals as students make progress through these tasks (Lester & Charles, 2003; Schoen & Charles, 2003). Teaching through problem solving has been shown to improve student problem-solving performance mostly due to the fact that this teaching method helps students to develop a deeper conceptual understanding of the material. Research also shows that teaching students specifically how to problem-solve has little impact on students becoming better at problem solving (Lesh & Zawojewski, 2007; Lester, 1994; Schoenfeld, 1985, 1992).

2. What sort of time is required to teach through problem solving?

It has been determined that the best way for students to become proficient at problem solving is for them to have lots of experience in solving problems (Cai & Nie, 2007; Gu, Huang, & Marton, 2004; Lester, 1994). In order for students to become successful problem solvers, they need to have consistent and long-term exposure to solving problems. The problem-solving skill is an emergent quality that comes from many quality and varied experiences with solving problems. It has been determined that problem solving is not a switch that students suddenly develop proficiency in; rather, problem solving is a long-term instructional goal for teachers to nurture through continuous problem-solving experiences.

3. What types of instructional activities should students engage in?

Research has shown that students show the most growth when presented with cognitively demanding tasks (Cai, 2014; Stein, Remillard, & Smith, 2007). Lappan and Phillips (1998) have generated a list of criteria for a cognitively demanding task, or a "good mathematics problem:"

- The problem has important, useful mathematics embedded in it.
- Students can approach the problem in multiple ways using different solution strategies.
- The problem has various solutions or allows different decisions or positions to be taken and defended.
- The problem encourages student engagement and discourse.
- The problem requires higher-level thinking and problem solving.
- The problem contributes to the conceptual development of students.
- The problem connects to other important mathematical ideas.
- The problem promotes the skillful use of mathematics.
- The problem provides opportunity to practice important skills.
- The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulty.

It has been shown that students who have been exposed to mathematics through cognitively demanding tasks improve in their ability to reason and communicate, promote their conceptual understanding, and captures their interest and curiosity (Cai, 2014; Hiebert, 2003; NCTM, 1991; Van de Walle, 2003).

4. What is the teacher's role in orchestrating a problem-solving classroom?

Student's opportunities to learn are not solely dependent on the type of mathematical task; it is also important to consider the discourse that takes place during and after the task. Considerable research has shown a connection between classroom discourse and student learning (Cobb, 1994; Hatano, 1993). Good discourse is that which encourages thinking and reasoning on the part of the student. This can be done simply by allowing students to struggle before coming to solutions on their own, rather than stepping in early to rescue a student in their difficulty. It is a teacher's role to ask questions rather than answer them, and to encourage students to participate in their own solutions rather than to demonstrate an efficient teacher solution (Rasmussen, Yackel, & King, 2003).

5. How can positive student beliefs be nurtured?

To support students in becoming effective problem solvers, it is important to cultivate positive attitudes toward mathematics in general and problem solving specifically. In classrooms where students are active participants in solving problems and posing problems, students show improved problem-solving performance and attitudes towards mathematics (Cai, 2003; Rosenshine, Meister, & Chapman, 1996). Teacher beliefs about mathematics also has an impact in student beliefs. Teachers who believe mathematics to be an open and creative field where there are multiple approaches and strategies to solving problems tend to teach in ways that promote these beliefs in their students (Philipp, 2007).

6. Does learning basic skills take a *back seat* when teaching through problem solving?

When teaching through problem solving, the focus is more on conceptual understanding than on procedural fluency. There have been concerns that procedural skills and knowledge may be lost when the teaching focus in problem solving. Research has overwhelmingly shown that students using problem-solving approaches perform as well or better than students using traditional curricula in the area of computational skills and procedural knowledge. Further, students using problem-solving approaches perform better than students in traditional curricula on tests designed to measure conceptual understanding and problem solving (Cai et al., 2011; Fuson, Carroll, & Drueck, 2000). Overall, research indicates that teaching math with a focus on problem solving can significantly improve students' problem solving and conceptual understanding, as well as produce improvements in basic computational and other procedural skills.

With this summary of the current state of research on problem solving, it is clear that students best learn how to solve problems by being immersed in problem-solving classrooms over the long term, solving cognitive demanding tasks. Classroom discourse and teacher beliefs also have positive impact on problem-solving performance. These are some questions that have been answered, but there are many more questions that have not yet been resolved through research. What does teaching through problem solving actually look like? What aspects of a classroom environment best contribute to learning problem-solving skills?

In addition to this summary, there is a study by Salminen-Saari et al. (2021), that uses similar methods and concepts found in this dissertation. The authors in this study used video data and other technology to track student gazes while solving problems and identified occurrences of joint attention. They described joint attention as a social phenomenon where two or more individuals are aware that they are attending to something in common. Their research aim was to understand the nature of joint attention in mathematical problem solving and its effect on the collaborative problem-solving process. The study emphasized the significance of joint attention for collaborative learning and introduced the concept of joint representational attention to mathematical problem-solving research. Rather than relying solely on visual cues or physical gestures, joint representational attention emphasizes the cognitive aspect of collaboration. It involves collaborators actively engaging with the representation, coordinating their attention and interpretation, and using it as a shared cognitive tool to facilitate communication, reasoning, and problem-solving processes. It also emphasizes that interaction and collaboration do not necessarily require a concrete joint visual target. Successful interaction and initiation of joint attention can occur when collaborators share an understanding of the concept through representations. This study has some similarity to this dissertation in their primary use of video data, their attention to student gazes during the problem-solving process and their attention to a process-oriented perspective; however, this study had a focus on small group collaboration, rather than large group or whole class, and they attended to one aspect in the problem-solving process – joint attention. In this dissertation, I focus more generally on the entire problem-solving process and how groups navigate a problem in collaboration within the group and outside the group.

Currently, there is limited knowledge about the dynamics of students' collaborative interaction in mathematics classrooms, and only a few studies have approached collaboration from a process-oriented perspective (Seidouvy & Schindler, 2020). In the context of collaborative mathematical activities, a process-oriented perspective seeks to examine and analyze the interactions, strategies, reasoning, and decision-making processes that occur during collaborative problem solving. It emphasizes studying how individuals engage, communicate, and work together in order to gain insights to make progress in problem solving during collaboration. This perspective goes beyond simply examining the outcomes or results of collaboration and delves into the details of the collaborative process itself. Hansen (2022) focused on three key aspects of students' interactions: collaborative processes, mathematical reasoning, and exercising agency. Kuhn (2015) emphasizes the importance of understanding these key aspects to investigate conditions that foster productive collaboration. By studying these aspects of student interactions separately and observing their interplay patterns, Hansen gained a better understanding of the fundamental processes of collaboration. Existing research on collaboration (Child & Shaw, 2018; Kuhn, 2015) and reasoning (Lithner, 2017) argues that learning occurs in both instances and highlights the necessity of insights into the underlying processes that drive these learning opportunities. Hansen (2022) found that when students engage in a dynamic manner, assuming equal roles and demonstrating authority over mathematical ideas throughout the problem-solving process, they have the potential to construct a shared understanding by merging a diverse interplay of ideas. This dynamic structure of collaborative pairs exposes crucial elements in students' mathematical communication (Sidenvall, 2019), enabling a deeper comprehension of the underlying mechanisms that foster fruitful collaboration (Child & Shaw, 2018; Kuhn, 2015; Seidouvy & Schindler, 2020). These identified conditions involve both participants actively promoting the problem-solving process, grounding their reasoning in mathematical properties, and engaging in various collaborative processes – this was termed bi-directional interaction.

> “Characteristics for the bi-directional interaction were students who engaged with similar roles; mutually attempting to understand each other’s ideas; making suggestions, listening, and negotiating mathematical properties; and mutually driving the problem-solving process forward” (Hansen, 2022, p. 822).

Such a bidirectional interactional pattern nurtures the learning of mathematics through high-quality interactions (Pijls & Dekker, 2011; Varhol et al., 2020).

The research on problem solving in mathematics education has shed light on various factors and approaches that influence effective problem solving. Studies have highlighted the importance of teaching through problem-solving activities rather than isolated strategies, as it promotes deeper conceptual understanding. Consistent and long-term exposure to problem-solving experiences has been identified as essential for developing proficient problem-solving skills. Cognitively demanding tasks have been found to foster growth in students' reasoning, communication, and conceptual understanding. The role of the teacher in orchestrating a problem-solving classroom involves facilitating discourse, encouraging student autonomy, and nurturing positive attitudes towards mathematics and problem solving. Research has shown that problem-solving approaches can enhance both problem-solving skills and basic computational knowledge. However, many questions remain unanswered, such as what are the specific characteristics that make up a problem-solving classroom and what are the optimal classroom environment for fostering problem-solving skills. Furthermore, the last two studies highlight a current interest in a process-oriented perspective, emphasizing the need to understand the underlying mechanisms and collaborative processes that contribute to effective collaboration during problem solving. This dissertation will continue the emphasis on a process-oriented perspective, as it attends to what students do as they solve problems in collaborative spaces. And it will shed some light on optimizing classroom environments to support collaborative problem solving.

Chapter Three - Theory

I will now present the theories that I will be referencing in the analysis for this dissertation. I have already introduced Schoenfeld's theory of resources as the necessary pieces of knowledge that students bring into the problem-solving process. The theory guiding my research is Schoenfeld's theory of resources mixed with Mason (1988) shifts of attention and Koichu's (2018) shifts and choices model. Koichu describes all solutions as having a "situated solver" and I will be leveraging this idea to describe progress within a group. Complexity Theory (Davis & Simmt, 2003) allows me to observe the group work at different levels, as it describes emergence as rising from the interactions between the organism and the environment - the organism is not necessarily the individual learner.

Mason and Davis (1988) began working on a theory called shifts of attention which attempted to explain the developing of awareness as the shift of perception, indeed, often several shifts. Among many examples provided, a student can shift from seeing an infinite sequence as an unending process to seeing it as a completed act, or a student can shift from being stuck on a problem to being aware of being stuck, and then free to be able to do something about it. They suggest that for students to learn, to master a technique and to develop an awareness, these require a shift of perception in the student. The shift is from a specific to a general understanding; and despite the action implied by using the word 'shift,' these shifts cannot be done or performed onto someone. An example of this type of shift is when a student begins to understand linear functions as an operation that can be repeated and produce a series of ordered pairs. A shift takes place when the student begins to see a function as a general operator on an input that can produce a two- or three-dimensional graph depicting this relationship. Shifts can only be occasioned, or you may attract it or focus it for students but shifts of attention cannot be forced. "The role of a teacher is to create conditions in which students experience a corresponding shift in the structure of their attention, in which they become aware of acts and facts of which they were previously unaware" (Mason 1998, p. 244).

Shifts can come about in essentially four ways: investment, examplehood, resonance and grace. Although each of these have a unique character, they do not necessarily occur in isolation; rather, several may be acting simultaneously before the shift occurs. Investment is when a shift occurs after someone esteemed or respected enters the room and enables you to see the problem from a fresh perspective. This attention outside produces an inner separation which is one form of a shift. Examplehood is when an outside experience spurs a noticing of a more meaningful net of connections and associations. Resonance is when something one hears or reads causes a sudden insight or change of viewpoint. And Grace is the final type of shift attributed to spontaneity, haphazard or chance, that might feel like a gift (Mason and Davis 1988).

Mason asserts that learning is the transformation of attention. Learning has taken place when people shift their attention and see something in a different way than previously seen. “Learning necessarily involves shifts in the form as well as the focus of attention” (Mason, 2010, p. 24). It is interesting that in this framework on learning, there is a diminished role for direct action by the teacher. The whole point about learning is that what teachers traditionally pointed out or tried to shift attention in their students, now must be made available to the learner as a choice or an option that comes to mind (Mason, 2010, p. 26). In this way of thinking, Mason is suggesting that teachers provide environments for students where the potential to shift attention is high, so that a shift of attention is occasioned and a new concept or approach may be internalized (Mason 2010, p. 42). In this way, shifts of attention is an appropriate framework to help understand how students make progress when individual resources are low or depleted in a problem-solving situation.

“The transformation of attention has, we propose, qualities analogous to physical change of state, with the role of latent heat being taken partly by stimulation from the environment, and partly by the self, working on automatizing and integrating awareness (Gattegno, 1962; Maturana & Varela, 1971, as cited in Mason & Davis, 1988).” This quote is interesting in this review for a couple of reasons: Firstly, it demonstrates the beginnings of a theoretical explanation for the moment of illumination in learning. Secondly, it is referencing, perhaps for the first time within mathematics education, Maturana and Varela, who would later become well known for their work described in the book “The Tree of Knowledge” and for laying the groundwork for Enactivism.

Enactivism is a theory of cognition rooted in biology. Maturana and Varela (1987) developed a biology of cognition influenced by the 1940's framework called cybernetics. The theory was developed to reflect how animals adapt to changing environments. An animal adapting to an environment can be thought of as learning at a very primal level. This evolution process describes the animal and the environment as changing together and changing continuously. An enactivist sees knowledge co-emerging from the interplay of the individual and the environment. An individual's new knowledge changes the environment (or the perception of the environment) and then results in changing the knowledge. This process is continuously cycling. For enactivists, cognition is "the enactment of a world and a mind on the basis of a history of actions that a being in a world performs" (Maturana & Varela, 1992, p. 9). Knowing comes from interacting with the environment. When this happens, each individual brings their own unique experience to the situation, and this experience affects the individual's perception of the environment which in turn affects the environment. Learning is not a sequence of steps, rather it is a continual interplay between observation and environment that results in the concept's co-emergence. "The enactivist claim, then is that cognition does not occur in minds or brains, but in the possibility for shared action" (Sumara & Davis, 1997, p. 415).

In a mathematics classroom, students respond to a stimulus according to their own individual experiences (structures). The student activity as a result of the stimuli changes the stimuli and the cycle repeats. Learners interact through their actions, thoughts and emotions, which are shaped and influenced by the environment that is also changing constantly. Proulx (2013) describes mathematical strategies as a "real-time product of interaction, of the meeting, of the solver and his environment, directly and continually influenced by both" (p. 313). Instead of seeing learning as a destination or an object to know, enactivists see learner and object as co-evolving or co-emerging. The learner brings their own unique experience to a problem, and based on this experience, the learner interacts with the problem and changes their perception of the problem.

Through enactivism, and later complexity theory (this will be described later), we are beginning to see a theoretical construct that explicates the generation of a key idea or moment of insight in the problem-solving process. These extra-logical processes that were critical to making progress in a problem were previously explained through patience, luck, and a-ha moments, are now being described as emergent events from a collective. These instances of illumination, generation of ideas, are coming from the situated context of collaboration. Enactivism emphasizes the embodied and situated nature of cognition, which means that the environment and the agent's interactions with it play a significant role in shaping cognition. Within this framework, insight moments or problem-solving breakthroughs are not solely the result of individual cognitive processes but emerge from the interactions between the agent, the environment, and other agents. Complexity theory (discussed in more detail later) expands on this idea by providing a framework for understanding emergent phenomena in complex systems. In this view, problem solving is a complex system with many interacting elements, including the problem itself, the individual or group working on the problem, and the broader social and cultural context in which the problem exists. Emergent events, such as insight moments, arise from the interactions between these elements and cannot be reduced to a single cause or factor.

Problem solving – In choice-affluent environments

In the early 21st century, problem solving continued to be a central focus in curriculums across the western world, in the classroom and in mathematics education. Researchers were focusing on how problem solving occurs and how to enhance problem solving in classroom settings. Schoenfeld (1985) showed that Pólya’s heuristics were not effective for novice problem solvers, and Mason, Burton, and Stacey (1982) were suggesting heuristics that rely on extra-logical processes such as illumination, insight and creativity in order to go beyond the resources that their repertoire of past experience afforded them. Where does this illumination and insight come from? Mason and Davis (1988) began to describe this phenomenon with their theory of shifts of attention which was developed around the same time as enactivism was taking hold. Complexity theory, then, builds on enactivism to describe how ideas can emerge from complex systems and Davis and Simmt (2003) integrate complexity theory into a mathematics classroom.

Complexity theory bears some resemblance to enactivism, radical constructivism, situated learning, and some versions of social constructivism and has only developed over the last 45 years. Radical constructivism is a philosophical perspective that holds that knowledge is not discovered but constructed by individuals through their own experiences and interactions with the world (Steffe & Kieren, 1994).

Social constructivism is a learning theory that emphasizes the role of social interactions and cultural context in the construction of knowledge and understanding (Rytila, 2021). Like enactivism, complexity theory arose from cybernetics and deals with themes of emergence, interaction between object and environment and adaptation.

Two key qualities are used to identify a complex system: adaptivity and emergence. Adaptivity is the change in the object and the change in the environment as the system interacts and evolves. Emergence is the self-organization of the individual agents into a collective with a clear purpose. Weaver (1948) described complexity by contrasting it with *not-complex*. Not-complex can be simple systems such as trajectories, orbits or billiard balls where actions and interactions can be characterized and even predicted in detail. As the number of variables increase, they become exponentially more difficult to predict and scientists move to new analytical methods such as probability and statistics to interpret, and the systems move from simple to disorganized complex. These two systems do not cover the range of possibility. It is when we recognize that objects within systems may not operate based on a set of known inputs/outputs (not deterministic), but rather their operations emerge in the interaction of the agents that we begin to think of it as a complex system. Perhaps it is this key quality of emergence that can describe the generation of a key idea or moment of inspiration in a problem-solving scenario.

In terms of mathematics education, Davis & Simmt (2003, p. 138) suggest that mathematics classes are adaptive and self-organizing. Behaviours and norms of the group emerge from the collective. Complexity theory represents a move toward understanding the collective as a cognizing agent (as opposed to a collection of cognizing agents). It is this levelling up or self-organization that is a key quality in a complex system and may be present within a mathematics classroom.

Davis & Simmt (2003, p. 145) outline five conditions of complexity as necessary but insufficient conditions for systems to arise and to learn:

(a) Internal Diversity – enables novel actions and possibilities.

(b) Redundancy – sameness among agents is "essential in triggering a transition from a collection of *me's* to a collective of *us*."

(c) Decentralized Control – locus of learning is not always the individual.

(d) Organized randomness – emergent behaviours are about living within boundaries defined by rules, but also using that space to create. Liberating constraints draw a distinction between proscription and prescription in tasks.

(e) Neighbour Interactions – there needs to be collaboration… not necessarily people to people but more for ideas to bump up against one another.

These five conditions are stated as *necessary but insufficient*, because a complex system cannot be forced or coerced into existence. Its very nature requires a randomness and a freedom amongst the individual agents for self-organization or emergence to occur. Complexity theory provides a new perspective on problem solving that challenges traditional, more linear views of how problems are approached and solved. According to complexity theory, the environment of problem solving is a complex system with many interacting elements, including the problem itself, the individuals or groups working on the problem, and the broader social and cultural context in which the problem exists (Davis & Simmt, 2003).

Complexity theory suggests that problem solving is not a linear process that can be solved by a single individual or group, but is instead a non-linear, iterative process that emerges from the interactions between the various elements of the problem. In this view, solutions to difficult problems are not predetermined or pre-existing but emerge from the interactions and feedback between the problem and the individuals or groups attempting to solve it. Complexity theory provides a lens to understand the processes in a complex classroom environment, but it is not yet a model for mathematical problem solving.

Koichu (2018) presents a model of mathematical problem solving (building on Mason's theory of shifts of attention (1989, 2008, 2010)) with the main idea being that in any problem-solving experience, the key solution idea results from a solvers' shifts of attention between the solvers' resources, peers and sources of knowledge. Koichu hints at characteristics of complexity theory with the idea that the solution process is made up of a series of choices that the problem solver negotiates. This is akin to organized randomness and neighbor interactions as described in Davis & Simmt (2003). Koichu's model is referred to as the Shifts and Choices Model (SCM) and relies on three premises:

1. Even when a problem is solved in collaboration, it has a situated solver: an individual who invents and eventually shares its key solution idea.
2. A key solution idea can be invented by a situational solver as a shift of attention in a sequence of his or her shifts of attention when coping with the problem.
3. Generally speaking, a solver's pathway of shifts of attention is stipulated by choices the solver is empowered to make and by enacting the following types of resources:
    a. Individual resources
    b. Interaction with peer solvers who do not know the solution and struggle in their own ways with the problem or attempt to solve it together, and
    c. Interaction with a source of knowledge about the solution or its parts, such as a textbook, an internet resource, a teacher, or a classmate who has already found the solution but is not yet disclosing it. (pp. 310-311)

Point three above also resembles complexity theory in that it describes the control in the solution being in the hands of the student solver and not the teacher (decentralized control (Davis & Simmt, 2003)).

Koichu (2018) decides to follow Schoenfeld's (2013) recommendation to move from a framework for studying problem solving to a model that would describe a theoretical structure for problem solving. This theoretical structure, the shifts and choices model, would connect our understanding of how problem solving occurs with our understanding of how to improve problem solving in classrooms. Koichu sheds the assumption of problem-solving as an isolated activity, suggesting that problem solving may improve for learners if they are situated within environments that provide many opportunities for the solver to shift amongst.

Despite stating that all problems are solved by a situated solver, Koichu introduces the possibility of the situated solver working with others as a result of shifts of attention and choices made in the solving process. In the center of this model, one sees the situated solver shifting attention among individual resources, resources from peers and resources from a solution source. These shifts are made in an environment of choices a problem solver is empowered to make: "Among endless conscious and unconscious choices that individuals face when solving problems … the model takes into account the following: a choice of a challenge to be dealt with, a choice of schemata for dealing with a challenge, a choice of mode of interaction, a choice of extent of collaboration, and a choice of an agent to learn from" (p. 309). Koichu presents a model for problem solving that describes how solvers shift attention from individual heuristics, interaction with peers and interaction with a solution source when resources are insufficient. He concludes by suggesting solvers best make these shifts in environments that are affluent in choices, and he describes these as *choice-affluent* environments.

In Koichu's model that describes how problem solving occurs and how to enhance problem solving, one wonders if students solve problems by shifting their attention amongst a myriad of choices, what might happen if we create learning environments that are abundant in these choices? What do choice-affluent learning environments look like, and how does problem solving improve when these choices increase?

Thinking classrooms as choice-affluent and complex environments

The Thinking Classroom framework is extensive, but the features that are relevant to this dissertation and provide a good conception of how these classrooms operate are:

- visibly randomized groupings
- vertical non-permanent surfaces
- rich tasks
- student autonomy

In a Thinking Classroom, students are put into randomized groups daily. This type of group organization is intended to break down social and cultural barriers, improve the mobility of knowledge, decrease the reliance on the teacher, and improve student engagement and enthusiasm (Liljedahl, 2016). The randomization occurs at the beginning of each lesson and it is done in a way that students can see. In my classroom, I wait for some students to arrive, I project the seating plan from last day, and then I hit a shuffle button. After this, the seating plan is shuffled and all the students are arranged randomly with their new partners. With the randomization being visible, the students have a greater buy in with the process, and most students are more willing to work with whatever their group ends out being. If the randomization was not visible, students would tend to think that the groups were being organized based on ability or behaviour. When groups are organized based on ability or behaviour, students tend to work in ways that cannot be considered collaborative (Clarke & Xu, 2008; Esmonde, 2009).

These randomized groups of students conduct almost all of their collaborative work and discussion while standing together at whiteboards. Liljedahl found that by having students work on vertical whiteboards, students were more willing to take risks and engage more readily in the task and student work was visible to all, catalyzing the movement of knowledge (improving porosity) and providing the teacher with cues on which groups to attend to. To facilitate discussion, there is only one pen per group, and this pen is often circulated through the group members to change the dynamic of the collaboration.

The third feature in a Thinking Classroom relevant to this discussion are the rich tasks. In a Thinking Classroom, all student activity revolves around rich tasks. Rich tasks are highly engaging, collaborative tasks that drive students to want to talk with each other as they try to solve them (Liljedahl, 2008). These tasks tend to be presented orally, and a series of two or three tasks will take students through the intended curriculum content for the lesson. In a Thinking Classroom, all of the mathematics that the students are exposed to and learn are presented through these rich tasks. As students are working through a task, the teacher is maneuvering through the room attending to groups to provide hints and extensions as needed. At the end of a task, the teacher leads a consolidating discussion usually using one of the group's board work in order ensure the desired content is achieved. This consolidation is important, as it is when the teacher draws attention to the intended learning for the task. It is where ideas are clarified, vocabulary is introduced, and students are able to ask questions about key ideas.

A fourth underlying characteristic that should also be mentioned is that students are provided autonomy, both in their actions and their thoughts. Student autonomy is important in a Thinking Classroom, because "if one values a classroom with high porosity, a classroom where knowledge moves around the room by means of all members in the room, then one needs to give students freedom to move and think when and how they wish" (Pruner, 2016, p. 78).

As mentioned earlier, these are just four of many features that make a Thinking Classroom, but they are the ones most pertinent to this discussion. "A Thinking Classroom is a classroom that is not only conducive to thinking but also occasions thinking, a space that is inhabited by thinking individuals as well as individuals thinking collectively, learning together and constructing knowledge and understanding through activity and discussion" (Liljedahl, 2016).

A Thinking Classroom is also a choice-affluent environment. Koichu (2018) describes a choice-affluent environment as:

> an environment in which students can at different times choose the most appropriate (1) challenge to pursue, from solving a difficult problem to comprehending a worked-out example; (2) mathematical tools and schemata for dealing with the challenge; (3) extent of collaboration, from being actively involved in exploratory discourse with peers of their choice to being independent solvers; (4) a mode of interactions, that is, whether to talk, listen, or be temporarily disengaged from the collective discourse, as well as whether to be a proposer of an idea, a responder to the ideas by the others, or a silent observer; and (5) agent to learn from, that is, the opportunity choose whose and which ideas are worthwhile of their attention. (p. 320)

From this description, I contend that through the classroom design and the pervasive undercurrent of student autonomy, Thinking Classrooms fulfill all five of Koichu's descriptors. Koichu refers directly to Liljedahl's Thinking Classroom as a candidate for a choice-affluent environment when he observed one group looking at another group's board work in order to make some progress in a problem. "The students engaged themselves in interactions exactly when they needed them and not when the teacher decided for them that they needed them. In terms of the definition of a choice-affluent environment, the students in Peter Liljedahl's class were empowered to make choices (2), (3), and (5) above" (Koichu 2018, p. 320).

Thinking classrooms are choice-affluent environments, but they are also complex spaces. A Thinking Classroom as a complex space can be justified through the five conditions of complexity described below (Davis & Sumara, 2014).

- Internal Diversity: The daily randomizing of the groups produces diversity within the groups.
- Redundancy: All students are at the same age and have similar prior mathematical experiences, they are members within random groups, they are all working at vertical spaces and all working towards completing a common task. This sameness helps to trigger the transition from individual learners to a learning community.

- Decentralized control: Thinking Classrooms are neither teacher-centered nor student-centered. Rather, learning is shared and emergent and control is distributed amongst the groups. Knowledge and ideas are not coming from the teacher alone; they ebb and flow through the classroom as progress is made or seen or heard from others.
- Organized Randomness: The tasks in a Thinking Classroom may have structure or provide constraints (proscriptive), but the ways that groups progress through the tasks is completely random and unstructured. The natural randomness of human cognition thrives in a Thinking Classroom and is what allows creative ideas to develop, evolve and migrate throughout the room.
- Neighbour Interactions: The very nature of the groups working side-by-side in a vertical and public medium facilitates the interactions within groups and among groups.

Because a Thinking Classroom has all five of the conditions for complexity, it is a learning space that is ripe for the collective emergence of understanding or creativity within a mathematical solution.

I finish this literature review and discussion on theory with Koichu's model for problem solving in choice-affluent environments and Liljedahl's Thinking Classroom framework as a complex space, and we see that these models are the current state of an evolution on the research in problem solving. Beginning with Pólya's heuristics and Schoenfeld's resources we saw the development of the research on problem solving with a static focus on the individual and problem solving by design. Mason shifted the research into the social and developed his theory on shifts of attention to help us understand how key ideas can emerge in the problem-solving process. While accepting that "there are probably no configuration or configurations of teaching decisions that would be optimal for enhancing problem solving for all" (Koichu, 2018), Koichu presents a model that both describes the problem-solving process and presents some ideas on how to design a problem-solving environment. This choice-affluent environment is a nice parallel for the environment that mathematicians engage in, in that mathematicians also have a multitude of resources to choose from – both social and technological.

Resources and Emergence

Putting these theories together led me to an expectation on how students and groups progressed in problem solving within the choice-affluent space of a Thinking Classroom. This expectation is a starting point in this dissertation, and I am placing it here to demonstrate my early understandings and how the theories are being implemented in this study.

When a task is presented to the class, there is usually an eruption of discussion and activity as they move to their boards in their randomized groupings. This discussion

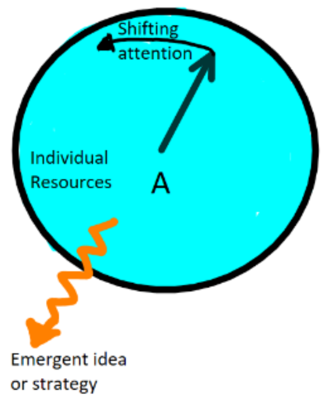

FIGURE 15: INDIVIDUAL PROBLEM SOLVING

signals the *understanding* stage of their problem solving. According to Schoenfeld's resources framework, this is when learners are drawing on their personal repertoire of resources that might be applied or useful in this problem. Their attentions are shifting among their individual resources until a strategy or idea emerges (fig. 15). In this diagram, the individual, A, has a repertoire of experiences and knowledge represented by the shaded region. While thinking about the problem, A's attention shifts through these resources until an idea or a strategy emerges. In a Thinking Classroom, this individual phase of understanding the problem and searching through individual resources is often quite short, because students start each problem within their group. So, it is expected that students will immediately begin dialogue with their partners to check to see if they are understanding the problem posed; students in Thinking Classrooms almost always begin their problem solving as a collaborative.

In collaborative problem solving, three students have the added benefit of shifting attentions between their shared resources and their partner's resources (fig. 16). In this diagram, I present the original individual, A, in the upper left circle is now interacting with the resource spaces of two other individuals: B and C. The emergent idea or strategy is now coming out of the confluence, overlap, of the individual's resource spaces. This is generally the reason for the noise and activity early after the initial problem posing.

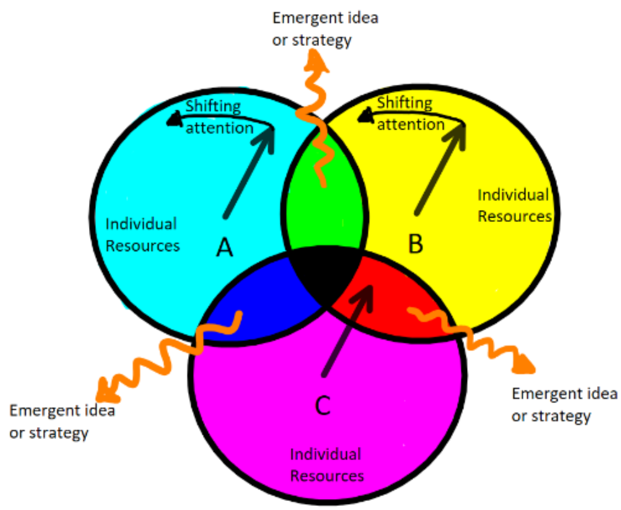

FIGURE 16: COLLABORATIVE PROBLEM SOLVING

Students are checking with their partners to see if their understanding of the problem posed aligns with others in their group. This is the shifting of attentions between their own knowledge and experiences as well as those of their partners. As a group of three, the sample space of resources is generally tripled, and ideas are more likely to emerge without any outside influence such as a teacher or even neighbouring groups. In this scenario the entire group is now considered as the cognizing agent, and so the situated solver could be considered the entire group. Through the shifting of attention through their individual resources and through their partner's resources, the group creates some possible strategies for beginning the problem-solving process. By attending to a specific feature in the problem, students share their perspective with others in their group and reach a mutual understanding of the problem to solve. This is typically how students move through the first stage, *understanding the problem posed*, in a Thinking Classroom, and if it is a legitimate non-routine rich task, meaning students do not have a specific process in their resources for a quick solution, students will undoubtedly become stuck.

Being stuck may be noticed in Thinking Classrooms when group activity and group conversation decreases dramatically. The whiteboard pen may be put down, their shoulders sink ever so slightly, conversation moves off-topic or tapers off and their gazes turn outward – the group has entered the Stuck-unstuck stage. In choice-affluent environments, the group's attention can shift amongst an abundance of choices in the room including, but not limited to, conversations with other groups, observations of other's board work, hearing conversations of others or interactions with the teacher (fig. 17). In this diagram, the key idea or strategy is emerging not only from the shared resources of the group, but it is also coming from the interactions with other choices within the room, such as groups, boards, technology, the teacher, and more. This shifting of attentions is serving a couple of purposes. It is providing an opportunity for confirmation and an opportunity for illumination. When groups are stuck, it may be because of an error made or it might be that they just do not know how to make it to another step in the solution. By shifting their attention outward, observing other boards in the room and possibly conversing with neighbouring groups, their resource space has increased dramatically. The *stuck-unstuck* group is now able to see if their work aligns with other group's work in the class, and their attention may shift onto a solution idea that allows them to become unstuck. In choice-affluent environments, the sample space of available resources is abundant, and this supports the autonomy and independence of the students as they start to make sense of the problem and it also supplies a continuous stream of resources for the students to attend to. This abundance of resources can stimulate any number of shifts or combinations of shifts from investment, examplehood, resonance or even grace. In this global scenario, the entire class is now considered as the cognizing agent, and so the situated solver could be considered to be the whole class.

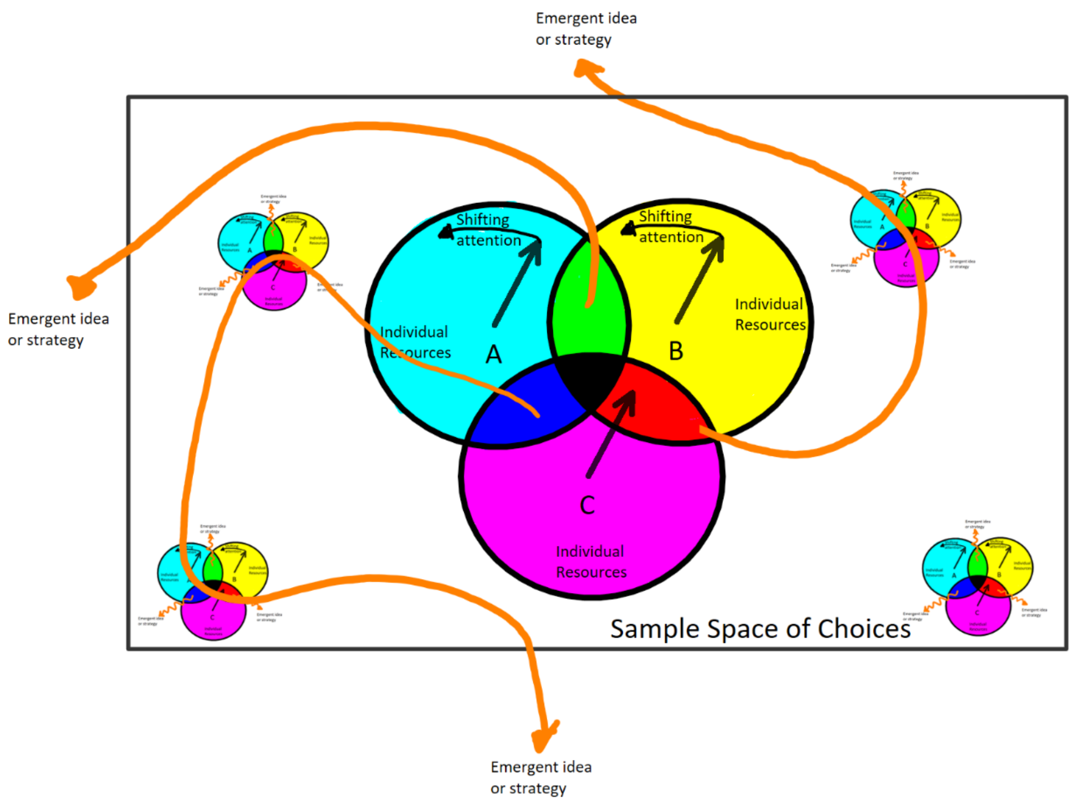

FIGURE 17: COLLABORATIVE PROBLEM SOLVING IN CHOICE-AFFLUENT ENVIRONMENTS

The literature review was introduced with Devlin's lament concerning students not experiencing mathematics (problem solving) in ways that mathematicians are engaging in the subject. Research on problem solving in mathematics education has reached a point that actually comes close to describing classroom problem-solving experiences that resemble real world problem solving by mathematicians, so what might these classrooms look like? A choice-affluent classroom is one where students "are able to choose the most appropriate (1) challenge to pursue, from solving a difficult problem to comprehending a worked-out example; (2) mathematical tools and schemata for dealing with the challenge; (3) extent of collaboration, from being actively involved in exploratory discourse with peers of their choice to being independent solvers; (4) a mode of interactions, that is, whether to talk, listen, or be temporarily disengaged from the collective discourse, as well as whether to be a proposer of an idea, a responder to the ideas by the others, or a silent observer; and (5) agent to learn from, that is, the opportunity to choose whose and which ideas are worthwhile of their attention" (Koichu, 2018). What does this classroom look like and how does it support problem solving for the learners? Devlin suggests that the ways that students engage in mathematics need to reflect and imitate how mathematicians solve problems, and students need to develop a skillset that provides them the knowledge and flexibility to choose from a multitude of tools and resources that can help in solving problems. Koichu's choice-affluent environment may be considered a response to this suggestion, but there are still many questions to answer with respect to creating these learning environments and studying their effects on problem solving in the learner.

What does emergence look like in a Thinking Classroom? Emergence in any classroom is when intellectual movements arise spontaneously and may quickly exceed the possibilities of any of the individuals – the knowledge, idea or understanding is a property of the collective. This is not only present in a Thinking Classroom, but it is amplified. As described above, ideas or progress in a problem will quickly move from an individual to the whole class. Due to a Thinking Classroom satisfying all the conditions for complexity, it is a fertile space for observing emergence. Emergence in a mathematics classroom is a key solution strategy, a unique diagram or method, an A-Ha moment of inspiration or a new understanding of a concept; in a Thinking Classroom, these moments can all be observed, and their emergence can be mapped because of students working in the public space of vertical whiteboards. This leads to the research questions for this dissertation:

1. What forms of interactions seem to support the movement of ideas and how do these interactions support problem solving?
2. What form do neighbour interactions take and how do they align with the theory of complexity theory?

Chapter Four - Method and participants

The data in this study come from a series of video recordings of students working through problems in my classroom at a high school in North Vancouver, British Columbia. The students are in a pre-calculus class in their senior year at high school. They are in the academic pathway of mathematics, meaning that most are planning on continuing their studies in a post-secondary institution in their following year.

The setting

In 2013, I had been teaching high school mathematics for 19 years, and I believed that I had it all figured out. Throughout my years in the classroom, I had accumulated many skills and implemented many systems which I had believed created a very positive and effective learning environment. With hindsight now, I see that most of my practices were designed with a focus on control. I felt that in order to best teach my students, I needed to control all aspects of their learning. I needed to control how the mathematical content was delivered, how it was received, and everything in the environment that might affect either of these two intentions. Before I describe my current classroom where the data was collected, I think it is important to describe my early classroom and my journey of change.

My old classroom

I began teaching high school mathematics in 1994 in North Vancouver, and I remember those early days of my career always being interested in teaching better. This is probably not too different from most teachers, but I recall every August, before the start of school, thinking carefully about how to change my practice, change my assessment, and possibly change the sequence of topics. I was always interested (and I still am) in improving the experiences of my students, so they could better learn the mathematics and perhaps feel more positive about the subject itself. In the 90's, I was also interested in technology, and this general interest merged into my professional career.

As a teacher, I enjoyed experimenting with different technologies in the classroom. Computers and the internet were improving at such a fast rate, and I knew that these tools could be used to help students learn. In the late 90's, I remember implementing online bulletin boards for my students to collaborate and get help from their peers at home. I began producing digital note templates for students to complete while participating in my lessons. And I began curating a classroom website where students could access the course outline, the digital note packages, and find some web links to mathematically interesting web sites. I started to become known in the school community as the *techie math teacher*.

In the 2000's, I continued changing my practice with a goal of improving experiences for my students, and I continued to experiment with different technologies in the classroom. I went back to school and earned a diploma in teaching and technology from Simon Fraser University, and this two-year program also influenced my teaching. It was early in this decade that I received my first tablet computer. I remember being dazzled with this amazing technology – I could actually write on my digital note templates with my computer pen! With this computer, I began capturing screen recordings of my lessons and completed digital notes that I would post on my class website. My website had a calendar for the school year, and on each day, I would include links to the lesson recording and the pdf notes for students to view. Later in this decade, I convinced my school to purchase a class set of I-clickers. This was an amazing technology that was being used in a lot of post-secondary lecture halls. I-clickers were small handheld remotes that allowed students to *anonymously* indicate answers to multiple choice questions. Throughout my lesson, I would ask questions like "Do you understand?" or "what do you think the answer is?", and students would use the I-clickers to respond. I would instantly see the results of the poll on a small device at my teacher pedestal. I was dazzled with how I-clickers improved class participation and provided me with immediate assessment information.

By the end of 2010, I was becoming quite confident in my teaching practices. I had over 15 years of experience, continuously evolving my practise to a point where I believed that I had it all figured out. I razzled and dazzled my students with high tech, and I likely misinterpreted this dazzle as learning and engagement. At this time, I began working with textbook companies on resource development, I began working with the Ministry of Education on provincial exam development, and I began leading professional workshops with teachers. I was an experienced mathematics teacher, and I had a lot to share.

For many years, I based my teaching approach on my own experiences as a student. My students typically sat in desks arranged in rows according to a seating plan that I carefully arranged at the beginning of the school year. I would start by organizing the seating alphabetically by last name, and then make adjustments over the first few weeks until I felt it was just right. This often meant placing "trouble-making" students close to me or surrounded by well-behaved students to create a calm and peaceful classroom environment. Once the seating plan was set, it rarely changed throughout the rest of the school year.

I also had various procedures in place to prevent potential issues, such as locking my door a few moments after the bell rang to discourage tardy students. Late students had to wait outside my door until I deemed it appropriate to allow them in, with the idea being that missing my lesson was a natural consequence for being late and would encourage them to arrive on time in the future. To promote student engagement, I implemented a reward system where students could earn stamps for participating in class or answering challenging questions. After collecting 10 stamps, they would receive a bonus mark. While this system was effective at rewarding students who were already participating in class, it did not necessarily motivate quieter or weaker students. I placed a great emphasis on taking thorough notes and maintaining a clean and organized notebook. I often reminded my students that if something was important enough for me to write on the overhead projector, it should also be recorded in their notebooks. I would also periodically check their notebooks and require students with incomplete or blank notes to stay after class or during lunch to copy the notes from the overhead.

While my teaching style may have seemed strict, I actually enjoyed explaining math concepts in an engaging and entertaining way and was deeply passionate about the subject. I genuinely wanted all of my students to succeed and believed that the measures I took, such as enforcing a structured classroom environment and checking homework, were all part of that effort. In fact, I marked homework partially as a way to motivate students to complete it and also to gather data on their work habits, which I needed to report on in student progress reports. At my school, we were required to include both a percentage mark and a work habit mark in these reports, and I preferred to base the work habit mark on objective measures like homework completion.

I followed a common teaching approach where I would explain a new math topic, demonstrate how to solve various problems, and then give students a chance to try similar problems on their own. This is known as the "now you try one" approach (Liljedahl & Allan, 2013a, 2013b). While I would walk around the classroom to help students during this time, I mostly ended up managing behaviour, reminding students to open their notebooks or get back on task, or showing individual students how to solve the problems. After a few minutes, I would ask for volunteers to come to the front of the class and demonstrate their work on the board, and the same group of students would typically volunteer and earn reward stamps. Once the "now you try one" portion of the lesson was finished, I would give the students their homework and expect them to work quietly at their desks until the end of class.

I also gave biweekly quizzes and around ten chapter tests per year, which I kept in a filing cabinet to reuse and standardize my assessments. Students could view their tests during non-class hours but were not allowed to take them home. I did experiment with allowing students to rewrite tests or demonstrate their understanding of basic concepts through what I called an "I-test," where "I" stood for "incomplete" and passing the test meant passing that part of the course. However, this was the extent of my innovative teaching practices. I was a passionate and controlling mathematics teacher, and I believed that as long as students listened carefully and thought the way I did, they would do well in my class. Despite my efforts to be more engaging and provide incentives to learn, I often felt frustrated by the lack of attentiveness, engagement, and motivation among many of my students. It seemed that only my honours students or top performers were truly invested in learning math.

Motivation to change

In 2011, I became an executive member of the British Columbia Association of Mathematics Teachers (BCAMT), a specialist organization for math teachers in BC. As part of this role, I coordinated registration for their teacher conferences, a position I still hold. In 2013, while coordinating registration at one of our annual New Teachers Conferences, I happened to be in the same room as the keynote speaker, Dr. Peter Liljedahl from Simon Fraser University. As I listened to his presentation, I realized that everything he was discussing related directly to my own experiences in the classroom.

Liljedahl presented on research he had conducted with a graduate student, Darien Allan, on "studenting," or everything that students do in a classroom, both good and bad. When student behaviour supports learning, it is considered good, but when it does not, he referred to it as "gaming," or undesirable behaviour. He showed that it was common for students to engage in gaming in a classroom, and one slide in particular caught my attention (fig. 18). It depicted observations from a classroom using the "now you try one" instructional approach, and I could easily see my own students fitting into the categories described.

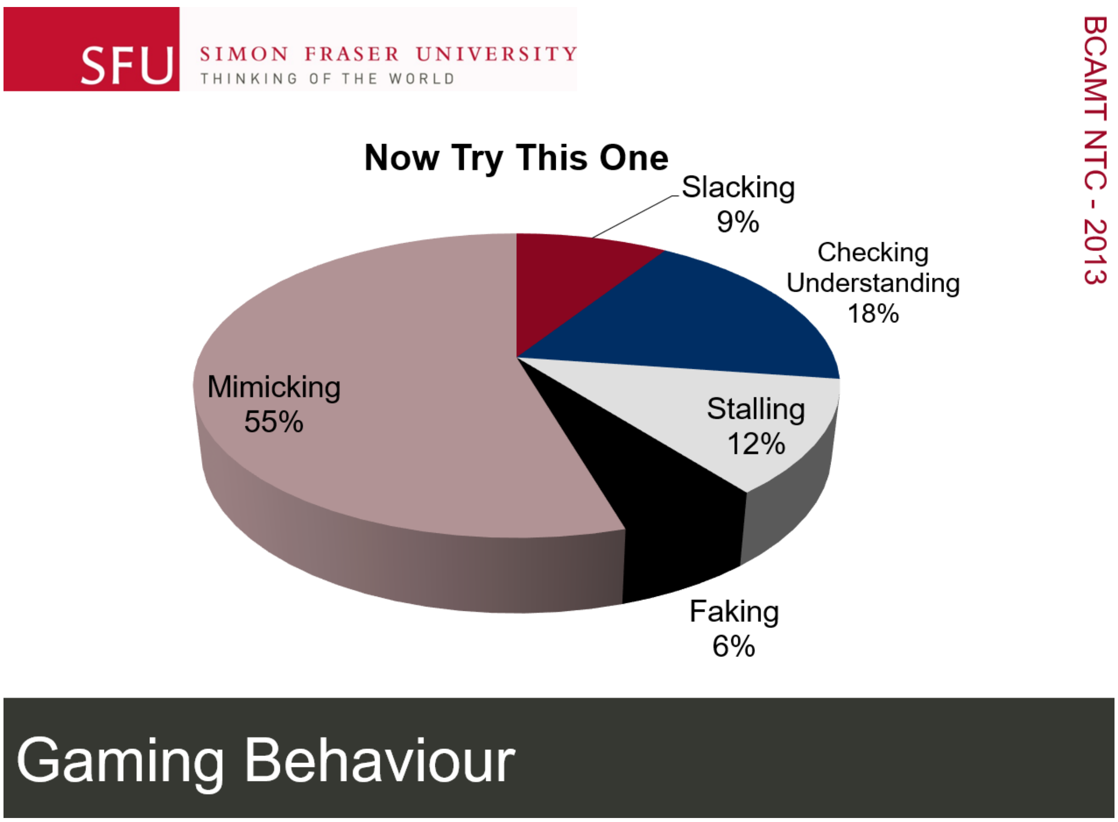

FIGURE 18 A SLIDE FROM DR. LILJEDAHL'S PRESENTATION (USED WITH PERMISSION)

According to the slide, about 12% of students were "stalling," rummaging through their backpacks or asking to leave the room for a break. Another 9% were "slackers," slouching in their desks and not engaging with the material. There were also "fakers," who pretended to participate in class activities but were really just waiting for the correct solution to be shown, representing about 6% of the class. The most striking category was the "mimickers," who made up 55% of the students observed and likely a similar proportion of my own students. These were the students who copied the math line by line from the teacher's demonstration, substituting numbers as needed, without actually thinking about the underlying mathematics. Liljedahl argued that these mimickers were not thinking critically and therefore not learning.

Granted, this dataset came from a single class, but the numbers resonated with my own experiences of teaching in this way. As I listened to Liljedahl speak, I too recognized that about 55% of my class were mimicking and about 25% of my classes were faking, stalling and slacking. This realization that about 80% of my students were not engaging in meaningful learning during my "now you try one" lessons was a profound moment for me. I had been teaching for 19 years and had spent that time building classroom procedures and refining my instruction, but I still had a significant proportion of students who were not learning. This led me to question my own teaching practices and consider alternative approaches.

Fortunately, Liljedahl's keynote also focused on strategies for creating a classroom culture that promotes thinking, engagement, and problem solving, rather than gaming. He suggested making three changes to our teaching practices to begin this transformation:

1. Use visibly random groups (VRG) every day. Group work is a well-known strategy that can improve student collaboration, communication, and learning, but in most groups, students quickly settle into roles. By randomizing groups daily, students are constantly required to reconsider their roles and are given opportunities to take on different responsibilities. By making this process visible, students trust that it is truly random and are more willing to work through challenges, knowing that the group will change the next day.
2. Have students do all of their work in their groups at vertical non-permanent surfaces (VNPS). Liljedahl found that when students work on VNPS, such as whiteboards or chalkboards, engagement increases, student work becomes visible (which is helpful for both teachers and students), and students are more willing to take risks with their thinking. This last point may seem counterintuitive, but he argued that it is the "non-permanence" of working at the whiteboards that increases students' willingness to take risks. Since

mistakes can be easily erased, students are less afraid to engage with problems.

3. Stop making students take notes. This is a natural consequence of implementing the first two strategies, but it was a major change for me. Liljedahl found that for most students, the act of taking notes serves as a proxy for learning, rather than actually promoting learning. Most students cannot keep up with note-taking, cannot process the discussion in class while taking notes, and never use their notes for learning later in the course.

After implementing these practises in 2013, I noticed immediate improvement in student's actions, behaviours, engagement and affect. Many students were excited to be doing their own mathematics, they persisted in solving problems, and they even seemed to enjoy the process: they would laugh more and celebrate their victories with others. Admittedly, this was not perfect. There were still students who were slacking and disengaged, but there was an enormous improvement from the old model. The whole environment was such a radical change from what I was used to as a teacher, that I was determined to never go back to my old practices – I was determined to continue my journey with Thinking Classrooms.

In 2013, I enrolled in a master's program in mathematics education at Simon Fraser University. I have fond memories of this experience, as I was able to get into deep discussions regarding pedagogy and the nature of learning with fellow teachers in my cohort every week. This was a two-year adventure in reading, researching and writing about mathematics education; at the end of which, I began work on my Master's thesis that was titled, "Observations in a Thinking Classroom" (Pruner, 2016). Through this thesis project, I analysed hours of classroom video from my own classroom, and I made some observations of what I noticed from this *fly on a wall* perspective. Results of the thesis indicated that high mobility of students and ideas, autonomous behaviour in students, and a significant amount of class time spent on tasks were some of the observations that were noticed in a Thinking Classroom. I defended my thesis and completed my Masters in the summer of 2016, and my journey with Thinking Classrooms has continued to evolve.

My Classroom today

Today, my classroom does not look or feel anything like it did pre-2013. Instead of desks, my room has 10 tall round bistro-styled tables, each with 3 stools to accommodate my typical 30-student classroom (fig. 19). On three of my four walls, I have large sections of whiteboard, and on my fourth wall, I have a window that is used as a student writing space and a projector screen for my computer.

FIGURE 19 MY NEW CLASSROOM TABLES

At the beginning of class, students drop their bags and backpacks off in the corner of the room and wait for the seating plan to be randomized. I project their seating plan on the screen and hit "shuffle" to randomize their seating, and then they make their way to their designated table. After this, I circulate the room, greet my students and distribute one whiteboard pen to each of the 10 groups. I begin most of my lessons with a fun game, puzzle or non-content based math task. I like to start classes like this because it is a nice way for students to begin working with their new partners, and these puzzles and games can really set the tone for future problem-solving tasks. As student regularly engage in math puzzles and games, their beliefs about mathematics slowly change over time, and they develop persistence, curiosity and different problem-solving strategies that can come in handy down the road.

After the opening task, I design the learning goal around two to three open tasks for the students to work on in sequence. Sometimes, we consolidate after each task, and other times, we consolidate after all of the tasks are complete. Consolidation usually has me gathering the class around some of the student's work and discussing the steps and ideas that are apparent on the boards. This is the stage that most closely resembles teaching from a traditional standpoint. It is in this consolidation space where I am keenly aware of my learning goal for the day, and I am orchestrating the discourse so that the learning goal emerges from the discussion: pointing to strategies, discussing new vocabulary and highlighting key ideas. At the end of almost every lesson, I arrange for some sort of independent self-check for the students. This can be an exit ticket that I collect and comment on, or it can be something less formal for each of my students to reflect on their own learning for the day.

Data gathering

I was interested in the interactions that support the movement of ideas in a Thinking Classroom. After years of noticing a seemingly autonomous action of these extra-logical moments arising in various and multiple locations within a classroom, I was interested in capturing the details of a typical classroom engaged in a typical mathematics task. During these lessons, I found that as a teacher, I was preoccupied with my usual teaching duties of classroom management and maintaining the flow of the tasks, and this made me somewhat blind to the minutiae of detailed actions that my students were engaged in. I decided that capturing video recordings of a lesson might provide me with the opportunity to study these detailed actions of my students.

After my experience of capturing video data for my master's thesis, I learned a few things. I learned that video quality was very important. Often, in analysing that older thesis data video, I was unable to see what students were writing due to the lack of resolution in the video. I also learned that it would be really valuable to capture student's conversation. In the old data, I did not have quality audio from the recording, so I was not able to actually hear the details of the student conversations. Lastly, I wished I had two different camera perspectives. Many times, when I was analysing that older data, I would feel that I just missed a student action such as a glance or a gesture, and I wished that I had another video perspective to add clarity to my observation.

In order to collect data in this classroom, two video cameras were placed; I chose two cameras to address the issues noticed from my previous experience of video data collection. The first camera was hanging from the ceiling in the middle of the room and pointed towards one set of whiteboards, capturing the work of three different groups. The second camera was placed on the actual whiteboard and pointing down towards one of the three groups.

The purpose of the first camera was to capture a global perspective of the three groups working. Through this lens, one can see how the individuals are moving and gesturing within their groups, what the students are writing on their whiteboards, and it also captured the direction of their gazes. The latter feature proved valuable in observing how students were looking at other groups or other boards.

The purpose of the second camera was for capturing audio. In previous 1-camera experiments, the data was missing most of the student dialogue. This missing dialogue made it difficult to understand the solution process. Adding this second camera on the whiteboard, I am now able to include most of the student dialogue in the data. Because it is just one camera capturing the dialogue of three groups, there are some instances where the dialogue in not discernible. This was exacerbated by the students wearing masks. The data was collected in the fall of 2019 when covid-19 was going strong, classes were in session, but covid protocols (like wearing masks) were activated. Unfortunately, there were a couple of students in the data who were ‘quiet talkers,’ and their voices were further muffled by their masks. For these students, I was not able to record their dialogue and simply wrote “unintelligible” in the data table.

These two video cameras were turned on at the beginning of class, and they were left running for the duration of the class. The classes were 2 hours and 20 minutes in duration, so usually the cameras turned off on their own when the batteries were depleted. The GoPro camera saved each video file as a 4GB MPEG movie that was 17 minutes and 42 seconds in length. At the end of each day, the video files were labelled with the period, the day and the activity that was mostly captured in that time period. After a week of recording, the files were viewed and reviewed, and a few optimal candidates were flagged. To become an optimal candidate, I was looking for videos that showed lots of examples of resource gathering, videos that contained intra-group collaboration, and videos that had high quality audio for transcription purposes. Because of the nature of a Thinking Classroom, these qualities were present in almost all of the videos, but some were better than others, and these were the videos that were put aside for more detailed analysis.

Data

The videos that I ended up choosing for this analysis consist of two different stories around two different tasks. In each story, there were three groups working side-by-side, and each group contributed a unique perspective for the analysis. Once this video was determined to be the focus, I then used video editing software to combine the two camera recordings into one. The video from the central camera became the main video, so one can see the writing on the whiteboard and the glances and gestures from each participant. Embedded in this video was a smaller video (picture-in-picture) capturing the closer up work of a singular group. This second inserted video was the one providing the best quality audio, so this became the audio for the combined video. I was careful to synchronize the timing of the two videos before the final video was produced. In the end, I produced two 17-minute videos capturing the work of three groups as they progressed through two different problem-solving tasks – two stories. From each of these videos, gaze-dialogue transcripts were made.

Gaze-dialogue transcripts were first developed by Liljedahl & Andrà (2014) who added gaze arrows to interactive flowchart transcription (Ryve, 2006; Sfard & Kieran, 2001) as a way of documenting what and who students are attending to. A gaze-dialogue was recorded for each of the three groups recorded. The transcript is recorded in six columns (fig. 6). The first column contains the time interval. This time interval is almost always 10 seconds; the reasoning for this interval length is described below. Sometimes, there are greater intervals of inactivity, so the time intervals were stretched for these circumstances. The second column, titled outside, serves two purposes. This column captures the dialogue from any person outside the group who is interacting with the group. This column is also a place holder for gaze arrows. If a person within the group is gazing outside of their group, the gaze arrow points to this column. The middle three columns represent the dialogue for the three group members, and they are labelled with letters corresponding to the individual in the group. An asterisk is used in these columns to indicate who is currently holding the whiteboard pen (there is only one pen per group). Solid arrows (→) are used to indicate a directed conversation among participants. A solid arrow with a point on one end (→) represents a student speaking to another student and a solid arrow with points on both ends (↔) shows when both students are speaking to each other. A double asterisks (**) is used in the table to indicate who is in possession of the whiteboard pen. In some of the transcripts, a seventh column was added to include a small picture of the board work at the specified time interval. This was found to be necessary at times when the notes were not able to give enough detail of the actual happenings in the interval.

Time	outside	A	B	C	notes
6:00 – 6:10					Student A grabs the pen and adds some clarifying steps. Student C is looking at incorrect solution on board to the right.
6:10 – 6:20					Student C is shifting glances from his board to the right board.
6:20 – 6:30				These two are different.	Student C points to the constant term in each expression.
6:30 – 6:40					Students A and B attend to the other group's work. Student B checks the graphs in Desmos.
6:40 – 6:50			They are exactly the same.		Student B holds his phone up to show Student A and C
6:50 – 7:00		**			Student A begins to verify their solution.

FIGURE 20 SAMPLE GAZE-DIALOGUE TRANSCRIPT

For example (fig. 20), one can see that student C is speaking directly to students A and B. Gazes are represented by dashed arrows (--→) to encode every time a participant is attending to something outside of their group. So, for example, in figure 6, students A, B and C are all gazing to whiteboards outside of their group at different times.

To capture shifts in attention the transcript breaks each recording into equal intervals. After some experimentation with multiple transcriptions across a variety of interval lengths, 10 seconds was chosen for the interval length. I found that when the interval was shorter, there were many intervals with little to no observable dialogue or shifts of attention. When the interval was longer, I found that too much transpired within the interval to effectively document within the gaze-dialogue transcript.

Within these intervals, I interpreted groups that were working as having adequate resources (Schoenfeld, 1985) in the form of knowledge and/or repertoire of past experiences. If a group showed some sort of productive activity (ie. gesturing, writing, or dialoguing with another student), I would consider this working. If a group was not working within an interval, I interpreted that group as having depleted their resources. I relate working to having resources, because, if students are working, they are likely making some progress (right or wrong) towards a solution. In this way, I assumed that individuals who were actively participating within a group that was working had access to the collective resources of that group. Likewise, I assumed that individuals that were not actively participating both lacked the individual resources to solve the problem and did not have access to the collective resources of the group.

The video clips being analysed for this dissertation captures three groups of students as they progress through the tasks described below. I was interested in how ideas moved between groups, and the videos are mostly clear in capturing how the individuals in the group gesture, who in each group is writing, where their individual attentions are drawn to, and what is said amongst the participants.

Data Analysis

At first sight, one might not see anything too special in how these groups are working towards their solution. However, through analysis of where students are attending and how they gesture, one can begin to see how ideas are originated, dispelled, or confirmed as progress towards a solution continues. One can also see how the groups increase their attention outside of each group later in the solving process as a means to affirm their current work or to aid in their progress. Gaze-dialogue transcripts were chosen for the analysis because they provide a document of where students shift their attention as they progress through a solution. At every 10-second interval, one can see whether a solution is progressing due to internal resources or due to external resources. The external resources are apparent when a student is in dialogue with another student, using their smart phones, or gazing at another board's work. Complexity theory describes neighbour interactions as one of the conditions for emergence. In a classroom, neighbour interactions can take the literal form of dialogue, gazing, and eavesdropping, and they can take a more figurative form of an external idea bumping up against an internal previous memory, experience, or prior knowledge. Gaze-dialogue transcripts provide a useful tool for documenting the literal form of these interactions.

The setting for each story

This classroom has 30 students and is operated under the Thinking Classroom framework. As described earlier, when entering class, students deposit their bags in the corner of the room and wait for the teacher to randomly generate the 3-person groups and the seating plan. After seeing their group and their corresponding table, the students move to their assigned location and begin a small conversation with their new partners. In the beginning of the year, the teacher would provide ice-breaker type questions to encourage the groups to get to know each other, but at this point, later in the year, the teacher just reminds students to take some time to say 'hi' and introduce themselves if necessary.

After this short introduction time, I begin the class by providing a series of open tasks for the students to work on. An open task is one that is proscriptive in nature. Compared to prescriptive tasks that instruct students to do one thing and then another, a proscriptive task begins with one goal or question, and students are free to choose their own strategies and pathways towards a solution. The tasks are designed to be open, so students will have choice in how to navigate the problem, and with a purpose, so that a specific learning goal is attained. Students are given 20 – 30 minutes to work on each task. After this time, the teacher orchestrates a consolidation: the whole class gathers around student work and discusses the solution. This discussion is often directed by the teacher, as the teacher attempts to reify the main learning goal of each task. In this consolidation phase, the teacher is 'teaching' by introducing vocabulary, correcting steps, or answering student's questions regarding the work. After the consolidation phase, students are given some time to take notes of their learning, and then another task is introduced – the process is repeated.

The task for lesson 1

The task that they were asked to work on was the following:

Write the following function in the form $y = \frac{a}{(x-k)} + b$ and then sketch the graph.

$$y = \frac{(4x-5)}{(x-2)}$$

FIGURE 21 TASK FOR LESSON 1

Students in this class had experience with rational functions that are presented in this form: $y = \frac{a}{(x-k)} + b$

Most students knew the general shape of the base function, y = 1/x, and they understood how the parameters, *a*, b and *k*, transform the original graph. In the previous lesson, the students were tasked with writing rational functions as a single fraction of polynomials. They did this by finding and creating a common denominator on the constant term, *b,* and then combining the two numerators. In other words, the students had experience with converting the functions in this way:

$$y = \frac{3}{(x-2)} + 4 \longrightarrow y = \frac{(4x-5)}{(x-2)}$$

FIGURE 22 STUDENT'S PREVIOUS UNDERSTANDING

They did not, however, have any experience in my classroom of converting the functions in the other direction:

$$y = \frac{(4x-5)}{(x-2)} \longrightarrow y = \frac{3}{(x-2)} + 4$$

FIGURE 23 GOAL FOR THIS TASK

The algebraic steps in this conversion are quite challenging; they generally involve one of two different methods:

1. creating a factor of (x – 2) in the numerator by subtracting 8 from the 4x and then compensating for that subtraction by adding 8.
2. using polynomial division to re-write the rational function as a quotient (b) and remainder (a).

As this problem was not previously encountered by students in this class, it is a prime candidate for studying how students problem-solve in choice-affluent environments.

The task for lesson 2

The task that they were asked to work on was the following:

Write as a single logarithmic expression and then state restrictions on *x.* $$\log_7 x^2 + \log_7 x - \frac{5\log_7 x}{2}$$

FIGURE 24 TASK FOR LESSON 2

Students in this class had experience with an introductory lesson on the properties of logarithms, where each property was considered and reasoned in isolation:

$\log A + \log B = \log(AB)$
$\log A - \log B = \log\left(\frac{A}{B}\right)$
$m \log A = \log A^m$

FIGURE 25 STUDENT'S PREVIOUS WORK WITH LOGARITHMS

There are several features to this task that made it a worthy candidate for problem solving in this lesson. First, students had not yet simplified a logarithmic expression that required multiple applications of the logarithm properties. Next, the final fractional expression in the first expression is challenging as many students were not familiar with this being similar to the final logarithm property on the list above. And lastly, students have not yet been asked to state restrictions on logarithmic expressions. They were aware of the domain for the base logarithm function, *y = log x*, but they have not yet considered how this might apply to larger more complicated logarithmic expressions. As this problem was not previously encountered by students in this class, it is also prime candidate for studying how students problem-solve in choice-affluent environments.

Chapter Five - Results and Analysis

Each of the two lessons have three groups working on the above stated tasks; therefore, there are three different groups to analyse for each of the two lessons. Each group contributes something different to the overall picture of how the problems were solved. In this analysis, I will focus on each lesson and the corresponding three groups by providing an overarching commentary on the entire video segment. Then, I will go into each of the three group perspectives. Each perspective will begin with the gaze-dialogue transcript and a written description of what was observed. After each description will be an analysis describing the results that I am beginning to notice. Then I will finish with a more detailed summary analysis on what is being revealed from the complete data from all three groups.

After viewing many of the classroom videos, I began to notice that there were details that were occurring at different levels of student activity. I was noticing that I could attend to the details of what the individual students were doing (the individual level), I could attend to the details of what the group on a whole was doing (the group level), and I could also attend to a global level – how the class (the three groups together) were making progress on the task. By discussing the results at these varying levels, I hope to get a better understanding on how the problem-solving process and student understanding is related to the level at which different activities are observed.

In the analysis of the video, I will be drawing on Koichu's work (Koichu, 2018) and attending to who the situated solver is at various times during the problem-solving process. I will be detailing my observations on how and where the situated solver shifts their attention (Mason, 1988) both literally, by attending to different spaces or people, and figuratively, by making a sudden change in their thinking or making some progress in the task. Schoenfeld's concept of resources (Schoenfeld, 1985) and its abundance will be leveraged as the fuel for students making progress. Conversely, the scarcity of resources is used as a reason for pausing in an activity and possibly being stuck. Taken together, I will be analysing the situated solver's progress in a problem as they shift their attention amongst locations and people in order to access resources both internally and externally to make progress on a problem. In any given problem, the situated solver will change from person to person within a collaborative, but the solver may also change level from an individual to a group or to a class. If a solution is being advanced by an individual, then this individual is working independently through a problem using internal or external resources. If the individual is in cooperation with a partner (or two partners), then the situated solver has moved up a level to a group. If the group is working in cooperation with another group, then the situated solver has moved up another level to the class.

It should also be noted that Davis & Simmt's (2003) conditions for complexity are all met in this classroom. There is internal diversity, because the students act as individuals with their own variety of experiences and personal resources. There is redundancy due to the students having commonality. They are similar in age, they have similar school back grounds, and they come from similar communities of learners. There is no direct teacher control over how students navigate their tasks. Students are provided freedom to explore each task and generate a strategy of their choosing – this is decentralized control. There is organized randomness, because students are placed into random groups so that they have different experiences with different students in every class. And lastly there are literally neighbour interactions due to the students standing side-by-side at the whiteboards, but a figurative neighbour interaction also occurs in the sense that ideas on boards are also bumping into one another.

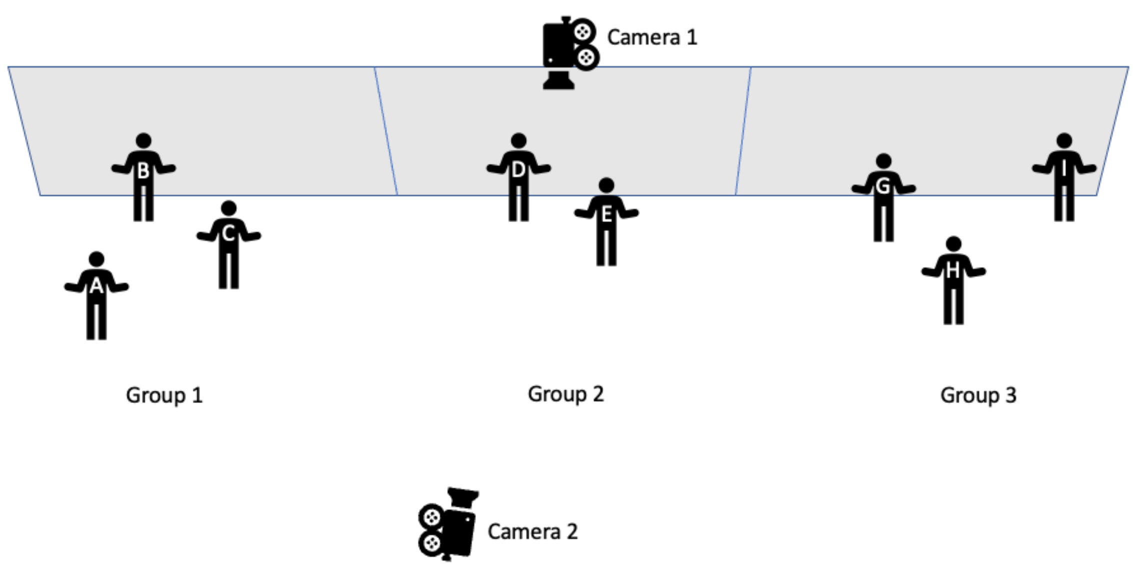

FIGURE 26 DIAGRAM OF THE THREE GROUPS AND CAMERA POSITIONS

I have included this diagram (fig. 26) to help the reader understand how these groups are generally positioned while working on the task in each lesson and to see how the individual student letters correspond to their positions within a group and at the whiteboard. In this diagram, camera 1 is the camera positioned pointing down at the whiteboard; it is this camera that is primarily responsible for capturing the dialogue within the groups, but it also plays a secondary role in capturing more detail in what group 2 writes on the board and where the students in all groups are gazing. Camera 2 is the camera positioned in the centre of the room; it is this camera that is responsible for capturing the global interactions of all three groups (gazes, gestures, and board work).

Lesson 1

In this first lesson, we see group 1 make most of their progress in the task using their collective resources. The situated solver changes within the group, but most of the progress is made because of the resource exchange between two group members. It is interesting to see what happens in this group after they solve the task.

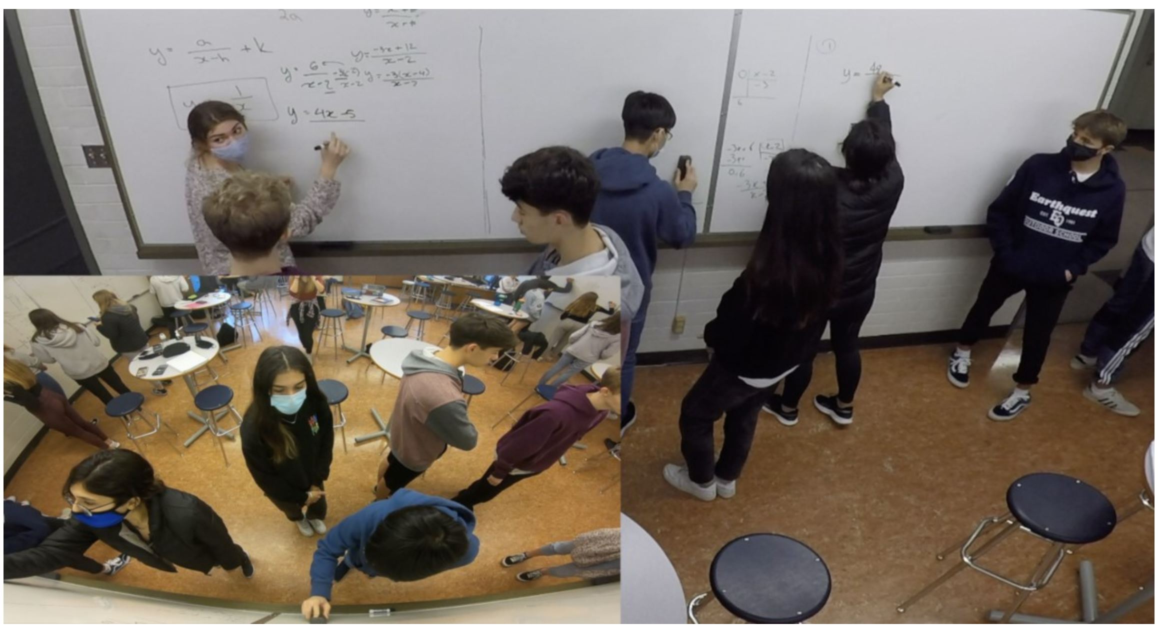

FIGURE 27 ACTUAL VIDEO IMAGE OF GROUPS FROM THE TWO CAMERAS

The second group (containing only two students) maintains only one situated solver for the duration of the lesson. This person works on the task mostly on her own, but she presents the result incorrectly. Over the course of the video segment, she is observed looking outside of her group and eventually noticing the error in her work from these outside observations.

The third group also has a single situated solver for most of the task, but this individual is stuck for much of the beginning of the video. She is seen looking outside for ideas for a large amount of time before inspiration finally hits and progress is made. Each of these groups express different trajectories of collaborative problem solving within choice-affluent spaces, and so each group will be analysed and discussed separately. In each analysis, the group’s actions will be summarized, supported by the gaze-dialogue transcript and analysed to bring to the forefront the details of the problem-solving process.

Before describing each group’s progress in more detail, I have restated the task below (fig. 25) for reference.

Write the following function in the form $y = \frac{a}{(x-k)} + b$ and then sketch the graph.

$$y = \frac{(4x-5)}{(x-2)}$$

FIGURE 28 TASK FOR LESSON 1

Group 1:

In this group’s video, Group 1 appears to rely heavily on their shared resources to accomplish the task at hand. Although the situated solver varies among the group, the majority of the advancement is due to the exchange of resources between two members. It is intriguing to observe the group's behaviour after successfully completing the task.

This group started out with student B having a strong idea for the solution. “I know how to do this… I know” was said multiple times by student B, and he tried a few attempts with the pen. The attempts were actually going in the right direction, but student A was not satisfied with his reasoning (fig. 29, 3:10). She grabbed the pen during a pause in the action and began a different approach. She first erased student B's work, and then started a solution that was almost a reverse engineering strategy. Eventually solving the problem (fig. 30, 5:30). Student B went to his table to grab his phone. He checked the solution in Desmos and the group was all satisfied that they had the correct solution. Student A then went and recombined the two fractional terms to verify the solution algebraically. Satisfied with this, she then labels this verification as a "check" on the whiteboard (fig. 31, 7:30), and the group appeared to be collectively satisfied in their work. Student A, looking around the room, noticed another student from across the room who was looking at their correct solution. A conversation began, and student A went over to the outsider's group to assist with the other group's work. Student B also left his board to assist another group in the room. Student C was left still puzzling over his groups work. After the teacher came and had a short discussion with this group, again verifying their work, Student A begins a conversation with Student C to explain the reasoning behind the solution. After this, student C is now nodding frequently.

Time	outside	A	B	C	notes
1:40 – 1:50		I'm leaving this here so I can cheat a little. **	Oh, I know how to do this.		
1:50 – 2:00		I think you factor out the 4… times the bottom **	Factor out the 4		
2:00 – 2:10			Oh! No no no, I remember how to do it… You have to add numbers **		
2:10 – 2:20			We want an (x-2) up here.. That's what we want. **		
2:20 – 2:30		Can I have the pen please? Sorry. **	Oh I remember.		
2:30 – 2:40					Student A writes: 4(x-2)n = 4x-5
2:40 – 2:50		I think it's something like that. **			Then writes: 4x-8
2:50 – 3:00		Wait. (taps finger on board). **			
3:00 – 3:10		Sighs. Mumbles some calculations. **			
3:10 – 3:20			This is what I remember. **		Student B writes: Y = 4x-5
3:20 – 3:30			And then you can… plus 3 minus 3. **		Student B writes: Y = 4x-5 + 3 - 3
3:30 – 3:40			So it doesn't change anything. And then it's x-2. **		
3:40 – 3:50			This might not be exactly right, but this is what I remember. **		Student B writes: Y = 4x-2 – 3 / x-2
3:50 – 4:00			This is still separated… those cancel… There's something I'm doing wrong. **		

Figure 29 Gaze-Dialogue Transcript Lesson 1, Group 1 (1 of 3)

Time	outside	A	B	C	notes
4:00 – 4:10		Ok, well, we don't have to do that stuff to get to this. We can just straight up split it. **	That's true.		Student A erases their work, and grabs the pen.
4:10 – 4:20		We know that up here, it doesn't split into a 6 and a 6. **			Student A writes: Y = 4x – 2 – 5 And gestures to work from a previous problem.
4:20 – 4:30		We saw that this value is this value. So what I'm thinking… **			Student A gestures to the coefficient and the horizontal asymptote
4:30 – 4:40		Is that we have this. We have an 'a' a plus '4', and we have an x – 2. **			Student A writes: Y = a/(x-2) + 4
4:40 – 4:50		What number do we have to multiply by 4 to get -5? **			
4:50 – 5:00					Big pause in the action as they contemplate this question.
5:00 – 5:10		That's what we do! **			Student A glances at the solution to the previous task that is still on her whiteboard and INSPIRATION!!!!
5:10 – 5:20		This makes a 4x – 8, so this is going to be a positive 3. **			Student A writes: (x-2)/(x-2) beside the 4, and replaces the a with 3.
5:20 – 5:30		Because we know to get from -8 to -5, we need to add 3. **			
5:30 – 5:40		This is going to be 3 over x-2 + 4. **			Student A then begins to write some missing steps for clarification. Then puts pen down.
5:40 – 5:50					All three students are staring at their solution.
5:50 – 6:00					Still staring.

FIGURE 30 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 1 (2 OF 3)

Time	outside	A	B	C	notes
6:00 – 6:10					Student A grabs the pen and adds some clarifying steps. Student C is looking at incorrect solution on board to the right.
6:10 – 6:20					Student C is shifting glances from his board to the right board.
6:20 – 6:30				These two are different.	Student C points to the constant term in each expression.
6:30 – 6:40					Students A and B attend to the other group's work. Student B checks the graphs in Desmos.
6:40 – 6:50			They are exactly the same.		Student B holds his phone up to show Student A and C
6:50 – 7:00		**			Student A begins to verify their solution.
7:00 – 7:10		**			Student A writes: Y= 3/(x-2) + 4(x-2)/(x-2)
7:10 – 7:20		**			Student A writes: Y= 3/(x-2) + (4x – 8)/(x-2)
7:20 – 7:30		**			Student A writes: Y = (4x – 5)/(x-2)
7:30 – 7:40					Student A labels the work "Check" and puts the pen down
7:40 – 7:50					Student A looks across the room at another student.
7:50 – 8:00					Student A goes to this student and engages in conversation gesturing at her board.
8:00 – 8:10					Other student comes into this group and Student A explains their solution/
8:10 – 8:20	OK	Sounds more complicated than it is			Student A finishes explanation. Other student nods in agreement.

FIGURE 31 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 1 (3 OF 3)

Analysis:

This perspective shows a group that is working well within their group – activity is high and there is lots of dialogue between student A and B (4:00 – 5:40). The group's internal resources appear to be sufficient, as there is no attention to anything outside of the group. The situated solver shifts from student A to B, but overall, it is student A. She relies on the affirmation of her partners. They use technology to confirm their solution, and then algebra to doubly confirm. When finished, students A and B both go into other groups to share their understandings (7:50 – 8:20). Eventually student A comes back and supports student C in his understanding of the solution.

In the data, I noticed many different pauses in the action of the problem-solving process. The pauses may signify a moment of being stuck, and it is interesting how the student/group transition from being stuck to being un-stuck.

The first pause occurs in the 2:10 – 2:20 segment when student B appears to be struggling to remember a pathway for this solution. Just as he appears to have an insight, student A reaches for the pen in a moment of impatience, makes a few markings of her own, pauses for a moment, and then student B grabs the pen again. In a short 90 seconds, the pen has switched hands three times. This shows that the situated solver is not an individual, but rather it is the group. Resources are being accessed rapidly between student A and student B, but it is not one individual who is making progress. Attentions are shifting between the two students, and progress is being made up until the 4:00 mark. It is this rapid interplay between the two members of a group that demonstrate the two students becoming the cognizing agent – the situated solver is at the group level rather than the individual level.

Just before the 4:00 moment, student B is pausing, and states "There's something I am doing wrong." At this moment, Student A reclaims the pen, and goes on a different trajectory in the solution. Seemingly starting the solution again, she writes most of the solution, pausing when she gets stuck determining the numerator of the final answer. Student B and C are now observing student A from the outside. This is another shift in cognizing agent – the level of the situated solver shifts from the group to the individual. There is a pause, as she looks intently at the work from the previous task that is still on the board. She then has a shift of attention likely due to resonance, saying "That's what we do!" And then she is able to complete the solution on her own. In this 90 second period, the situated solver is only student A, because for this whole interval, she is making progress on her own and the attention shifts are made by her only. She is attending to her own resources and reasoning and shifts to her previous work to gain inspiration (resonance). In these first three minutes, one observes the situated solver transitioning from the group to the individual, due to the solitary actions of student A making progress.

It is interesting to note what these students do when their solution is complete and the intensity of solving dies down. All three students are observed to glance around the room at other boards. Student C is attending to a difference between his own board and group 2's board to the right (this board's solution is incorrect). Student B checks their solution using technology and student A enters into a dialogue with an outside group (a group to the left of the video capture) about their solution strategies. One could consider this a shift again from the situated solver at the student level to the situated solver at the whole class level, as the shifts are now happening on the global scale: Student C is attending to the group's work on his right, student B is attending to the external resource of digital technology, and student A is attending to other groups in a supportive role. In this 9-minute sample, by attending to the pauses, one can see the situated solver transition from a group to an individual and then to a whole class. In complexity theory, emergence does not necessarily arise from an individual, but it can arise from the whole system. In this situation, the emergence is the solution to the problem, and it does begin at the individual level, but through the shifts and interactions, the emergence is demonstrated first by the group and then to the whole class.

This sample of video also demonstrates two different actions on resources. First, the group is accessing resources, individually and cooperatively, as they make progress and eventually complete the task. This is the action of pulling in resources. Next, the groups is observed working with and supporting other groups in the room, and this can be thought of as the pushing out of resources. Once the group has successfully solved the problem, they become an additional resource for the rest of the class of problem solvers.

Group 2:

During the lesson, the second group, which comprises of only two students, utilizes a single situated solver who primarily works independently. However, this solver incorrectly solves the task. Throughout the video segment, the solver is seen gazing beyond her group and eventually discovers her mistake through external observations.

Student D writes out the question and then pauses, putting the lid back on the pen. Student E begins a unique solution performing long division on the expression. She uses an algorithm that is not part of the class's experience, having brought this into the class from her home culture (the student is from Iran). She completes the division algorithm and then writes the solution incorrectly on the board. It was incorrect, because she mixed up the quotient and the remainder from the division algorithm, demonstrating more of a superficial understanding of the algorithm. This incorrect solution remains on the board for quite some time. The pen switches back to student D, as he begins to sketch the graph represented by the incorrect solution. At this point, student E steps back and surveys the neighbouring boards. She appears to be in deep contemplation, as her gaze is regularly shifting across all three boards (groups 1, 2 and 3). The teacher steps in for a brief conversation and incorrectly verifies their solution (4:40). A moment later, another student from outside of these three groups comes into their group to question their thinking (5:40). Student E describes their solution method to this outsider and how it resulted in their final answer. The outsider nods at this description and goes back to his group with what might be an incorrect understanding. Student E then decides to step back in and re-write the long division using the class's more familiar representation. This takes some time, but eventually finishes with the same result as done earlier – the quotient and the remainder are the same. However, their final solution remains unchanged and incorrect. She passes the pen back to student D, and steps back again, surveying the neighbouring boards (9:30). This survey by student E is quite lengthy. Seeming dissatisfied, student E approaches the board and tries to change their solution to align with the neighbouring boards. She is not satisfied with this, so she opens up her phone to review a video of polynomial long division. She scrolls along the video to a spot where she suddenly has an illuminating moment, and sees the original error made. "That's my bad," she says as she quickly makes the adjustment and fixes their solution (11:20). Student D then fixes his sketch and verifies their solution with a graphing calculator.

This group progressed rapidly through a unique solution strategy. But they made a tiny misstep by mixing up the quotient with the remainder. This misstep was only noticed after student E spent some time surveying other's work around the room. This surveying indicated that there was a problem and caused her to look into her phone for a video tutorial of long division. After this, the correction was made.

Time	outside	D	E	notes
1:40 – 1:50		Can I erase this? **		Student D proceeds to wipe the board clean.
1:50 – 2:00		**		Student D Writes out the question.
2:00 – 2:10		**		Student D Writes: Y = 4x-5/x-2
2:10 – 2:20		Um… sure. **	I'm going to use it like that if you're ok with that?	Student D puts lid on the pen and steps back a bit. Student E gestures to a long division algorithm from the previous task.
2:20 – 2:30			I write it out in a different way… sorry about that. **	Student E begins to write a division algorithm.
2:30 – 2:40			**	Student E writes:
2:40 – 2:50			**	Student E finishes the algorithm and then moves back to the question.
2:50 – 3:00			**	Student E writes the solution incorrectly from her long division.
3:00 – 3:10		Oh no no no. How do you do this?	No? **	Student E begins to erase her long division work.
3:10 – 3:20			You have to have 4x right? 4 times x is 4x. 4 times -2 is -8. **	Student E describes her long division steps.
3:20 – 3:30			So like basically this will be your.. Here.. This will be here… and this will be here. **	Student E circles the remainder, quotient, and dividend and points to the corresponding parts in the solution.
3:30 – 3:40		Ok. I think I get it.	This is easier for me. I can just re-write it. **	Student D begins to erase the circles
3:40 – 3:50			**	Student E erases the answer and re-writes it. Student D steps back.
3:50 – 4:00			**	They both begin looking at other boards. Student D goes to get his graphing calculator.

FIGURE 32 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 2 (1 OF 4)

Time	outside	D	E	notes
4:00 – 4:10			**	Student E writes equations for asymptotes. And passes pen to student D.
4:10 – 4:20		**		Student D begins to sketch the graph. Draws axis. Student E steps back and makes two glances at boards on either side.
4:20 – 4:30		**		Student D continues with sketch.
4:30 – 4:40		**		Student D continues with sketch.
4:40 – 4:50	Teacher: So now it's paying off.	**		Student D continues with sketch. Teacher approaches student E.
4:50 – 5:00		**		Conversation between teacher and student E is indistinguishable.
5:00 – 5:10	Teacher: and you're doing division differently than I've ever done it before. That's ok… so x-2 goes…	**	That's because I'm middle eastern.	Student D continues with the sketch.
5:10 – 5:20	…into this 4 time. 4 times is 4x-8, the remainder is 3. And there you go. Cool. We are going to go over this one.	**	Ok.	Teacher affirms the long division steps and does not notice the error in the final answer.
5:20 – 5:30		**		Student D finishes the sketch and puts the pen down. Student E looks at the board to her left.
5:30 – 5:40		Ok… now we just need…		Student D begins to check in his graphing calculator.
5:40 – 5:50	Another student approaches: I don't know what we're doing.		I don't think anyone does… It's basically long division. You're basically dividing this by that.	Student E begins to talk with outside student.
5:50 – 6:00	Other student: so you pull out x-2 as a factor?		Well.. You divide the numerator by the denominator.	Student D notices that the curve is incorrect and makes a change.

FIGURE 33 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 2 (2 OF 4)

Time	outside	D	E	notes
6:00 – 6:10		**	And then your remainder will be here and this will be here and this will be here.	Student E gestures to the long division and the solution.
6:10 – 6:20	Outside student laughs… oh… oh	**		Student D fixes the sketch.
6:20 – 6:30	Outside student: And where did you get the 3? Oh.	**	Oh… that's the remainder.	Student D begins to label some ordered pairs.
6:30 – 6:40	Student C observes the board and begins asking his group about their solution. Outside student goes back to his group			An outside student is looking at this board.
6:40 – 6:50			How do you do long division.	
6:50 – 9:30	There is a small amount of time here where student E teaches D how to perform long division using the classroom representation.			
9:30 – 9:40				Student E erases some rough work. And steps back beginning to look around
9:40 – 9:50				Student D is checking his graphing calculator.
9:50 – 10:00				Student D looks at his graph, picks up the pen, then puts it back down again.
10:00 – 10:10				Student D continues to punch in his calculator and look at his board.
10:10 – 11:20				Student E continues to look at other boards. Focusing more on the boards on either side.
11:20 – 11:30			I think it is like the other way… I think it is 4 plus… can I? **	Student E steps back to the board.

FIGURE 34 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 2 (3 OF 4)

Time	outside	D	E	notes
11:30 – 11:40			**	Student E does the long division one more time. Student D goes to his table to get his notebook.
11:40 – 11:50			**	After completing the algorithm, student E puts the same incorrect answer up again.
11:50 – 12:00			**	Student E quickly erases her algorithm.
12:00 – 13:00			**	Student E opens up her phone and scrolls through her previous notes. She eventually goes to a video and scrolls to a certain location on the video.
13:00 – 13:10		←	That was my bad. **	Student E erases the incorrect numbers and replaces them with the correct ones.
13:10 – 15:20	← - - -	** - - -	- - -	Student D fixes their graph finally to match the correct answer. Student E continues to look at other boards

Figure 35 GAZE-DIALOGUE TRANSCRIPT Lesson 1, GROUP 2 (4 OF 4)

Analysis:

In group 2, Student E is the situated solver for most of the activity. There are times when this shifts to the group, and to the whole class, but these are not often and not for long durations. At the 3:00 mark (fig. 33), her solution is complete and incorrect, and this is when her partner steps in asking for clarification, “How do you do this?” This marks a shift in the situated solver from the individual to the group. After their discussion, student D appears satisfied, and there is a pause as they both begin looking at other boards. There is a lull in the action as student D is completing a sketch for their solution and the teacher steps in for a short conversation. The situated solver shifts to the classroom when a student from another group steps in and inquires about their solution. Student E and the outside student engage in a conversation where she describes the reasoning behind her strategy. Throughout this time (fig. 33, 3:50 – 5:30), student E is seen staring at other boards on many occasions. She appears to be noticing that her solution is different from those around. This type of shift would be considered examplehood, as it is outside experiences that spurs a noticing. This pause in her action is lengthy; in fact, she continues to look at other boards through to 11:20. Within this long pause, student D continues with his work sketching, and student E even re-writes her long division to a more familiar form. Unsettled with the different answer on her board, student E eventually shifts her attention to an outside resource (examplehood again) which appears to be a tutorial video on her smart phone, and this video appears to provide the illumination necessary to correct and complete her solution.

Student E is primarily the situated solver throughout this story. When there are pauses in her action, her attention shifts to other boards, and this is when she realizes that there must be something wrong. This struggle continues for over 10 minutes until she finally shifts to the outside resource of technology. Within this lengthy pause, student E enters into discussions with her own partner, the teacher and another classmate. During these times the situated solver shifts from the individual to the group and then to the whole class. These shifts are subtle, and it is hard to say whether they are helpful overall, but they do occur. It is possible that these shifts are necessary to keep student E engaged in the solution and to eventually correct her error.

Group 3:

In this perspective, we observe a group not making much progress on their own and relying on other's work for inspiration. The third group also has a single situated solver for most of the task, but this individual is stuck for much of the beginning of the video. She is seen looking outside for ideas for a large amount of time before inspiration finally hits and progress is made. After the completion, group understanding is the goal, and then student I goes out and has conversations with three different groups around the room. The first two conversations are resource transfers, and the third was an affirmation of ideas.

This group has a very slow start after student G writes the question on their board. Student I steps back from the board and watches the group 2's board quite intently. There is no work on their whiteboard for some time. Her attention shifts from group 2's and to her own board quite frequently. She is then observed to do finger writing on her own palm before approaching the board and beginning their solution. She stops part way and steps back again. There are a lot of pauses in this group's work. When she steps away, mid-solution, she gazes intently at the group to her right. She steps back in, makes some minor changes, and pauses again. She now engages in a dialogue with student H regarding some possibilities. Student I looks again at the group's work to her right (4:20). She goes back to her board and makes some more incremental progress. She then completes the solution and student H steps in again to verify it. The two students engage in a conversation that is hesitant but appears to give them confidence in their solution (Student H is nodding). At this point, student G steps in, and the whole group is in conversation with Student I describing the solution details. Student I leaves the group to grab her phone and check the solution with Desmos. At this time, Student G re-writes the solution, apparently checking her own understanding of it. Desmos has provided verification, and Student I steps back and observes other groups (mainly to her left). Student I takes a photo of their work, adding it to her digital notes. Student H also takes a photo. Student I now steps away and engages in a conversation about their solution with a student from another group, nodding and gesturing to the board. The other student is nodding as ideas are being exchanged. The other student says "this makes sense, it just takes a while for my brain…" (10:30). There is now a pause in the action, as student G begins to sketch a graph. Student I then engages in a conversation with a different student from another group, gesturing to the work on Student I's board (12:30). Again, reasoning and sense making is apparent by the gestures and nodding. Resulting in an "OOKKK… I get it." from the other student (12:55). Student I then moves over to the first board on her left and begins a conversation with student A. Both students are nodding and affirming each other's thoughts as they detail their own steps in their thinking. Student I now being out of her group leaves Student G and H to discussing their solution. Student G points to another board and nods while discussing her graph with student H.

Time	outside	G	H	I	notes
1:40 – 1:50		**			Student G cleans board.
1:50 – 2:00		**			Student G writes the question. Student H already starts glancing at other boards.
2:00 – 2:10		**			
2:10 – 2:20		**			Student I is looking at other boards.
2:20 – 2:30		**			Student H is drawing her thoughts with her finger on her palm
2:30 – 2:40		**			
2:40 – 2:50		**			
2:50 – 3:00			I have an idea. Maybe… question mark. **		Student H takes the pen and approaches the board.
3:00 – 3:10			**		Student H writes: Y=4x-3-2/x-2
3:10 – 3:20			**		Student H writes: =-3/x-2 and pauses
3:20 – 3:30			**		Student H still pauses
3:30 – 3:40			**		Student H steps away from the board.
3:40 – 3:50			**		Students H and I gaze intently at board to their right.
3:50 – 4:00			**		Student H erases part of the numerator in their work and pauses.
4:00 – 4:10			**	What about if you move this up…	Student I gestures to the solution.
4:10 – 4:20			**	And you get like x squared or something… I don't know. You get like a 4 x squared and then just an x	
4:20 – 4:30			**	But I guess I don't know what I'm doing.	Student H glances over to the board at the right again.
4:30 – 4:40			**		
4:40 – 4:50			**		
4:50 – 5:00			**		Student G looks to left board.
5:00 – 5:10			**		Student H writes: (x-2)/x-2
5:10 – 5:20			**		Student H looks at board to left.
5:20 – 5:30			**		
5:30 – 5:40			**		There is a lot of gesturing with the pen in this 30 second pause.
5:40 – 5:50			**		
5:50 – 6:00			**		Student H very slowly writes a 4 in the numerator

FIGURE 36 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 3 (1 OF 2)

Time	outside	G	H	I	notes
6:00 – 6:10			**		Student H writes: 4x-5 then places a three in the missing numerator on the previous line.
6:10 – 6:20			**		Student H writes: -8 + 3 in the numerator
6:20 – 6:30			**		Student H erases her last line: 4x-5
6:30 – 6:40			**		Student H quickly writes: 3/x-2 + 4
6:40 – 6:50			**	Unintelligible.	Student I asks a question gesturing to the top of the solution.
6:50 – 7:00			Unintelligible **		Student H responds to I's question. Student I is nodding.
7:00 – 7:10			I guess you know there has to be a 4 out there. **		Student H gestures to the 4 outside the brackets.
7:10 – 7:20			**		
7:20 – 7:30		Unintelligible **			Student G gestures to the work and asks a question.
7:30 – 7:40		**			Student G begins to write under the solution. Student I looks at board to the right
7:40 – 7:50		**			Student H goes to get her phone
7:50 – 8:00		**			Student G erases what she wrote
8:00 – 8:10		**			Student H is checking Desmos
8:10 – 8:20		**			Student H shakes her fist in success.
8:20 – 8:30		Unintelligible **	Unintelligible		Student H gestures to their solution then shows her Desmos graph to student G
8:30 – 8:40		Unintelligible **	Unintelligible		Student G then puts a check mark next to their solution. Student G continues an algebraic check.
8:40 – 8:50					Student H looks to the left board
8:50 – 9:00					
9:00 – 9:10					Student G erases her algebraic check. Student H takes a photo of their solution.
9:10 – 9:20					
9:20 – 9:30					
9:30 – 9:40					Student H checks Desmos again
9:40 – 9:50					Student H looks around at other boards
9:50 – 10:00					Student H engages in a conversation with an outside student explaining this solution.
					At the 12:30 mark, student H explains their solution to another different outside student
					In the rest of the video, many other students are pointing to this board and talking about this solution.

FIGURE 37 GAZE-DIALOGUE TRANSCRIPT LESSON 1, GROUP 3 (2 OF 2)

Analysis:

It is clear that student H is the situated solver, in control of her group, as her partners are passive and to the side for much of the solution process. She does seem to rely on outside resources once at the beginning of the solution (4:20 – 5:20), as there is no dialogue and one can see her gazing at the board to their right. This long gaze appears to be enough, as student H then proceeds to slowly complete the solution. After it is complete (6:40), student H then begins to explain and justify her work to her two partners. There does appear to be continued uncertainty with her work, as she is seen gazing at the board on her right 4 times after she has explained her work to her partners.

Summary analysis for lesson 1

Through the analysis of these three groups, there are some details that are beginning to emerge. Resources are not only being used to make progress in a problem, but they are also used to verify or look back on a solution. Resources are moving in two directions; they are not only taken in, but they are also being pushed out. And, the problem-solving process looks very similar at each of the three levels for the situated solver.

In all three groups, there is a verification stage in their solutions. In group 1, the group uses technology (fig. 31, 6:40) and a conversation with an outside student (fig. 31, 7:50) to verify their solution. In group 2, the verification is when the student first notices that her solution is not the same as the solutions around her. She is using other group's boards as resources for this verification. Ultimately, she uses technology to verify that her solution is indeed incorrect (fig. 35, 12:00), and then she uses this resource to amend her work. In group 3, inter-group conversation plays a role in the verification process (fig. 37, 6:40), and technology is also used (fig. 37, 8:00). Resources are not only necessary in the progression of a solution, but they are also necessary in the verification of a solution. It should also be noted here that all groups even participated in a verification. By engaging in solution verification, the groups are demonstrating a desire for understanding a solution. If their goal was only to get an answer, then there would be no need for verification.

Resources are not only taken in when solving problems, but this data also shows resources being pushed out. This pushing out of resources could be considered as an example for some of the conditions for complexity. Pushing out resources shows a decentralized control amongst the students and could also be considered as an example of a neighbour interaction. There appears to be a culture of responsibility in this Thinking Classroom where students are responsible for their own understanding, and they are also responsible for their colleague's progress and understanding. Group 1 engages in a conversation with an outsider at 7:50 (fig. 31), group 2 supports an outsider at 5:40 (fig. 33), and group 3 works with two different groups at the end of their solution (fig. 37, 9:50). If it only occurred in one group, then it might not be a pattern; but resources were pushed out by all three groups. It appears that individual understanding is not the only priority in a Thinking Classroom, there is also a culture for whole class understanding that motivates the actions of students.

Lastly, the problem-solving process looks very similar at each of the three situated solver levels. When the situated solver is at the individual level, progress is made while using internal resources and progress is paused when those internal resources are not sufficient. The solver then looks externally for resources. The external search includes but is not limited to looking at other boards around the room, using technology, and dialoguing with a partner. If these outside resources help, then the individual solution continues. If not, then the situated solver may move to the group level. At the group level, the situated solver now makes progress in much the same way. As a group, they make progress in the problem relying on their collective resources, and progress is paused when these group resources become insufficient. The group then looks externally for more resources. The problem-solving process looks very similar at all levels.

Lesson 2

The task for lesson 2 is re-stated in fig. 38 for reference.

Write as a single logarithmic expression and then state restrictions on *x.* $\log_7 x^2 + \log_7 x - \frac{5\log_7 x}{2}$

FIGURE 38 TASK FOR LESSON 2

In this lesson's group 1 perspective, we see a three-person group where the solution is completely solved by just two of the students. The third student is a quiet bystander for most of the time, only stepping in at the very end. In the second group's perspective, we see three weaker students struggling to make sense of the task. This group only makes progress after an outside injection of resources. And in the third group's perspective, we see an almost solo contribution by one group member who is relying heavily on contributions from around the room. Again, each group will be analysed separately, because they capture different aspects of collaborative problem solving and decision-making. For each group, I will summarize the group's perspective and use the gaze dialogue transcript to support the analysis and highlight the specifics of the problem-solving process.

FIGURE 39 IMAGE OF ALL THREE GROUPS WORKING ON LESSON 2.

Group 1:

The audio quality for this group was quite weak, so most of the dialogue is marked as unintelligible. In this perspective, students A and B appear to be the most active in the problem-solving process. Student C is out of the video frame for the first minute, and even after he steps into the frame, he does not appear to contribute to the solution. Student C plays more of an active role at the very end of the video.

In the first minute (fig. 40, 0:40 – 1:40), students A and B are in discussion, they are both seen gesturing to the board, the pen switches possession from A to B but no visible progress is made. At 1:40, the teacher makes an announcement, "table 2 has a great idea," and both of these students gaze over to table 2's board (table 2 is across the room, not to be confused with group 2). Student A is heard saying "ohhhh," and then is instructing student B on how to progress. Progress is much more rapid at this point, but they are still unsure; there uncertainty is seen through their many erasings and re-writings of the short three-line solution. After this outside influx of resources, the initial solution to the simplifying problem is finished by 3:30 (fig. 41). At this time, student C steps into the group, and student A is seen describing the solution to student C. This is visible through the many gestures by student A to different spots in the solution on the board and culminates in both students nodding. The pen switches into student A's possession as they begin to work on the second part of the problem, stating restrictions on the variable. Again, this part of the solving is carried mostly by students A and B, as student A eventually writes $x \neq 0$ on the board (fig. 41, 4:40). Student A and B leave the frame, leaving student C behind. Student C is observed to be staring intently at the solution. Student B comes back a short time later (fig. 41, 5:20) and makes a change to the restrictions, erasing and then writing $x \neq < 0$. Student C continues looking through the solution on his board until the 7:00 point. At this point, student C is observed being the most active that he has been for the entire solution process. Student C is seen explaining the solution to an outside student gesturing to different parts on the board. Student B is also at the board but does not appear to be part of the conversation.

Time	outside	A	B	C	notes
0:40 – 0:50		**			Student A writes out the question, and student B is helping.
0:50 – 1:00		**	Unintelligible		No writing. Pausing. Student C is not in the frame.
1:00 – 1:10		Unintelligible **	Unintelligible		Student A is gesturing to the question, but not writing.
1:10 – 1:20		**	Unintelligible		Student B begins to gesture
1:20 – 1:30			**		Student B takes the pen
1:30 – 1:40	Teacher makes a whole class announcement "table 2 has a great idea"		**		Student B begins to write and then is interrupted by the announcement
1:40 – 1:50		Ohhh	**		Student A erases the final denominator student B re-writes as seen from "table 2"
1:50 – 2:00		Unintelligible	**		Student A gestures to move the 5/2 into the exponent. Student C joins the group.
2:00 – 2:10		Unintelligible	**		Student B writes as a single log expression
2:10 – 2:20		Unintelligible	**		Student B combines the powers in the numerator
2:20 – 2:30			**		Student A leaves to get a calculator and returns.
2:30 – 2:40		log base 7 of 1 over 2.... Oh no.	**		Student B erases the argument and student A seems a little confused. Student A is typing in her calculator.
2:40 – 2:50		Unintelligible	**		Student B begins to write ½, then erases. Student A is gesturing and re-stating the problem. There is no writing. Student B writes an x.
2:50 – 3:00		Unintelligible	**		Student B erases the x. Then quickly writes x^3/x^5/2

FIGURE 40 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 1 (1 OF 2)

Time	outside	A	B	C	notes
3:00 – 3:10		3 minus 5 over 2 is 1 over 2	**		Student B has stopped writing and student A is gesturing to the work.
3:10 – 3:20		And then we can move this to here.	**		Student B writes log of x^1/2. Student A gestures to indicate moving the exponent to the front.
3:20 – 3:30			Unintelligible **		Student B writes the ½ in front but does not alter the rest.
3:30 – 3:40		Unintelligible	**		Student B erases the exponent. Student A is explaining the work to Student C.
3:40 – 3:50		Unintelligible **			Student A erases the final step and re-writes
3:50 – 4:00		Unintelligible **			Student A re-writes the final line as she is speaking with Student C
4:00 – 4:10		Unintelligible... because dividing by 2 is the same as timsing by 1/2 **			Student a nods and student C nods in agreement.
4:10 – 4:20		Unintelligible **			Student A draws a box around the answer.
4:20 – 4:30		Unintelligible **	Unintelligible		Student A writes x = Pauses and gestures to the board
4:30 – 4:40		Unintelligible **	Unintelligible		Student A writes $x \neq 0$
4:40 – 4:50					Student A and C take a photo of their work. Then Student A leaves the frame.
4:50 – 5:00					Student B also leaves the frame and speaks with Student H. Student C is gazing at the board work.
5:20 – 5:30					Student B re-enters the frame and makes a correction to the restriction without consulting with partners.
7:00 +					Another student enters group and Student B and C discuss their solution. Student C participates in the discussion.

FIGURE 41 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 1 (2 OF 2)

Analysis:

This perspective is interesting for a couple of reasons. Students A and B are the primary solvers for most of the work, and there is a definitive moment when they appear to move from being stuck to being unstuck. This moment occurs when they are encouraged to look at the work on an outside board – this would be considered resonance because it is something seen that causes a sudden insight. It is also interesting to observe student C. Student C appears to contribute almost nothing to the solution, not even participating in any of the solving dialogue; however, after a considerable amount of time (more than 7 minutes), student C is observed to describe the solution to an outside student (fig. 41).

The situated solver for much of this solution is at the group level, however, this changes to the individual frequently. The vast majority of the dialogue arrows are coming from student A, but student A does not have the pen. For most of the solution, it is student B who has the pen and is doing the writing. Students A and B are generating a solution in concert – the pair have become the cognizing agent. Due to the dialogue arrows between student A and B and student B doing most of the writing, the situated solver is changing levels from the individual to the pair many times throughout this section (fig. 40 and 41, 0:40 – 3:30).

Student A and B appear to be stuck for the first minute due to the observation that no writing is taking place. They become unstuck after shifting their attention to an outside board at 1:40 (fig. 40). This indicates that possibly the resources of the pair were not sufficient, and the situated solver became the whole class just for a moment as resources were acquired from another group. This resource collection appears to be sufficient for students A and B to continue with the rest of solution on their own.

It is interesting to observe student C. Student C appears to not understand much of what is happening, as he is positioned out of the video frame for much of the solution. And when he finally appears in the video, he does not contribute anything to the work. On first view, I assumed he was not at all understanding what was happening, but the data brings something else to light. There is a short period in the middle of the solution where student A is seen speaking directly to student C and gesturing to the work on the board (fig. 41, 3:30 – 4:00). Perhaps more importantly, when student C is in the video frame (fig. 41 and 42, 1:50 – 7:00+), he is always staring intently at his partner's work. If he was not getting something out of the solution, he would likely move to off-task behaviour, such as chatting with a friend or looking at his phone. But he remained on-task for the entire solution; he just did not participate with his partners. At the very end of the video clip, we see that he does appear to understand the solution, as he is observed speaking with an outside student and gesturing to different parts of the board.

Group 2:

In this group's perspective on the same lesson, the whole group makes little progress on the task even after the teacher announces a hint at the 1:30 mark. Student E seems to be the only member of the group who even hears the announcement, as you can see him glancing at table 2's board five times between 1:30 and 3:00 (fig. 42). It is finally on this fifth glance that something clicks, and student E announces, "they got a good idea over there," and progress begins to be made. Student E makes one more glace at table 2's board at 3:15, but this final glance seems to only affirm the work that they are doing, and they complete the initial solution by 4:00 (fig. 43). The solution happens to be incorrect, because student D incorrectly applied an exponent property in the final step, using his calculator. Instead of subtracting the exponents for division with common bases, he divides the exponents. This error goes unnoticed for the entire time. After writing the incorrect solution at 4:10, they devote their attention to determining the restrictions. Students D and F have a lengthy conversation about a possible restriction, but this does not result in any progress. At 5:20 (fig. 43), the pen is put down and there is a pause in the action. Students D and E begin surveying other boards and student F appears to have given up, as she is seen attending to her phone. Student D notices something on the board to their right, says "look what they did there," and then starts working on his calculator. He uses his calculator to confirm that the input of a logarithm cannot be zero, and then uses the calculator again to confirm that the input cannot be negative. He then picks up the pen (fig. 44, 5:50) and finishes the solution on the whiteboard.

Time	outside	D	E	F	notes
0:40 – 0:50			**		Student F is reading out the question and E is writing it.
0:50 – 1:00		ok um…	oh, ok, ok, the first thing is we multiply these two **		
1:00 – 1:10		That would be a log 7 x 3.	**		
1:10 – 1:20			**		Student E writes a new line on the board
1:20 – 1:30			Right **	Don't we have to do this first right	Student F is gesturing to the fractional part of the last expression
1:30 – 1:40	Teacher makes a whole class announcement "table 2 has a great idea"	hmm? No, we can't do that because it is not an equation.	We can get rid of this by multiplying each side by 2. **		Student E gestures to the fraction and the other term with a cross motion.
1:40 – 1:50		We don't have anything to compare it to, so we can't do that. Why don't we start with this minus sign?	ok. **		Student E looks over a table 2 two times. Student F looks at the projected question.
1:50 – 2:00		What's the rule for that again? Would it be that over that?	**	hm hmm.	Student D is gesturing to each of the terms.
2:00 – 2:10		5 log.	**		Student E looks at table 2 again and student D looks at the projector screen.
2:10 – 2:20		That would mean x to the power of 5 right?	That's right actually. **		Student E looks again at table 2. Then begins to write as directed by student D. Student F is glancing at her phone.
2:20 – 2:30			**		Student E writes the third line of the solution
2:30 – 2:40			**		Student E finishes with logx^5/2
2:40 – 2:50			**		Students D and F are playing with their calculators and student E looks at table 2 again.
2:50 – 3:00			Ohhhhh. Ok ok ok. They got a good idea over there. **		Student E looks at table 2 again and then has an idea. Students D and F are still using their calculators.

FIGURE 42 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 2 (1 OF 3)

Time	outside	D	E	F	notes
3:00 – 3:10			you just turn it into minus 5 over 2 log 7 x. **		Student E is gesturing to the first line of their work
3:10 – 3:20		yeah, that will work because the 5 is just 5 times.	Get rid of this and turn it into 5 over 2. That's what they did over there and her said it was a good idea. **		Student E re-writes the end of the second line of their work.
3:20 – 3:30		log 7... that's a power now...	Makes it into... **		Student E is re-writing the third line. Student D is describing what to write.
3:30 – 3:40			**		Student E is writing the next line.
3:40 – 3:50		over	**		Student E finishes the 4th line of the solution. Student D begins to use calculator again.
3:50 – 4:00		ahh... one point two four	**		Student D hold up his calculator for others to see. Student E writes another line of the solution.
4:00 – 4:10		6 over 5 or 1.24	I'm just going to put 1.2 **		Student E finishes and puts pen down.
4:10 – 4:20		Well, that's good for that one. We don't know what x is so we can't do that.			
4:20 – 4:30		**		We have to state restrictions on the... So, anything to the power of one is itself. So therefore, x has to be...	Student D picks up the pen and student F gestures to the final answer.
4:30 – 4:40		**		... greater than itself.	
4:40 – 4:50		no but what if x was 0.5, then it would be less than itself. **			Student E has stepped back to listen to the conversation.
4:50 – 5:00		I think you're thinking of x as the exponent **		hm? No, because 1.2 is the exponent.	
5:00 – 5:10		Yeah, but it's the power of 1.2. **		Everything to the power of 1 is itself.	
5:10 – 5:20		If x equals 0.5, then the number would go down. **		yeah... ok.	The pen is put down and all three step back from the board

FIGURE 43 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 2 (2 OF 3)

Time	outside	D	E	F	notes
5:20 – 5:30					Students D and E glance at other other boards. Student F is on her phone.
5:30 – 5:40		Look what they did there.			Student D gestures to the board to their right. Student D starts punching into his calculator
5:40 – 5:50		So it turns out that you cannot have a log of zero. So I guess that would be why…			Student D gestures to the right board again. Student D starts punching more into calculator
5:50 – 6:00		Hmmm! Can't log a negative either. **			Student D grabs the pen.
6:00 – 6:10		**			Student D writes x>=0 next to their solution.
6:10 – 6:20					
6:20 – 6:30					
6:30 – 6:40					
6:40 – 6:50					
6:50 – 7:00					
7:00 – 7:10					
7:10 – 7:20					

FIGURE 44 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 2 (3 OF 3)

Analysis:

In this perspective, the group's resources are not sufficient to make progress in either stage of the solution. The group is fully engaged in solving the problem, but they do not make any progress until one of the members fully processes a hint that is on another board (resonance). This does take some time, as the student looked at the board containing the hint five times over the course of 90 seconds (fig. 42, 1:30 – 3:00). The group also becomes stuck in the second stage of the problem, stating restrictions on the variable. This time it is student D who gleans some ideas from a different group's board. The idea is not just taken, but student D feels the need to verify it first with his calculator. This indicates a desire for understanding the solution over just getting an answer on the board.

Based on the observations of the continuous dialogue between student D and E for most of the initial solution, this is a situation where the situated solver is at the group level. Resources are not sufficient in this group as can be seen by the eventual pause in writing and in dialogue between 2:20 and 3:00 (fig. 42). In this period, student E is seen gazing at table 2's board five different times, due to the hint given by the teacher. This demonstrates that simply viewing a board is not necessarily sufficient for making progress in the problem. It seems that student E needed some time to understand the implications of the hint before incorporating it into their solution. Student D exhibits similar behaviour in the second stage of the problem. He is observed gazing at the board to their right apparently noticing the other group's solution to the restriction part of the problem. But he also does not immediately transfer this solution idea onto his board. Instead, he takes some time to verify the idea with his calculator before incorporating it into their solution. This shows that in this environment, students are not simply motivated to achieve answers in their problems; rather, they are motivated to understand the problem, and the resources are gathered to gain understanding towards their own personal solutions. For this second stage of the solution, it is apparent that the situated solver has shifted to the individual level, because most of the progress was made solely by student D; and although there was some dialogue, this appears to be more a conversation with himself.

Group 3:

In this third perspective on the second lesson, the situated solver is almost entirely student H. It appears that she is able to solve the majority of this problem using her own internal resources. She is holding the pen and writing the solution from the beginning, but she is also speaking to her partners, as she writes each step. Student G does seem to contribute a little to the first stage of the solution (fig. 45, 1:00 – 1:10), but again, this is more just affirming the work that student H is already doing. Student H notices that the third expression can be written as 5 over 2 just moments before the teacher announces the hint at 1:30. They all glance at table 2's board, but this is just affirming student H's idea. Student H spends extra time explaining this step to student I (fig. 45, 1:50 – 2:10), to student G a little later (fig. 46, 3:30 – 3:50), and then again to student I (3:50 – 4:20). There is a significant amount of time (3:00 – 3:30) where they are all three staring at their work. It is possibly this time that generates student G and I's further questions on the 5 over 2 step. After the initial solution is complete, Student G takes the pen and begins to work on the restrictions. Even though student G is doing the writing, student H is contributing the ideas to be written. At the very end of the completed solution, student H asks both of her partners if they do understand the solution, and they both acknowledge that they do.

Time	outside	G	H	I	notes
0:40 – 0:50			**		Student H is writing out the question
0:50 – 1:00			**		Student H finishes writing out the question
1:00 – 1:10		We can move that to the front	**		All three are staring at the question. Student G gestures to the exponent in the first expression
1:10 – 1:20			that can be simplified and we can do that at the same tie as bringing these together **		Student H is gesturing to the board while talking.
1:20 – 1:30			because that will just be x squared times x which is x cubed **		Student H writes log 7 of x cubed.
1:30 – 1:40	Teacher makes a whole class announcement "table 2 has a great idea"		minus… that's 5 over 2 log x. **		Student H says that's 5 over 2 before the teacher shares the hint.
1:40 – 1:50			Oh yeah, so you can do log 7 of x to the 5 halves **		Student H finishes writing the second line.
1:50 – 2:00			it's because that 5 over 2 can be written as 5 over 2. **		Student H writes an intermediary step to explain her thinking.
2:00 – 2:10			**	oh ok.	Student H erases the intermediary step.
2:10 – 2:20			Now we can divide them **		
2:20 – 2:30			Because dividing, what do you do? Subtract the exponents? So 3 minus 5 halves. **		Student H writes another line and reaches for her phone to calculate the subtraction.
2:30 – 2:40			**		Student H is using her calculator
2:40 – 2:50			**		Student H writes out another line of the solution.
2:50 – 3:00		Can we move this to the front?	Oh yeah. **		Student G gestures to the radical in the logarithm. And student H writes the final line.

FIGURE 45 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 3 (1 OF 2)

Time	outside	G	H	I	notes
3:00 – 3:10			**		All three students are staring at their solution.
3:10 – 3:20			**		All three students are staring at their solution.
3:20 – 3:30			**		All three students are staring at their solution. Student H glances at the projector.
3:30 – 3:40		Can you tell me how you got this?	So you can re-write this like that. **		Student G gestures to the 5/2 exponent in the second line of work. Student H erases the long division line in the first line with a short 5/2.
3:40 – 3:50			When it's dividing the whole thing by 2, that's still dividing the whole thing by 2. **		
3:50 – 4:00		ok	**	I guess I am wondering why its not also under the x.	Student I gestures to the 2 in the denominator in the first line.
4:00 – 4:10			Well you gotta divide by 2 and that happens here. **	I guess	Student I glances at the projector.
4:10 – 4:20			dividing by 2 is like timsing by 1 half. **	oh ok.	
4:20 – 4:30			It looks the same as what table 2 has too. **		All three glance at table 2's board. Student H holds her pen up for someone else to take it.
4:30 – 4:40		So what about the restrictions? **			Student G takes the pen
4:40 – 4:50		So like this? **	you cannot square root a negative		Student G writes x>=0.
4:50 – 5:00		**	It can't be less than zero, because you cannot square root a negative number.		All three stare at their solution. Student H speaks with Student B.
5:00 – 5:10			So does this make sense?		Student G puts pen down
5:10 +				Yeah.. It makes sense now, not sure I could do it on my own though.	All three students spend the next two minutes talking about the solution.

FIGURE 46 GAZE-DIALOGUE TRANSCRIPT LESSON 2, GROUP 3 (2 OF 2)

Analysis:

In this group, there appears to be no outside resources necessary for making progress in the problem. Student H seems to complete the entire solution relying only on her own personal resources. Due to the collaborative nature of the room, she is not just completing the solution on her own; rather, she is observed at every step checking with her partners to make sure that they are in agreement and that they understand the steps. She is even observed at the end (fig. 46, 4:50) speaking with an outside student (student B) to explain how to determine the restrictions.

In this group, the situated solver appears to be exclusively student H. She is writing with the pen and describing her steps as she makes steady progress through the task. There are three moments where student H is observed looking outside her group at other boards, but each of these appear to be to affirm her work and not to generate ideas. There does appear to be a genuine desire for understanding by her two partners. In the critical step of changing the denominator of 2 to a fraction of 5 over 2, both partners push student H to explain that step. Student G does this once at 3:30, and Student I does this twice at 1:50 and at 3:50 (fig. 45 and 46). This perspective is interesting to analyse, as it shows what happens in a choice-affluent space when the situated solver is one person for the entire problem. In this case, the solver is still seen to use her outside resources to affirm her work, but also she plays a teacher role for her partners and other students in the room.

In this collaborative space, even when the progress is made by one individual, the situated solver changes levels. Student H appears to make all of the necessary steps on her own to solve the problem, but within this solution, she is frequently in dialogue with her partners and also gazing outside of her group. When she is in dialogue with her partners, the situated solver is changing to the group level and when she is gazing outside, the situated solver is changing to the class level. The progress is being made on the whiteboard solely by student H, but during this time, student H shifts her attention amongst a myriad of outside resources. It is impossible to know what is gained through these external shifts, but they do happen, and they may be providing subtle incites to student H allowing her to progress.

Summary analysis for lesson 2

It is becoming apparent that the change amongst situated solver levels is not sequenced nor well defined. I expected that the situated solver would move from the individual to the group when the internal resources of the individual became insufficient; and then move from the group to the class when the shared resources of the group became insufficient. The data supports that these changes do occur, but they do not necessarily occur in this sequence. I also expected that the transitions between these levels would be well defined. I predicted that there would be a noticeable pause, indicating that the internal resources had become insufficient, before transitioning from the individual to the group level. Although there is evidence of these pauses in the data, the transitions between levels also happen without pauses. Sometimes, these transitions are observed to happen frequently within one or two 10-second intervals (fig. 40, 1:30 – 1:40, 2:00 – 2:10; fig. 46, 4:20 – 4:30). It is becoming apparent that the change of levels in the situated solver is more unpredictable and less linear when groups solve problems.

Through the analysis of these three groups, one becomes aware that the situated solver is changing many times throughout each solution. As mentioned earlier, the situated solver can be considered at three different levels:

> The individual – the situated solver is at the individual level when the main actor in the solution process is the individual and progress is made through the use of internal resources.
>
> The group – the situated solver is at the group level when two or three group members are working in concert and progress is made through the use of the group's shared resources.
>
> The class – the situated solver is at the class level when solution progress is due to resources acquired outside of the group.

I thought it might be interesting to analyse the gaze-dialogue transcripts themselves and see how they might inform us on the frequency and durations of problem solving happening at each of these levels. To accomplish this, I interpreted inner group dialogue arrows to indicate a group level situated solver, gaze or dialogue arrows outside indicate a class level situated solver, and lack of arrows would indicate an individual level situated solver. Each 10-second interval was recorded as one of the three possible levels for the situated solver according to the presence and direction of the arrows. Admittedly, this is not precise, but on the whole of two lessons and three groups per lesson, this provided an interesting layer to add to this analysis.

TABLE 1 SITUATED SOLVER LEVEL DISTRIBUTION

Lesson 1	Individual Level	Group Level	Class Level	total	Individual Level	Group Level	Class Level
Group 1	8	25	7	40	20%	63%	18%
Group 2	14	14	18	46	30%	30%	39%
Group 3	23	10	17	50	46%	20%	34%
Lesson 2							
Group 1	6	18	4	28	21%	64%	14%
Group 2	10	18	12	40	25%	45%	30%
Group 3	5	18	6	29	17%	62%	21%
Aggregated	66	103	64	233	28%	44%	27%

Table 1 (above) shows the situated solver level distribution for each group and the aggregated level distribution for all groups.

This table confirms some of the more detailed observations made when describing the results of each group. For example, in lesson 1, group 3's progress was made mostly due to the efforts of one of the students (student I). In this table, one can see that the situated solver was at the individual level for 46% of the problem-solving time. Contrast this with group 1 in the same lesson. In group 1, students A and B were highly collaborative for much of their solution, and the table supports that group 1 was at the group level for the situated solver for 63% of the problem-solving time. Based on these and other comparisons, the table accurately summarizes the observations made of each group in each lesson.

This table indicates that the problem-solving process in choice-affluent environments, such as Thinking Classrooms, may involve acquiring resources at all of the levels of the situated solver all of the time. The data is collected from six different groups working on two different tasks, and the data appears to be quite different in the individual group stories. The data includes groups that are highly collaborative and highly resourced, groups that have individual solvers, and groups that are low resourced; and in all of the groups, the problem-solving process includes the situated solver at all three levels. The aggregated results suggest that most of the problem-solving time is spent at the group level (44%), and the other two levels are split somewhat evenly: individual (28%) and class (27%). But even when the group level is the highest (lesson 2, group 1 (64%)), the group is still changing to the other two levels: individual (21%) and class (14%). In choice-affluent environments, the level for the situated solver will shift through all three possibilities regardless of the group composition or task.

Chapter Six - Discussion

Having taught in Thinking Classrooms for over 10 years, I had some initial predictions and expectations on what this data was going to reveal (see Chapter Three). I expected that the situated solver level would advance as the individual or group ran out of internal resources, and I expected these advances to be sequential or linear in nature. Going into this study, I expected that the data would confirm these predictions and answer my research questions succinctly. In the end, I did notice these changes; however, they were not necessarily sequential nor predictable.

The initial results show that the situated solver can be considered on different levels. There is the individual, there is the small group collaborative, and there is the global level representing the whole class of learners. Resource acquisition occurs at each level; sometimes there is a pausing that may indicate the condition of becoming stuck. When the situated solver is at the individual level, the resource acquisition is from the internal resources of the individual. When the situated solver is at the group level, the resource acquisition is from the shared resources of the group. And when the situated solver is at the class level, the resource acquisition is from outside the group. At each of these levels, when the resources become insufficient, the situated solver may enter the condition of being stuck. To transition from stuck to un-stuck, the situated solver shifts to a different level to access different resources. At the individual level, this condition of being stuck may be noticed when the person holding the pen (the situated solver) reaches a point where the pen stops moving and the problem-solving pauses. At this point, the solver has become stuck, and begins to access external resources by dialoguing with other group members or passing the pen to a partner. If these actions do not result in progress, the situated solver is now at the group level and the group solver has become stuck. The group will then access external resources in the form of gazing at other group's work around the room, dialoguing with members from a neighbouring group, reaching out to the teacher in the room or accessing external resources through their access to digital technology. Each of these moves is observed in the data to be occurring both in sequence with respect to the individual level and simultaneously when the situated solver is the small group or the whole class. As the analysis of the data continued, it became apparent that in these choice-affluent environments, the shift amongst levels is occurring spontaneously and non-linearly as the task is being solved.

By spontaneous, I mean that the shifts appear to occur without cause; and by non-linear, I mean that they do not appear to fall within a structure of a sequence of problem solving as was predicted before the study began. The individual will shift to the group, then back to the individual, or it may shift from group to the whole class, then back to the group again. It is in these shifts of perspective where outside resources are being acquired and progress is being made.

In the initial description, the levels of the situated solver and the state of being stuck all appear to be very well defined and simple in nature. On deeper analysis of the data, it became apparent that the switching of levels was not so simple. When groups solved problems in the data, the transitions amongst levels were not always predicated on being stuck; rather, the transitions were often occurring spontaneously and non-linearly. There are situations when progress was being made and the group does not appear to be stuck, and I observed the situated solver changing levels (lesson 2, group 3). And there are situations when groups are stuck, resources in neighbouring groups are abundant, and the situated solver does not shift levels to acquire those external resources (lesson 1, group 2). Although it is a nice model for describing some of the processes for solving problems in choice-affluent environments, the data shows that it is not always this simple – problem solving in choice-affluent spaces is more complicated, perhaps even complex.

Phases of Progress

In these two lessons of video data, it became apparent that the transitions of the situated solver among levels was a complex process. It was complex in the sense that it did not obey a simple construct – I was not able to describe all transitions in the same way. When an individual situated solver depleted their internal resources and became stuck, sometimes this would lead to a change in level to a group and sometimes it would not. When a group was seen to be making good progress on a solution (group resources were abundant), sometimes this group would still seek resources from the outside. As is the case with many real-life situations, ideas can be simple and organized, but real data can be messy. The complexity of this Thinking Classroom environment is an important result to be taken out of this analysis, but there is still value in describing some of the simpler interactions and observations gleaned from the data.

One can see evidence of a group shifting attention amongst individual resources, resources shared by the group and resources from external sources to make progress in a problem-solving activity. Prior to the analysis, I had anticipated 3 phases that groups would move through when solving problems in choice-affluent environments. Focussing on lesson 1 and group 1 (fig. 44 and 45), one can see a nice example of all three of these phases:

Phase 1 (fig. 44, 4:00 – 5:30): Students A and B are making progress in the solution. Student A is writing and describing their steps to student B as progress is made. In this phase, the situated solver is at the group level, because students A and B are working together to make progress.

Phase 2 (fig. 44, 5:30 – 6:00): Students are stuck. Student A puts the pen down, and both students stop activity and stare at their own work.

Phase 3 (fig. 45, 6:00 – 7:00): Student C is observed looking at another board, and they draw their partner's attention to this board. Students A and B also look at the board, bringing in the outside resources. At this point, the situated solver moves to the class level, as resources from outside of the group are being accessed.

Phase 1 (fig. 45, 7:00 – 7:30): Student A picks the pen up and continues progress on the problem.

Time	outside	A	B	C	notes
4:00 – 4:10		Ok, well, we don't have to do that stuff to get to this. We can just straight up split it. **	That's true.		Student A erases their work, and grabs the pen.
4:10 – 4:20		We know that up here, it doesn't split into a 6 and a 6. **			Student A writes: Y = 4x – 2 – 5 And gestures to work from a previous problem.
4:20 – 4:30		We saw that this value is this value. So what I'm thinking… **			Student A gestures to the coefficient and the horizontal asymptote
4:30 – 4:40		Is that we have this. We have an 'a' a plus '4', and we have an x – 2. **			Student A writes: Y = a/(x-2) + 4
4:40 – 4:50		What number do we have to multiply by 4 to get -5? **			
4:50 – 5:00					Big pause in the action as they contemplate this question.
5:00 – 5:10		That's what we do! **			Student A glances at the solution to the previous task that is still on her whiteboard and INSPIRATION!!!!
5:10 – 5:20		This makes a 4x – 8, so this is going to be a positive 3. **			Student A writes: (x-2)/(x-2) beside the 4, and replaces the a with 3.
5:20 – 5:30		Because we know to get from -8 to -5, we need to add 3. **			
5:30 – 5:40		This is going to be 3 over x-2 + 4. **			Student A then begins to write some missing steps for clarification. Then puts pen down.
5:40 – 5:50					All three students are staring at their solution.
5:50 – 6:00					Still staring.

FIGURE 47 GROUP BECOMES STUCK AT 5:30

Time	outside	A	B	C	notes
6:00 – 6:10					Student A grabs the pen and adds some clarifying steps. Student C is looking at incorrect solution on board to the right.
6:10 – 6:20					Student C is shifting glances from his board to the right board.
6:20 – 6:30				These two are different.	Student C points to the constant term in each expression.
6:30 – 6:40					Students A and B attend to the other group's work. Student B checks the graphs in Desmos.
6:40 – 6:50			They are exactly the same.		Student B holds his phone up to show Student A and C
6:50 – 7:00		**			Student A begins to verify their solution.
7:00 – 7:10		**			Student A writes: Y= 3/(x-2) + 4(x-2)/(x-2)
7:10 – 7:20		**			Student A writes: Y= 3/(x-2) + (4x – 8)/(x-2)
7:20 – 7:30		**			Student A writes: Y = (4x – 5)/(x-2)
7:30 – 7:40					Student A labels the work "Check" and puts the pen down
7:40 – 7:50					Student A looks across the room at another student.
7:50 – 8:00					Student A goes to this student and engages in conversation gesturing at her board.
8:00 – 8:10					Other student comes into this group and Student A explains their solution/
8:10 – 8:20	OK	Sounds more complicated than it is			Student A finishes explanation. Other student nods in agreement.

FIGURE 48 GROUP IS CHANGING LEVELS AT 6:30

In phase 3, the group's attention shifts outside. This group attended to a board on their right in the room, the board apparently showing a different solution. This outside resource appeared to be sufficient for the group to move back into phase 1 and make some contributions towards a solution. This third phase may also be characterized by individuals conversing with other classmates, dialoguing with the teacher, and sometimes even going onto their phones to seek resources from outside of the class. Phase 3 continues until the group is satisfied that their resources are adequate, and they can move back into phase 1 – sometimes solving the problem and sometimes getting stuck again. Prior to the analysis, I expected that these groups would progress through these three phases in a linear fashion:

1. Progress in a problem is made using internal resources of the situated solver.
2. Situated solver becomes stuck. Activity decreases and progress in problem stagnates.
3. Situated solver moves up a level. This consists of glances, conversations, group visits, or dialogue with teacher, depending on the level of the situated solver. This phase continues until:
   a. Problem is solved
   b. Progress is made and the situated solver is able to move down a level and back to phase 1.
   c. Progress is not made and situated solver becomes stuck, phase 2.

When solving problems collaboratively in choice-affluent environments, I expected groups to move linearly through three phases (fig. 49). They would begin working within their group, and there would usually be a single situated solver at the individual level. This individual would shift their attention amongst their individual and internal resources until they solved the problem or until they became stuck. Phase 2 is being stuck. In this phase, they would remain engaged and on task, but noticable progress would decline, boardwork would stop and the pen may shift to other members (or be put down altogether). When the individual shifts their attentions to outside resources, they would move into phase 3. This is the benefit of choice-affluent

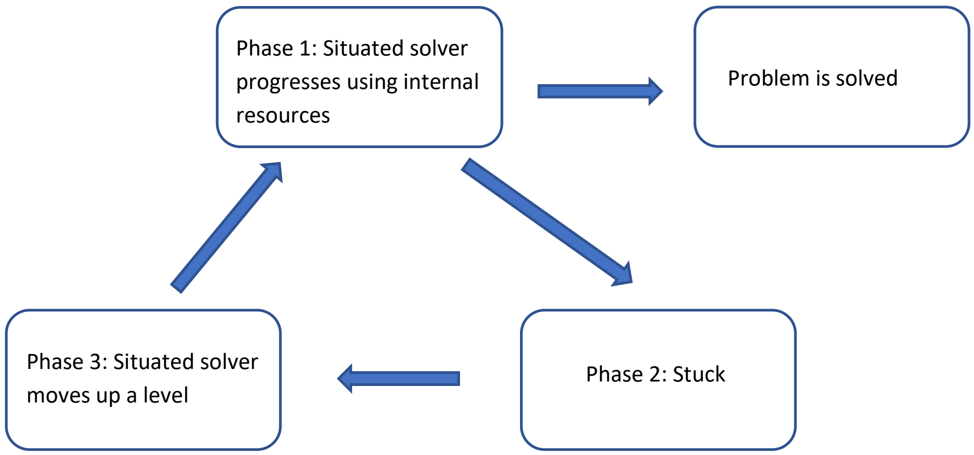

FIGURE 49 LINEAR DESCRIPTION FOR COLLABORATIVE PROBLEM SOLVING

environments. In classrooms where problems are solved individually or collaboration is contained within groups, phase 3 requires the singular action of accessing the teacher as an outside resource. In choice-affluent environments, like Thinking Classrooms, the outside resource space is abundant and heterogeneous. In phase 3, students may shift to the group level and collaborate with others within their group, or it may shift to the class level and observe other group's work (inter-group). At the class level, they may converse with other groups or the teacher, and they may use technology to access additional resources (extra-group). Students have many different choices to shift their attentions and emerge a key solution idea that will allow them to move back into phase 1.

After the analysis, I noticed that these expectations were far too simple. Groups were observed to move through these phases in sequence, and the condition of being stuck was noticed at times; however, there is an abundance of data that does not follow this sequence, and levels of situated solver often change without entering phase 2 (being stuck). After the analysis, it became apparent that the problem-solving process in choice-affluent environments like Thinking Classrooms is far more complex.

In Thinking Classrooms, and indeed in this classroom, the conditions for complexity (Davis & Simmt, 2003) are all met:

Internal diversity – the students act as individuals with their own variety of experiences and personal resources.

Redundancy – despite their diversity, the students all have a lot of commonality. They are similar in age, they have similar school back grounds, and they come from similar communities of learners.

Decentralized control – there is no direct teacher control over how students navigate their tasks. Students are provided freedom to explore each task and generate a strategy of their choosing.

Organized randomness – each class, students are placed into random groups so that they have different experiences with different students in every class.

Neighbour interactions – literal neighbour interactions occur due to the students standing side-by-side at the whiteboards, but a figurative neighbourly interaction also occurs in the sense that ideas on boards are also bumping into one another.

These conditions are described to be necessary, but not sufficient for creating a complex space and for learning to emerge, meaning that just because the conditions are met does not guarantee complexity or emergence. The observations made through this analysis more clearly show what complexity looks like in a classroom environment. The shifts in levels of the situated solver were frequently unpredictable and continuous in nature. The data shows a more undefined and uncaused switching amongst levels as progress is made. There are instances where groups move through all three phases in sequence and the condition of being stuck is noticed, but there are many more examples of groups shifting spontaneously and non-linearly as a solution is made. Even when a solution is made, activity does not cease. Students are observed sharing their knowledge, pushing their newly gained resources out to other groups, and checking their own individual understanding. In this complex space, groups are learning together, but understanding individually, and it is in this tension of purpose that activity and involvement remains high within a problem-solving episode and after.

The data suggests that although each level of the situated solver is represented, there is no defined order or sequence to how emergence of an idea occurs. Figure 50 (from Chapter Three) is probably the better representation for this process, as it suggests that even when the situated solver is at the class level, and ideas are emerging from the intersection of different groups, the arrows within the individual circles are still shifting and scanning the internal resources, and the individuals within each group are intersecting and sharing resources. The situation is best described as *fuzzy* – there is no defined sequence and no repeating pattern in how ideas emerge in these choice-affluent spaces.

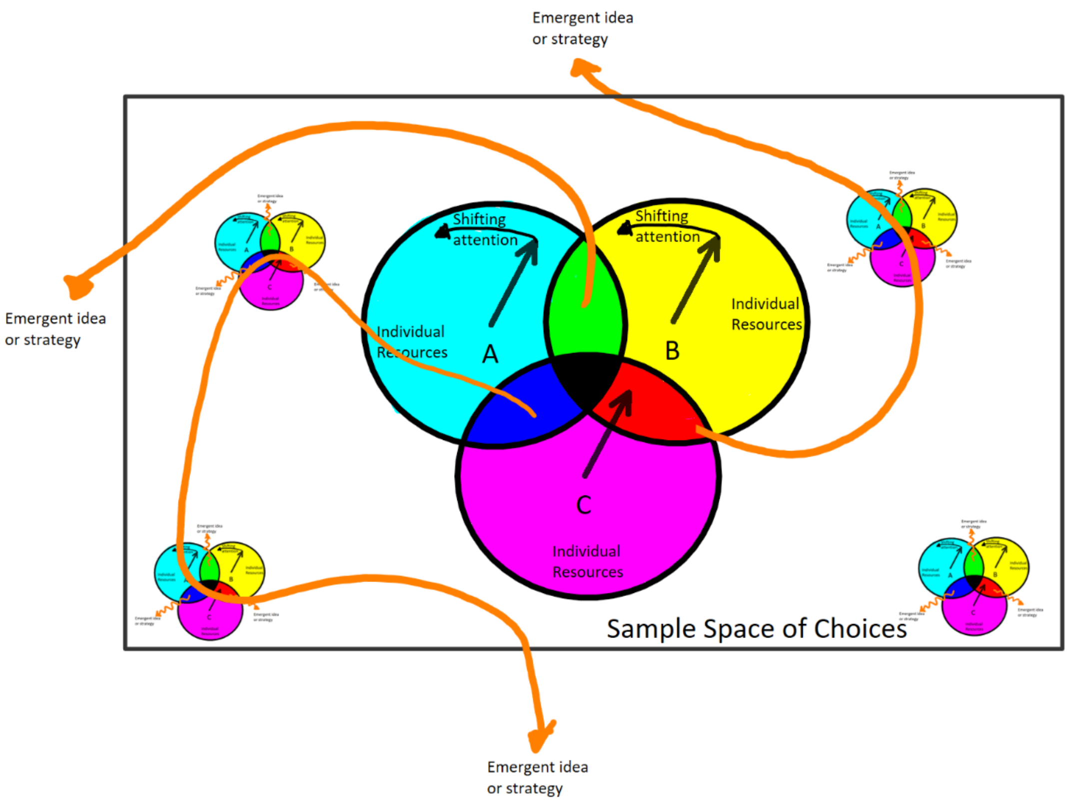

FIGURE 50 COLLABORATIVE PROBLEM SOLVING IN CHOICE-AFFLUENT ENVIRONMENTS

As groups eventually solve the problem, they are always encouraged to keep asking questions. This has become enculturated in my Thinking Classroom starting from our first day together in September. Students know that when they think they have finished their problem, they are to keep asking questions, such as:

- Does everyone in my group understand the solution?
- Is the solution correct? How do I know our solution is correct?
- Can this problem be solved in another way?

- Does the mathematics on our board represent and communicate our thinking effectively?
- What next? Are there any extensions that might be interesting to explore?

By participating in these types of reflections, learners are shifting their attentions from the particulars of the solution steps from the problem-solving process to the general. Mason (1998) writes: “the process of generalising, of seeing how a number of previously disparate items can be subsumed under one generality, is the occasion for a sense of pleasure, of expansion of awareness, of release of energy” (p. 254). Indeed, I often observe groups shouting in joy and giving each other high-fives as they move into this third stage; the positive affective experience plays a subtle yet powerful role in their overall feelings towards mathematics. Furthermore, this generalizing shift is the time where learners are adding to their repertoire of resources. Consciously or unconsciously, their experience has been enriched and, likely, their mathematical knowledge has increased. This is apparent for me, as an observer, many times when I hear students say at the beginning of a new task, “this is similar to the task we did last week.” The *reflection* stage is a time for students to apply a sort of meta-thinking about the problem that they have solved. Through engaging in the reflection questions, they are shifting their attention to the problem as a whole, and in doing so, they are adding to their personal resource inventory.

Choice-affluent environments, such as Thinking Classrooms, are spaces where the problem solving is enriched and accelerated, the affluence of choice acts like a catalyst in the problem-solving process. Without a choice-affluent environment, when problem-solving is a solitary activity, the learner is using a reduced resource space and is more likely to rely on grace as a source for a shift of attention that may (or may not) spur progress in the problem. To use another metaphor, problem-solving can be thought of as the process of mixing chemicals – a chemical reaction. If you have one chemical, this is like a solitary problem solver, very little happens. When you have a group of chemicals, like a group of people, reactions start to take place, new chemicals are formed; things are getting interesting. When you have a whole lot of chemicals, and you add some heat, and some kinetic energy, like a mixer, well then, you had better watch out; things are about to get exciting. Problem-solving in choice-affluent environments is like the last case; learners have access to a potpourri of outside resources, and their attentions autonomously shift amongst them like the mixing of the chemicals resulting in emergent ideas or strategies that are necessary to solve the problem.

The research questions

At the end of chapter three, two research questions were stated:

1. What forms of interactions seem to support the movement of ideas and how do these interactions support problem-solving?
2. What form do neighbour interactions take and how do they align with the theory of complexity theory?

What became apparent after the analysis is that answering one of these questions actually answers both of them. The forms of interactions that support the movement of ideas are, broadly speaking, neighbour interactions; and, what do neighbour interactions look like? They are essentially those interactions that support the movement of ideas.

Ideas are the stepping-stones that allow a solver to progress through a problem. In a math class, ideas can be a strategy, a diagram, a representation, or anything that allows a solver to move from being stuck to being unstuck. In many classrooms, if ideas are not generated by an individual solver's shift of attentions through past experiences and personal knowledge, then they are reliant on the teacher, a fellow student, or a moment of inspiration.

All classrooms are complex systems where learning can be considered as emergent; however, in choice-affluent environments, such as Thinking Classrooms, the conditions for complexity are strong, and emergence is more common and more likely. In the data for this dissertation, one sees these emergent events stemming from 3 different situated solvers: Individual, group, and class.

Intra-group communication is when a group is making progress without any outside resources. The situated solver may be at the individual level if they are working independently using internal resources, but it often shifts to the group level as partners engage in dialogue and progress continues. The individuals in the group may have sufficient resources for emergence to result. Inter-group communication is when the group's resources are not sufficient, and they need to rely on outside resources. The situated solver is shifting to the class level, as they may see an idea on another board, receive an idea from a conversation with an outside member, or hear an idea from a nearby student conversation. Extra-group communication is when a group makes progress by retrieving a resource outside of other groups or other boards. Extra-group communication can be from a teacher, or from any form of technology: websites, graphing calculators, YouTube videos, etc…

These are the interactions that support the movements of ideas. Ideas move passively within groups through direct dialogue with other group members, and they are also moved into groups through gazing at other's whiteboards, overhearing other conversations. Sometimes ideas are moved actively. Ideas are moved intentionally by a teacher providing a hint, or by a student from outside the group coming in and providing an idea. Thinking classrooms have a framework in place that supports and encourages ideas moving in all of these ways.

In complex systems, ideas move through these three levels of situated solvers. Because of the nature of complexity, these actions cannot be forced or planned. Its very nature requires a randomness and a freedom amongst the individual agents for self-organization or emergence to occur.

Davis & Simmt (2003, p. 145) outline five conditions of complexity as necessary but insufficient conditions for systems to arise and to learn: (a) Internal Diversity – enables novel actions and possibilities. (b) Redundancy – sameness among agents is "essential in triggering a transition from a collection of me's to a collective of us." (c) Decentralized Control – locus of learning is not always the individual. (d) Organized randomness – emergent behaviours are about living within boundaries defined by rules, but also using that space to create. Liberating constraints draw a distinction between proscription and prescription in tasks. (e) Neighbour Interactions – there needs to be collaboration… not necessarily people to people but more for ideas to bump up against one another. This leads to the second question: What do neighbour interactions look like within a complex system?

As stated earlier, the interactions are the conversations with fellow colleagues, outside group members, or the teacher. They are the glances at any board work in the room (including that belonging to the group). They are the overhearing of nearby group conversations or louder conversations from around the class. And they are also the multitude of interactions with outside sources through the available digital technology on most student’s smartphones.

In choice-affluent environments such as Thinking Classrooms, neighbour interactions are the actions necessary for ideas to move. In the data, there are many different forms of these actions. It is through these interactions that progress is achieved when solving problems, and it is the lack of these interactions which will therefore prohibit progress in less choice-affluent environments. These interactions are necessary for students to gain resources that they do not have from their own experience and knowledge. The classroom itself is the smartest learner in the room; therefore, it is necessary for the individual learner to have access to this resource when they are no longer able to make progress in a problem.

Through analysis of the data, and answering the research questions, I have learned something more. I have learned that problem-solving is not a deterministic process. It seems that all of my attempts to predict or describe problem-solving behaviour have counter-examples within the data. I can describe themes and specific scenarios, but in this complex space, the behaviours are more random (i.e., a student will appear to be fully independent in her solution, but she is still observed to be glancing and conversing with others to gain resources; or, students appear to be stuck, and they do not access the good resources right next to them.) Despite all of these contradictions, progress is still made, and problems become solved. I believe this is happening because of the randomness and the richness of the system as a whole.

Chapter Seven - Conclusions

Problem solving is an activity that is valued in most professions and vocations in our society (NCTM, 2000), and it also plays a central role in many mathematics classrooms and in most mathematics curriculums. Research in mathematical problem solving as a collaborative activity has mostly been studied within small groups (see, for example, Ryve, 2006), with not much attention devoted to problem-solving where resources are accessed outside of small groups. Koichu (2018) presented a framework for studying problem-solving in environments that more closely match problem-solving in our modern society; he termed this *problem-solving in choice-affluent environments*. Thinking classrooms are examples of these choice-affluent environments where students have an abundance of resources due to the public nature of conducting their work on whiteboards positioned around the classroom. In this study, I was interested in how and when students access external resources when solving problems in these choice-affluent environments. Using Schoenfeld's (1985) theory on resources and Mason's (1988) theory on shifts of attention, I analysed classroom video to better understand the *how* and the *when* question.

Gaze-dialogue transcripts were created from actual recordings of students problem-solving in a Thinking Classroom. After analyzing some of the video and the transcripts, I noticed that problem-solving may begin with students accessing a mix of individual and intra-group resources. This appears to continue until either the problem is solved, or the group becomes stuck; at which point, students will begin to look for resources outside of their group. It is this extra-group (or inter-group) resource acquisition that makes choice-affluent environments, such as Thinking Classrooms, so unique. By accessing outside resources, such as other groups, the teacher or the world wide web, students are more accurately engaging in real-world problem-solving, building their own problem-solving skills and adding to their individual repertoire of resources and experience – ultimately becoming better problem solvers.

However, after looking at the data on a whole, it became apparent that these processes were occurring, but they were not occurring in the expected order or with the expected precursor events. The data supports that collaborative problem-solving in choice-affluent spaces is non-linear and less predictable than anticipated. Ideas are moving within Thinking Classrooms in a myriad of ways. The progression of one group's solution will not likely happen again in a similar group in a similar problem. In choice-affluent spaces, resources are accessible from a variety of sources, and the situated solver will make use of these resources in different ways every time. If we would like our students to gain experience in solving problems, then choice-affluent spaces appear to be quite successful in providing these opportunities.

This study provides a method for analysing classroom video of problem-solving activity. The gaze-dialogue transcript honours what an observer would see in the classroom. Short of actually watching the video or observing the classroom live, the gaze-dialogue transcript tells the story of the problem-solving episodes by not only describing the written work of the students, but also their actions, attentions, and directions of their gazes. This type of transcript codifies everything that is observed in a concise and analytical form that allows for future analysis and discussion of the actions that transpired.

This study falls short in a few areas. The distinction between accessing individual resources and group resources is blurred. This could be remedied in future studies by interviewing students shortly after the problem-solving episode to distinguish between these two different kinds of resources being used. Another shortcoming is with the quality of the video data. Unfortunately, due to the technology used, the audio recording of the student dialogue was difficult (sometimes impossible) to discern. With better audio recording, the gaze-dialogue transcript can be more detailed, and subtleties of intra-group collaboration may come to light. Lastly, when groups access resources from other groups that are not in the video frame, one is not able to see the actual board work that is being attended to. This could be improved in future studies, by simultaneously recording each and every whiteboard in the room. Obviously, this presents some technological challenges, but I believe having access to this data would add some merit to the ideas being discussed.

We know that problem solving is dependent on heuristics, experience, and personal resources and that in real life, problem solvers rely not just on their own resources but also on external resources: peers, technology, social networks, etc. As such, I argue that authentic problem-solving is best learned and practiced in choice-affluent environments where the situated solver makes shifts of attention within spaces that afford an abundance of choice for engagement. I present here an opportunity to envision what this might look like in practice. In Thinking Classrooms, students have been observed access resources from many different sources when their group's resources are diminishing. Thinking classrooms provide one potential model that demonstrates choice-affluent environments for students to exercise and build their problem-solving resource repertoire.

Reflections

Writing this dissertation is a process that has taken more than three years. Over this time, I have become aware of aspects of my own classroom and teaching that are not directly related to my research questions, but I thought the reader might appreciate me sharing these learnings, nonetheless. I will break this reflection into three categories:

1. What I have learned about Thinking Classrooms.
2. What I have learned as a teacher.
3. What I have learned as a researcher.

What I have learned about Thinking Classrooms.

After studying hours upon hours of classroom video data, I have learned a few things about the Thinking Classroom framework. In a high functioning class, students do quite well with pushing out resources, and in all Thinking Classrooms there exist students who are off-task.

I knew that there were times, in Thinking Classrooms, where students were supporting other students, but I did not expect this behaviour to be so prevalent. There were so many examples in the video data of students supporting other students within their group and outside of their group. I mentioned it in the summary analysis of lesson 1 as a "pushing out of resources," but this action was observed in other classroom video as well (not included in the data). It appears that there is a responsibility observed amongst students in Thinking Classrooms to support their peers in their learning. I knew this was happening, but I was not expecting it to be so common.

Another observation made was not so positive – there is a lot of off task behaviour in the video data. Off task behaviour shows itself in primarily two ways. Any student in any group was observed to be off-task a variety of times during the task. This off-task behaviour included, but is not limited to, checking their phones, chatting with another student, getting a drink or something to eat, etc… In this first description, these students would be off-task for a short time, and it seemed inconsequential, possibly even beneficial, to their progress in solving the problem.

There was another type of off-task behaviour observed in most of the video data, and this is the student who appears to never be on task. In each of the two videos selected, these off-task students were not part of the three study groups, but I could see them in the periphery of the video. Moreover, in all of the additional video footage that was not included in the analysis, there were many examples of these students. These students are either sitting at their tables or standing at their whiteboards and they are completely engrossed in their smartphone or they are engaged in a non-content conversation with another student. As a teacher in these classrooms, I was aware that this behaviour was present, but I was not aware that it was so prevalent. Unfortunately, it is a necessary by-product of providing students so much autonomy in a Thinking Classroom.

Typically, when I see this type of distraction in a classroom, I do attend to it, but often times it may go unnoticed. While watching the many hours of video data, I did see myself attend to some of these students, but I saw many more examples that went unnoticed. I do think that, in any group of 30 teenage students, there will often be students who choose not to participate productively in the activities of the lesson. But with the freedoms provided in a Thinking Classroom, these behaviours may be exacerbated and difficult for some teachers to cope with.

What I have learned as a teacher.

Over the past three years, I have become more aware of the lack of direct control I have over my student's progress in problem solving and, more generally, over their learning. At first, this revelation was frightening, but, over time, I have found it to be a release from the high-pressure expectation for student learning and the personal responsibility that is associated with this expectation.

I have a science background and a scientific approach to life. By this I mean that I believe in cause and effect and that once a system is well understood, predictions can be made effectively. This belief was buoyed by my love for mathematics and my experience as a teacher of mathematics. I found that the more familiar I became with mathematical content the more capable I became as a problem-solver and a doer of mathematics. Applying this to teaching, I believed that the more I knew about how a student and how a class learns, the better I would be able to teach. It feels like a rather simplistic view now with the benefit of hindsight, but I truly thought that I could manage each student's learning by implementing an appropriate cause into the system.

The foundations for this belief were rocked over the course of my analysis of the classroom video. I came into this with an expectation for how students would navigate their problem solving (see chapter three). I expected it to fall quite readily into a flow-chart of stages culminating in a problem solved, a happy student, and an increase in personal resources. Although this clean progression was observed within the data, it was not the norm. Most students made progress through their problems in a more haphazard approach. Some would search for resources when needed, some would not. Some would scan for resources even when they did not appear to require any. The flow-chart for solving problems was rarely followed from beginning to end; and yet, problems were being solved.

Through this, I learned that student progress in a choice-affluent space simply happens. Any attempt to cause it, predict it, or manipulate it does not necessarily have the desired effect. This was quite alarming at first. In becoming more aware of the lack of direct control I have on my student learning; I began to wonder what role I do have as the teacher in the room. It was alarming because it was pushing up against my previously held beliefs of teaching. Perhaps effective teaching doesn't come from the perfectly planned lesson, or the perfectly managed class, or the perfectly orchestrated discussion. Perhaps teaching is messier than this.

I cannot say that I have it all figured out, but I realize now that the role of the teacher is in setting up the classroom to increase the potential for emergence. I find this encouraging, because learning does not result directly from my well-laid plans, nor is it ruined by my mistakes or missteps. Learning will occur when students are immersed in a quality task, in a choice-affluent space, and in a community where they feel safe to share ideas and take ideas from others. So, my role as an educator is more of a periphery role. I need to set the sequence of tasks for an intended learning goal, and I need to monitor and encourage student progress as they navigate their own trajectories towards the learning goal. My role is that of an organizing agent in the complex environment of a Thinking Classroom.

What I have learned as a researcher.

The biggest learning that I take away from this project as a researcher is that research takes time. I came into this study with some early expectations as to what might result, and for a large part of my data collection, transcription, and early analysis, I was trying to fit my results into my expectations. I even had a significant portion of this dissertation written to align with these early expectations, but it just didn't feel right. I am grateful for the iterative process of submitting drafts and receiving feedback, because all of this takes time. I believe it was a combination of the time wrestling with the results and discussing the feedback I was given that can be credited to shifting my results for this study away from my expectations.

I believe that the conditions for complexity in writing a dissertation has a lot in common with conditions for complexity in choice-affluent classrooms. Over time, I valued the many interactions I had with peers in my program, professors in my course work, and my advisors for this dissertation, not to mention the interactions with the literature that I came across. I was working alongside students with similar interests but a variety of backgrounds and experiences. Lastly, the doctoral program at Simon Fraser University thrives on de-centralized control, as each student is provided with autonomy in choosing an area of study, in choosing a question to investigate, in selecting a method for the study, and in deciding the timeline for the dissertation. Because of these conditions, much like the group work in my Thinking Classroom, the results for this study emerged over time and through collaboration with my peers and advisors.

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