TEACHING SECONDARY SCHOOL MATHEMATICS THROUGH STORYTELLING

by

Chandra Balakrishnan
B.A., Simon Fraser University 2000

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Chair: **Title and Name**
[Assistant/Associate] Professor of/Department of

____

Title and Name
Senior Supervisor
[Associate/Assistant] Professor of/Department of ????

____

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____

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ABSTRACT

It has long been understood that stories can be used to teach elementary and middle school students mathematical ideas. The objective of this research was to examine how stories can be used to teach mathematics at the secondary level.

DEDICATION

For my father and mother, Perumal and Gowry Balakrishnan.

Everything I am is a result of the sacrifices you made and the unfailing love you surrounded me with. I am blessed beyond measure to have you for my parents. You made me feel like I can accomplish anything.

For my children, Sage, MacKenzie, and Elle.

Thank you for inspiring me with your stories, your brilliance, and your wonderful senses of humour. Thank you for your love, patience, and understanding. You are the loves of my life and all of this was for you.

For my brothers and sister, Roger, Desi, and Anita.

Thank you for being there for me when I really needed it.

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1 INTRODUCTION

Flourishes of language, trappings of the imagination, and sheer nonsense garnish the stage for the presentation of my initial thoughts on the use of stories in the teaching and learning of high school mathematics. Here, within the pages of this document, I present to you, the reader, a mathematical extravaganza—a potpourri, if you will, of childish delights juxtaposed about the solemn works of some of the greatest minds history has ever known. Now, if you have not already put this paper down, then I applaud your charity, your willingness to abbreviate your logical underpinnings—for I am well aware that the kind of person who might read a discourse on high school mathematics is typically irreverent of such literary excursions. After all, mathematics is normally written in both a curt and precise manner, and you, my dear reader, have already been subjected to nothing of the sort. In a more direct, though somewhat pretentious fashion, I ask you now to consider the following set of familiar mathematical concepts: exponential growth, uniform motion, division by zero, radical operations, and linear equations. What images came to mind as you read these words? Perhaps you envisioned a few graphs and/or numerous abstract symbols and equations? You may even have pictured two airplanes travelling between distant cities or the advancement of bacteria observed under the lens of a microscope. At the same time, what emotions, if any, were attached to these ephemeral images? Did your imagination take flight at the mention of even a single item from this list? The more earnest question for the instructor of mathematics however, is this: What images—and connectedly emotions—might your students associate with these concepts? If it is anything less than warring Samurais, hideous aliens from Uranus, nefarious gems of uncertain origin, love-struck ogres, or bloodthirsty buccaneers, then the rest of this document may be of some interest to you. For it is only through engaging students' imaginations that we can bring emotional value and meaning to the array of seemingly irrelevant facts, rules, and algorithms they encounter at school (Egan, 2005).

In the next fifteen chapters, I show how stories can be used in various ways to do this very task. In the second chapter, I look at the cognitive tool of story and its explicit ties to the imagination. The imagination plays a fundamental role in the understanding of mathematical concepts—ranging from the most basic of structures, such as counting or variables, to more abstract ideas, such as the notion of higher dimensions or the concept of infinity. There is an intricate connection between the imagination and the emotions, and when students are emotionally involved with the material, they are using their imaginations. I argue that because stories are also tied to the imagination, they have the ability to elicit particular emotional responses. Additionally, I show how narratives are intricately connected to the way the mind works—we think, dream, and perceive the world in terms of stories. This, and the fact that children are especially predisposed to using stories, suggests stories play a significant role in the establishment of meaning. In chapter three, I explore this further from an educational perspective. Using Zazkis and Liledahl's classification of stories (2005), I examine six basic story forms that can each be used in the classroom context with different educational purposes in mind. These range from stories that are purely "ornamental" and do little to help establish mathematical meaning, to those where the mathematical content is practically inseparable from the elements of the story (Mazur, 2007). Additionally, I show that stories assist in the transition from solving concrete representations of problems to more abstract forms (Koedinger & Nathan, 2004; Koedinger, Alibali, & Nathan, 1999).

In chapter four, I consider how stories can be shaped in order to maximize their imaginative potential. For this, I look to certain cognitive tools that are readily available to students and that each affects the imagination in unique ways (Egan 1997, 2005; Zazkis & Liljedahl, 2005). The first tool is that of binary opposites, which plays a primary role in establishing conflicts and moving the story along (Egan, 1986). Students become emotionally involved with the characters in a story and their particular concerns when postured within a binary structure. Another cognitive tool that can be used to shape our stories is metaphor (Egan, 1986, 1997, 2005). Metaphor is pervasive in both language and thought and if used carefully can be instrumental in capturing the imaginations of students (Gentner, Bowdle, Wolff, & Boronat, 2001). This is especially significant considering the ability to recognize and generate metaphors increases slightly during early adolescence, before going into a steady decline thereafter (Egan, 2005). I argue that using metaphor, as in Edwin Abbott's, *Flatland* (1998), where mathematical objects take on human characteristics, may be an effective way to engage students' imaginations with the mathematical content from a new and very different perspective. Knowledge and human meaning is the next cognitive tool I consider (Egan, 2005; Zazkis & Liljedahl, 2005). "Origin stories," as they are often called, seek to represent the mathematical topic in terms of the emotions and motivations involved in its discovery. Additionally, I argue that we must bring out the most extreme elements of these stories if we are to capture our audience's attention and hold their interests (Egan, 2005).

Connectedly, I look at the contexts of our stories, and consider whether fantasy or reality settings are better suited for imaginative engagement at this level. The literature suggests a blending of the two may be fitting (Egan, 2005). One of the most powerful cognitive tools I will look at is rhyme, rhythm, and pattern. Recognizing the patterns found in most mathematical structures satisfies to a large degree, our innate search for beauty and meaning in the world. More importantly, when stories highlight these patterns, the mathematical ideas become more imaginatively appealing. Students use their imaginations to identify and predict elements in the patterns they encounter. Connectedly, the cognitive tool of jokes and humour relies on the deviation from cognitive patterns or incongruities in our mental frameworks. A story infused with humorous elements can quickly bring to the forefront the mathematical content we want our students to explore in a very poignant way. Finally, I look at the delivery of our stories and discuss how the orally told story may be more effective than the read story at meeting the specific needs of our students as well as producing the desired emotional responses.

In chapter five, I present the two research questions that I will attempt to answer in the nine subsequent studies. The first question examines the practical ways stories can be integrated into regular classroom practice, while the second question looks at how stories can be specifically shaped for imaginative engagement at the high school level.

In chapter six, I describe the participants in the study, the setting of the study, the types of data collected, and the theoretical frameworks used in the treatment and analysis of the data.

In chapters seven through fifteen, I present nine different studies, each consisting of a story or story assignment that addresses a particular topic in the secondary school mathematics curriculum. Additionally, each assignment was uniquely designed with specific cognitive tools and presentation methods in mind in order to speak to the two research questions described above. In the first study, I introduced students to the concept of exponents using a gruesome tale about two rival samurai warriors. In examining the stories students wrote, I show how the binary structuring and attention to pattern in *The Legend of Chikara Ni Bai,* made the idea of exponential growth come alive in the minds of students and take on real meaning. Additionally, I relied on the cognitive tool of jokes and humour to accentuate the mathematical patterns found in the story. In the second study, *The Pythagorean Problem*, I introduced students to the concept of irrational numbers using the context of a real historical event involving a secret cult-like organization. With a focus on the cognitive tools of knowledge and human and meaning and jokes and humour, and again placing the story within a binary structure, I show how a normally dry topic from the curriculum can be transformed into one shrouded with mystery and suspense. In *The Four 4s*, I demonstrate how a story can be used to introduce an exercise in order of operations with rational numbers. Additionally, I show how a story may not necessarily be a factor in the conceptualization of a mathematical concept, particularly when it is not intertwined with the mathematics involved. In the fourth study, *Skull Island*, I show how a story can do more than simply introduce an activity on graphing linear equations, but can also provide the context for a meaningful exploration of the subject. Once again knowledge and human meaning, jokes and humour, as well as binary opposites played key roles in this process. In *The Skippers*, I show how familiar word problems can take on real meaning in the minds of students when they are transformed into even the simplest of stories. In this study, I relied primarily on the cognitive tool of jokes and humour to shape my story. The immediate results showed teacher-modelling plays a key role in predicting the kind of stories students write. In *Kelloggs Needs Your Help*, I again used a story to provide the context for an engaging activity, this time examining the concepts of volume and surface area. Although, students still demonstrated misconceptions about two and three-dimensional space, they took several “imaginative leaps” in the process—something they might not have entertained in a traditional approach to the subject. I relied primarily on the cognitive tools of rhyme, rhythm, and pattern, binary opposites, and jokes and humour to engage students with the concept of division by zero in *The Wicked Sons*. Results from this study suggest that the story played a significant role in the way students conceptualized division by zero. In *Mathematically-Structured Story*, students wrote emotionally-charged stories based on the prime factorizations of the first twenty natural numbers. Here, a lack of simple polynomial patterns may account for the way students approached the topic with such interest and wonder. From this study, I show how the “aesthetic flow” can begin with the mathematics. In the last study, *The Story of Princess* $\sqrt{3}$, I used the cognitive tool of metaphor to engage students’ imaginations with the concepts of radicals. This study shows great promise as a new way to approach curriculum content—from the perspective of a mathematical object. In the final chapter, I give a conclusion to my research, mostly elaborating on some of the themes that arose and their implications for teaching mathematics at the secondary level. I now begin where all great stories begin—with the imagination.

2 STORY AND THE IMAGINATION

2.1 Imagination and Mathematics

Robert Osserman (Hoffman, 1998) recalls a story about the famous mathematician David Hilbert:

> [Hilbert] one day noticed that a certain student had stopped attending class. When told that the student had decided to drop mathematics to become a poet, Hilbert replied, “Good—he did not have enough imagination to become a mathematician” (p.95).

The imagination is typically not associated with the learning of mathematics at the secondary level. Instead, it seems to have found its place in “the arts,” while the “real work” of mathematics is both theoretical and logical, devoid of any human perspective or emotion (Egan, 2005; Hoogland, 1998). The 18th century economist and philosopher, David Hume once wrote, “Nothing is more dangerous to reason, than the flights of the imagination” (as cited in Rose, 1966, p.348). Indeed, mathematics instructors tend to spend most of the time developing their students’ logical and analytic abilities and fail to take into account the most powerful faculty their students possess—the imagination.

Hilbert’s contempt for his former student seems appropriate considering the very nature of mathematical activity involves numerous concessions to the imagination: We imagine when we ascribe a number to an object or collection of objects; we imagine when we perceive a variable to be something other than a letter or symbol; we imagine when we consider the magnitude of an infinite set (Pappas, 1989); we imagine when we regard numbers as points on a straight line (Bazzini, 2001); and we imagine when we use our familiarity with two dimensional space to grasp the idea of other dimensions (Dialogue, 2005). According to Rose (1966), taking a point of view outside of the imagination gives only a partial view of experience. We are restricted to the objective here and now. Understanding mathematical concepts often involves seeing things in new ways and venturing beyond our limited perspective. According to Mazur (2003), there are times when some new mathematical concept, image, or viewpoint resists emerging and we have to "angle for it" in the realm of the imagination (p.5). As Egan (2005) so elegantly puts it, "Engaging the imagination is not a sugar-coated adjunct to learning; it is the very heart of learning" (p.36).

2.2 Imagination and Emotions

The imagination is tied to the emotions in complex ways (Egan, 2005). According to Rose (1966), even the slightest act of the imagination is accompanied by a charge of emotion. Lakoff and Johnson (as cited in Gadanidis and Hoogland, 2002, p1-2) state that, "emotion is inextricably linked to perception and cognition, and is better understood as the tension or excitement level produced by the interaction of brain processes of perception, expectation, memory, and so forth." A discussion of the intricacies of the imagination is beyond the scope of this thesis, but what is crucial here is that, if we are to engage students with the subject matter, there must be an emotional dimension to our teaching. Mathematics education, however, is not generally concerned with eliciting emotional responses from students (Hoogland, 1998). Yet, the knowledge we intend to impart to students is characteristically human—derived from human hopes, fears, and passions. If we are to make knowledge meaningful for students, then we must introduce it in the context of these human emotions—and the imagination is the best tool for accomplishing this task (Egan, 2005).

2.3 Story as a Cognitive Tool

According to Egan (1997, 2005) and Zazkis and Liljedahl (2005), there are several different cognitive tools available to students that can be used to engage their imaginations and shape their emotional understanding of the content. Of these, the cognitive tool of story plays an especially significant role (Zazkis & Liljedahl, 2005). In this chapter, we will explore the very nature of this powerful cognitive tool and its potential to incite the imagination. We will first look at how stories have been used to pass down important cultural information from one generation to the next. They help us understand our roles in society and how we should interact with its members. This is largely due to the way stories orient the hearers' emotions in a particular way—they tell us how to feel about their content. In this regard, stories play a significant role in establishing meaning. Following this, I will argue that stories are the fundamental way the mind works—we think, dream, hope, fear, and love in narrative. Finally, I contend that children are especially predisposed to using stories and that using stories may be a natural and familiar way to pass on important information to our students.

2.4 Story and Culture

Since its earliest uses, story has been a fundamental method for transferring knowledge and culture from one generation to the next. Fables, fairy tales, allegories, poems, and nursery rhymes were more than just vehicles for providing entertainment, they communicated important cultural information in a memorable way. "Embedded within the colourful tales of the storyteller, one can find valuable lessons in history, religion, philosophy, science, and mathematics" (Gandz, 1936, p.93). These stories reflected what information that society deemed important, the beliefs of that society, and what values its members were to adhere by (Schiro, 2005). Stories informed people what their place was in society and how they were to interact with it. A good story should "include the values we want them [the listeners] to hold, the characteristic traits we want them to exhibit, the views of the world and of themselves we want them to cultivate" (Jackson as cited in Hoogland, 1998, p.82). Jackson further asserts, "a story is successful only if its listeners or readers adopt its values as truth and change their behaviour accordingly" (Hoogland, 1998, p.83). In other words, stories have the power to not only *inform*, but also to *transform* the listener.

2.5 Story and Emotion

The power of stories to transform lies in their ability to engage the listeners with the emotions they elicit (Hoogland, 1998). Imagine for a moment we are children again hearing the story of *Little Red Riding Hood* for the first time (Perrault, 1996). What thoughts might go through our minds as we learn of the young girl's plan to travel through the dark and foreboding woods? Even though we are unaware of what awaits her, we feel apprehension. Nevertheless, with our imaginations, we reluctantly follow Little Red Riding Hood hoping she will avoid almost certain danger. Our apprehension turns to fear when she meets the wolf. Perhaps she will perceive the danger and turn and run home? Instead, to our further trepidation, she begins a conversation with this conniving character. The wolf says, "You know it is not safe to walk in these woods alone." Our hearts begin to race. Still, Little Red Riding Hood, unaware of the immediate danger—and our mounting fears—innocently tells the wolf where she is going. Finally, to our horror, the wolf devours Little Red Riding Hood and her grandmother.[1] We feel despondent and lost. Our world has become a terrifying and unstable place. Clearly, our emotions are engaged with the events of the story and we are eventually compelled to feel a certain way—fearful of dark and foreboding woods, and more importantly, fearful of trusting strangers (Egan, 2005). This is the true power of stories—they not only solicit an emotional response from the listener, but they solicit a particular emotional response. Stories tell us how to feel about their content and how to respond (Egan, 2005).

2.6 Story and Meaning

We ascribe emotional meaning to the events in a story (Egan, 2005). A story tells us what it means to be safe or in danger, a hero or villain, good or evil, courageous or cowardly (Schiro, 2004). Good stories stir our emotions and offer us opportunities "to use our mind in fresh experimental ways, to flex our emotions, to enjoy, to learn, to add depth to our days" (McKee, as cited in Gadanidis and Hoogland, 2002, p.3). Traditional stories offer solutions to the conflicts they raise—that is, they end. It is this sense of ending that provides meaning (Egan, 1986). Stories help us make sense of the world we live in and see alternative ways of being in that world (Roach & Wandersee, 1995; Hoogland, 1998).

[1] More recent versions of this story tend to end with Little Red Riding Hood and her grandmother rescued by a lumberjack—a more pleasant ending, but the true emotional impact of the original story is lost (Perrault, 1996).

2.7 Story and Individuals

We use stories in our daily lives as well to give meaning to our experiences.

> We dream in narrative, remember, anticipate, hope, despair, believe, doubt, plan, revise, criticize, construct, gossip, learn, hate, and love by narrative. In order really to live, we make up stories about ourselves and others, about the personal as well as the social past and future (Hardy, as cited in Schiro, 2004, p.50).

We are much like actors in real-life stories, where "we walk on stage into a play whose enactment is already in progress—a play whose somewhat open plot determines what parts we play and toward what denouements we may be heading" (MacIntyre, as cited in Gadanidis & Hoogland, 2002, p.2).

Many argue that storytelling is not a conscious or deliberate activity, but the basic way the mind functions (Schiro, 2004). According to Fuller (1999) the brain organizes bits of information into what he calls "story-engrams", or miniature stories. Unlike computers, humans build complicated cognitive structures with story-engrams that make possible the understanding of relationships and meanings. Independently, bits of information cannot function as thinking units--that is, unless we fit them into a narrative structure.

Frank Smith (as cited in Schiro, 2004) elaborates this when he writes:

> Thought flows in terms of stories—stories about events, stories about people, and stories about intentions and achievements...We learn in the form of stories. We construct stories to make sense of events. Our prevailing propensity to impose story structures on all experience, real or imagined, is the ultimate governor on the imagination... The brain is a story-seeking-creating instrument. We cannot help thinking in terms of stories, whether we are recalling the past, contemplating the present, or anticipating the future... When we say we cannot make sense of something, we mean that we cannot find the story in it or make up a story about it. We look at life in terms of stories, even when there is no story to

> be told. That is the way we make sense of life: by making stories. It is the way we remember events: in terms of stories (p.50).

What is important here is that stories are not *one* way we make sense of the world—they are the *only* way we make sense of it (Egan, 2004).

2.8 Story and Children

It may not be surprising then to realize that children are especially predisposed to using stories. We see this in the way they play, assuming characters from the stories they hear and re-enacting scenes that capture their imaginations (Teaching Storytelling, 2007). "By creating narratives, they act out concepts and ideas that confuse them or that they find fascinating. Examining the world in this way they are able to explore and relate to the variety of topics and subjects that the world throws at them" (Black & Lee, 2005, p.1). Within the context of their fantasy play, children explore their own emotional needs, trying on new identities and different voices, sometimes subverting existing power relations. Finding ways to ease Little Red Riding Hood's fears may help children deal with their own fears (Hoogland, 1998). After listening to stories, children want to write their own stories and tell them to others (Teaching Storytelling, 2007). Students entering high school come with pressing emotional needs, and since they already know how to shape their thoughts and feelings into stories, lessons and exercises involving stories may be a natural and familiar way to introduce students to new concepts and ideas (Hoogland, 1998).

3 THE ROLE OF STORY IN THE LEARNING OF MATHEMATICS

3.1 Renewed Interest in Stories

In recent years there has been a renewed interest in the role of stories in mathematics education, both as a means of disseminating knowledge, and as a cognitive tool for understanding mathematics (Senechal, 2005). Frucht (as cited in Lipsey and Pasternuk, 2002) insists, "Using mathematics to tell stories and using stories to explain mathematics are two sides of the same coin. They join what never should have been separated: the scientist's and the artist's ways of uncovering truths about the world" (p.1). This renewal coincided with Andrew Wiles' solution to Fermat's Last Theorem in 1995. His prolific proof received international public attention and was followed by the best-selling book, *Fermat's Enigma* by Simon Singh. With mathematics suddenly becoming fashionable, several works, both fiction and non-fiction have appeared that draw their subject from the field of mathematics (Tomlin, 2005). Zero, π, and irrational numbers became the subject of many popular books, while films such as *Proof, A Beautiful Mind*, and *Good Will Hunting* deal with such advanced topics as number theory, combinatorics, and graph theory (Senechal, 2005). Even more indicative of this cultural trend is the popular prime-time television show, *Numb3rs*, in which a young mathematician helps the FBI solve crimes using mathematical theories and equations (Tomlin, 2005). At a recent conference at Mykonos in Greece, mathematicians, mathematics educators and researchers, historians, philosophers, writers, and artists came together to discuss what role stories and various other narrative forms can play in the learning of mathematics at the secondary level (Senechal, 2005).

Stories can be used in the classroom in very practical ways to serve different educational goals. As an art form, the ultimate goal of a story is the telling of the story itself, and everything else contributes to that end (Mazur, 2005). In mathematics education however, story is a powerful cognitive tool that can assist in providing meaning to the seemingly irrelevant facts, algorithms, and procedures students learn in class. In some cases, a story may be loosely tied to the mathematical content, while in others, the story and the mathematics are one in the same. In this chapter, I will look at the many ways stories can be used to engage students' imaginations with the secondary mathematics curriculum. For this purpose, I rely on the classification of stories laid out by Zazkis and Liljedahl (2005). First, I will look at stories that ask a question, or "word problems" as they are more commonly called. I will argue that presenting students with problems using familiar natural language and situations, may assist students in making the transition to working with more abstract representations (Koedinger et al., 1999, 2004). Next I will look at how some stories simply accompany a subject in mathematics and have little to do with establishing mathematical meaning. In most cases, these are just brief diversions to alleviate some of the monotony of the "real work". Following this, I will show how stories can be used to introduce a new topic or idea. These types of stories often place the mathematics within a human context, delineating the hopes, fears, and dreams of those who first discovered it. Another type of story I will look at is one that intertwines with a subject. Here, the story and the mathematics are almost inseparable—each one depending on the other. This type of story may also take the form of “pure mathematical narrative,” where the mathematics itself is the story (Mazur, 2005). Next I will look at stories that explain a concept. These types of stories can give meaning to the facts, algorithms, and rules associated with a particular concept. This is often accomplished by recognizing the patterning nature of the mathematics involved. Finally, I will show how stories can be used to introduce activities. Whether writing stories or creative project work, activities can reinforce the ideas presented through stories. According to Schiro’s hierarchy of engagement (2004), students gain a better understanding of the mathematics they encounter in stories when they have to do the mathematics themselves. I will now examine each story type in greater detail paying particular attention to its potential for imaginative engagement (Zazkis & Liljedahl, 2005).

3.2 Stories to Ask a Question

One type of story is a story that asks a question (Zazkis & Liljedahl, 2005). Students are already familiar with “word problems” as they are more commonly called. These rudimentary stories were introduced into mathematics lessons as an effort to demonstrate to students that the facts and rules they encounter at school have relevance in the real world (Cotti & Schiro, 2004). It was believed that if students saw mathematics in some real-life situation—something external to the development of mathematics itself—it would give some meaning to it (Mazur, 2005). However, these word problems had little influence on students. This is due in part to the way school mathematics was perceived. It was believed that the essence of school mathematics is learning symbolic, objectifiable facts, algorithms, and processes (Schiro, 2004). Because mathematics is so often presented in this way, oral and written communication about mathematical ideas is not always recognized as an important part of mathematics education (NCTM Standards, 2000).

The word problems students encounter at school still reflect this belief. Devoid of any character development, dramatic plot lines, or attractive settings, word problems are concise, lack cohesion from one another, and have little emotional impact (Schiro, 2004). “What is left is an empty shell of a story with emphasis on the question mark at its end” (Zazkis and LIljedahl, 2005, Chapter 1, p.7). Word problems hardly resemble the stories they evolved from.

> Mathematics instruction tended to remove the story from mathematics, the story that described the historical origins of mathematics, the story of mathematicians’ practical achievements, and the story of how people have imagined mathematics might be used in the real everyday world and in their dream and fantasy worlds (Schiro, 2004, p.47).

1. Students find word problems uninteresting and irrelevant and by the time they come to them after the “really important” symbolic work, they are often exhausted and bored (Schiro, 2004). This immediate focus on “symbolic” representations of problems may be unwarranted. Research by Koedinger and colleagues (1999, 2004) shows that presenting problems first using familiar natural-language and situated within familiar contexts, as word problems often do, may actually better equip students to approach abstract representations of problems. According to Schiro (2004), problems can be represented using language that corresponds to varying degrees of mathematical abstraction. “Grounded representations” or “natural-language representations”, like word problems, contain familiar words and reference familiar objects and events. “Abstract representations”, like symbolic equations, on the other hand, are concise, easy to manipulate, but have no familiar frame of reference in the real world (Koedinger et al.,

1999). The symbols and notation of abstract representations at the high school level are quite different from the words and syntax commonly used in English or in the simpler arithmetic learned in elementary school (Stein, 1996). This might account for why many of the same students who have difficulty in mathematics are also described as interested and motivated by English and history teachers (Halpern & Halpern, 2006).

3.2.1 Instructional Implications

So how can word problems be used effectively in the secondary classroom in a way that will impact student learning?

First, word problems need to be revamped—they need to return to the roots from which they came. I suggest that mathematics instructors infuse their word problems with dynamic characters, exciting plots, and elaborate contexts—that is, make them into stories. In this way, students will look forward to reading them and finding solutions to the mathematical problems embedded within.

Second, because students enter high school with weak algebra-comprehension skills, mathematics instructors might present students with problems first represented at a natural-language level, and then systematically move to higher levels of abstraction (Schiro, 2004). This approach complies with the NCTM Standards (2000) that discourages teachers from prematurely imposing formal mathematical language:

> Students need to develop an appreciation of the need for precise definitions and for the communicative power of conventional mathematical terms by first communicating in their own words. Allowing students to grapple with their ideas and develop their own informal means of expressing them can be an effective way to foster engagement and ownership (p.62).

The teacher could start students with word problems and then proceed to problems with only familiar natural language that have no situational cues, and then finally move to abstract equations. This approach is quite different from the traditional way teachers incorporate word problems into their instruction (at the end of the theoretical number work)—if even included at all (Schiro, 2004).

Another approach would be to have students engage in activities where they represent problems in both natural-language and abstract form (Koedinger & Nathan, 2004; Schiro, 2004). These activities might include: matching abstract equations to analogous word problems; translating abstract equations to word problems and solving them both; summarizing the solution to a word problem in abstract form; and showing students the symbolic nature of their own verbal strategies and developing general principles in abstract equation form (Koedinger & Nathan, 2004). This approach considers the fact that students at the secondary level are only beginning to think theoretically (Egan, 2004) and require more opportunities to further develop these more complex cognitive processes.

3.3 Stories to Accompany a Topic

3.3.1 Archimedes

Another type of story is a story that accompanies a subject (Zazkis & Liljedahl, 2005). Most students will remember the ancient Greek, Archimedes of Syracuse from their mathematics or science class. There are two familiar stories about his life. The first account is given by the Roman architect, Vitruvius, and describes how he discovered the principle of buoyancy (Dunham, 1994):

> When he went down to the bathing pool he observed that the amount of water which flowed outside the pool was equal to the amount of water of his body that was immersed. Since this fact indicated the method of deciding whether the King's crown was made of gold, he did not linger, but, moved with delight, he leapt out of the pool, and going home naked, cried aloud that he had found exactly what he was seeking. For as he ran he shouted in Greek, eureka, eureka (p.27).

The other story is an account of the dramatic way Archimedes met his demise. The Roman general, Marcellus, wrote that one of his soldiers killed Archimedes in 212 B.C. (Hawking, 2005). Tradition says, the soldier came upon Archimedes who was studying circles he had drawn in the sand. Archimedes was so taken with his problem, that when the soldier told Archimedes to stop what he was doing, Archimedes arrogantly told the soldier to move out of the way so that his shadow would not interfere with his drawings. Offended by Archimedes' disrespect, the soldier ruined Archimedes' figures and ran him through with his sword (Zazkis & Liljedahl, 2005).

Both of these colourful tales will engage the imaginations of the listeners. What student is not fascinated with people running around naked in public or an execution by sword? In terms of mathematics, the first story illustrates the sheer ecstasy of discovering a mathematical truth, so much so, that rational thinking is ignored. The second story shows the degree to which someone might be engaged with mathematics—even to the point of ignoring the dangers around them (Zazkis & Liljedahl, 2005). However, both stories are what Mazur (2005, p.4) calls, "raisins in the pudding." They are stories or story fragments meant to provide some relief from the dreariness of the main task of mathematical exposition. These "anecdotal digressions" neither help bring students closer to understanding the mathematical content, nor do they "fit in as part of the structure of the argument presented" (Mazur, 2005, p.4). There are several familiar stories that are ornamental in nature, many of which are derived from the history of mathematics, and that often focus on the lives of mathematicians (Galois killed in a duel, young Gauss' boredom in math class, or Hippasus drowned at sea are just a few) (Mazur, 2005; Zazkis & Liljedahl, 2005; Stein, 1996). Though not required, any lesson that does not contain these ornaments runs the risk of being mundane and lifeless. The greater risk however, is that these "raisins" often outweigh our primary purpose—to help students understand and find meaning in the mathematical content (Mazur, 2005). As Gadanidis and Hoogland, (2002, p.4) suggest, we need to "reverse the direction of the aesthetic flow so that it originates in the mathematics." Archimedes' *Eureka* story is no longer a "raisin" when it is attached to the more mathematically rich story of King Hieron's crown (Mazur (2005). In this part of the story, the king of Syracuse suspected that the goldsmith, who he had commissioned to make a beautiful gold crown, had secretly removed some of the gold and had replaced it with the same weight in silver. The king asked Archimedes to find a way to prove his suspicions. This eventually leads to the Archimedes' bathing epiphany mentioned previously (Guzman, 2000). As opposed to the second story that ends in Archimedes' death, the "Eureka" story leaves the door open for mathematical exploration into the concepts of volume and buoyancy. This is what Zazkis and Liljedahl (2005) call, a *story that intertwines* with a subject.

3.4 Stories to Introduce an Idea

Stories can also be used to introduce a mathematical idea or concept (Zazkis & Liljedahl, 2005). "Origin stories" are commonly used to introduce new topics and answer the question of how the mathematician came to work on the material and how the ideas were originated (Mazur, 2005). Goldstein, author of, *The Mind-Body Problem,* sees mathematicians as crystallizing the human predicament. “The mathematician is the classic hero, seeking entrance to the realm of immortals, a juxtaposition of transcendence and cluelessness” (as cited in Senechal, 2005, p.9). According to Egan (2005, p.80), “All knowledge is human knowledge” and when students see a concept through the emotions that were involved in its discovery, they will grasp its deeper human meaning. It is a “blending of the drama of the discovered with the drama of the discoverer” (Senechal, 2005, p.8). The facts and algorithms we teach our students will no longer be viewed as meaningless symbols and abstract ideas, but as the product of human passions, hopes, and fears (Egan, 1997). Research by VanSledright and Brophy (1992) reveals that elementary age students are particularly interested in the past, and are concerned with human motives and cause-effect relationships. This propensity for historic understanding does not go away as students develop Egan’s various kinds of understanding. Instead, it becomes a search for patterns and overarching schemas in order to make sense of the present and how they fit into the world around them (Egan, 2005).

3.4.1 Descartes’ Fly

Many textbooks begin an introduction to graphing with a brief account of how the French mathematician, Rene Descartes, developed the Cartesian coordinate system. In most cases however, this part of the lesson is either skipped or given little attention. Instead, students are quickly ushered into the theoretical number work—*x* and *y*-coordinates, plotting points, graphing lines, etc. In a fanciful version of the story, Descartes is lying in bed one day pondering life, when he notices a fly walking on the ceiling.[2] After watching it for some time, he thinks about how he can describe the fly's position at any given moment. He soon realizes that he can easily do this by determining how far it is from each of the two adjacent walls—hence the discovery of the Cartesian coordinate system (although we know this type of positioning system was already in use by the ancient Greeks for mapping and geometry). Here, the teacher could ask students to think about where they might have already seen such a system (i.e. maps, theatre seating, Battleship, etc.). Students could be invited to further explore how the distance between two flies on the same horizontal or vertical line might be calculated, and then use this data to come up with a way to determine the distance between two flies no matter where they are. Students will eventually arrive at the Pythagorean theorem. They might next consider how certain geometric figures can be identified on the plane, such as a horizontal or vertical line or even a circle (NCTM Standards, 2000). Taking it further, they could consider how a fly might be identified on a spider's web as an introduction to alternate coordinate systems. Though they may not, on their own, derive a formula for the equation of a line, or re-invent polar coordinates, their explorations will only enhance their intuitive understanding of coordinate geometry. The development of the Cartesian coordinate system is significant in the history of mathematics because it is the moment at which two seemingly distinct and unrelated fields of mathematics—geometry and algebra—were finally united (Devlin, 1997; NCTM Standards, 2000). Descartes commented about his discovery, "I would borrow the best of geometry and of algebra, and correct all of the faults of the one by the other" (Mazur, 2003, p.78). Without this narrative context however, students may never realize the true significance of this branch of mathematics. Furthermore, by introducing students to this material in the context of

[2] Adapted from Glass and Walz (1998).

how these ideas first originated gives them an opportunity to discover for themselves the interconnectedness of the facts, algorithms, and processes they are learning. Recognizing these connections does not always come naturally for students. According to the NCTM Standards (2000):

> Students are unlikely to learn to make connections unless they are working on problems or situations that have the potential for suggesting such linkages. Teachers need to take special initiatives to find such integrative problems when instructional materials focus largely on content areas and when curricular arrangements separate the study of content areas such as geometry, algebra, and statistics. Even when curricula offer problems that cut across traditional content boundaries, teachers will need to develop expertise in making mathematical connections and in helping students develop their own capacity for doing so (p.358).

According to "conceptual change theory" (Roach & Wandersee, 1995), the more a student knows about a concept—including the history of the ideas—the more connections they can make with other ideas, and thereby establish meaning.

3.4.2 Karl Freidrich Gauss

Another familiar story involves the young Karl Freidrich Gauss (Zazkis & Liljedahl, 2005):

> Young Karl often misbehaved in his math class. This was because he would complete his assignments much sooner than the other students and would become bored. One day the teacher grew impatient with Karl's behaviour and gave him an arithmetic problem to solve as punishment. The problem was to add the numbers 1 to 100. The teacher assumed the calculation would keep young Karl busy for some time, but to the teacher's embarrassment, Karl shouted out the answer even before the teacher finished writing the assignment on the chalkboard.

How was Karl able to do this? The teacher, at this point, can ask students to discover Karl's strategy. Eventually, students will arrive at the "Gaussian Pairing Method". In this strategy, the numbers 1 to 100 are arranged in 50 pairs: 1 and 100, 2 and 99, 3 and 98, and so on. Each of the 50 pairs sums to 101, so the result is given by 50×101 which equals 5050. From here, the teacher can ask students to sum the first 200 numbers, or the first 1000 numbers. With a little guidance, students will come up with a more general formula for this calculation, and in the process, discover the formula for the sum of the elements in an arithmetic sequence (Zazkis & Liljedahl, 2005).

There are many stories that can be used to introduce rich mathematical content and that will engage the imaginations of students. Not all of these are origin-stories, as our discussion up to now might suggest, but many are derived from various sources. Stories like, *The Bridges of Konigsberg* can be used to introduce Euler paths and circuits, *The Tower of Hanoi* to explore mathematical induction and logic, and *The Story of Tan* to study geometric shapes, area, and polynomials (Zazkis & Liljedahl, 2005; Knill, Dottori, Timoteo, Baxter, Fawcett, Forest, Kennedy, Pasko, & Traini, eds., 1996; Schiro, 2005). Many of these stories however, rarely find a place in the formal mathematics curriculum. Instead these stories and the mathematics embedded within them are usually only associated with colourful introductions, enrichment assignments, or “brain-teaser” type activities, that are rarely given a second glance (Zazkis & Liljedahl, 2005). I must inform the reader however, that the purpose of this thesis is not to simply show how stories can be used to engage the listeners’ imaginations, but to show how stories can be used to engage the listeners’ imaginations with the material of the mathematics curriculum. Each of the stories described above are rich in mathematical content and have the potential to, not only introduce ideas, but also become an integral part in the learning of those ideas. In other words, the story is not left behind when the mathematics takes place, but is intertwined with it (Zazkis & Liljedahl, 2005).

3.5 Stories to Intertwine with a Mathematical Topic

The transition from a story that introduces an idea, to one that intertwines with a subject is not necessarily a difficult task. As with Archimedes' "Eureka" story, and those described above, many of these stories already have rich mathematical concepts just below the surface (Mazur, 2005). Consider an exploration of area and perimeter concepts. The NCTM Standards (as cited in Zazkis & Liljedahl, 2005, Chapter 3, p.9) identify the following as important to developing an understanding of area and perimeter:

> Understand such attributes as length, area, weight, volume, and size of angle and select appropriate type of unit for measuring each attribute.
>
> Explore what happens to measurement of a two-dimensional shape such as its perimeter and area when the shape is changed in some way.
>
> Develop, understand, and use formulas to find the area of rectangles and related triangles and parallelograms.

Research shows that a common misconception among students is the belief that a rectangle with a larger perimeter will have a larger area. For example, a 2 by 11 rectangle seems like it would have a larger area than say a 4 by 6 rectangle. Of course, this can easily be disproved by presenting students with several rectangles of various perimeters and having them calculate their areas. However, this somewhat "dry" approach has little imaginative attraction. Instead let me introduce and explore the problem in the context of an imaginative tale (Zazkis & Liljedahl, 2005).

3.5.1 Princess Dido

Consider the plight of the Phoenician Princess Dido from classical Greek mythology. In the *Aeneid*, Virgil tells the story of how Dido and a group of companions sailed across the Mediterranean and landed on the Northern coast of Africa. Virgil explains:

> Here they bought ground; they used to call it *Byrsa*,
> That being a word for bull's hide; they bought only
> What a bull's hide could cover (as cited in Dunham, 1994, p.103)

Dido made a deal with the local ruler of that place, King Iarbus, to give her as much land as she could enclose with the hide of a bull. Although the King wasn't too excited about giving up his land, he figured this was a minor request. Cunningly, Dido tore the bull's hide into long thin thread-like strands, sewed them together, and proceeded to arrange them into the shape of a large semi-circle. From this extraordinary story we learn of the founding of the great city of Carthage (Pappas, 1998). More importantly, however, we discover the origins of the well-known *isoperimetric* problem (*iso*=same, *perimetric*=boundary) that has applications in several fields of higher mathematics, including the calculus of variations, differential geometry, discrete and convex geometry, probability, Banach spaces theory, and many more (Ros, 2001; Dunham, 1994). But for our purposes, "Dido's Problem" affords students an opportunity to access area and perimeter concepts, proportional reasoning, properties of regular polygons and angles, and even binomial products and factoring a difference of squares in a unique and memorable way. According to the NCTM Standards (2000):

> Students develop a much richer understanding of mathematics and its applications when they can view the same phenomena from multiple mathematical perspectives. One way to have students see mathematics in this way is to use instructional materials that are intentionally designed to weave together different content strands. Another means of achieving content integration is to make sure that courses oriented toward any particular content area (such as algebra or geometry) contain many integrative problems—problems that draw on a variety of aspects of

> mathematics, that are solvable using a variety of methods, and that students can access in different ways (p.288).

Consider how area and perimeter concepts, algebraic equation manipulation, and proofs can be intertwined with *The Story of Princess Dido*. As I am doing the mathematics, I will stay with the story, instead of leaving it behind (Zazkis & Liljedahl, 2005).

Before telling students how Dido cut the hide, the teacher might ask students to come up with their own ideas for how to do this. Remember the trick where you tell someone you can make one cut in a letter size paper so that you can walk through it? The teacher might show this trick to students after they have considered the problem for a while. The real task though, is how to maximize the area of the shape given a fixed length (the long thin strand of bull's hide). Students might imagine they have a 600 m length of bull's hide and then figure out various configurations. The teacher could tell students that the chieftain gave Dido the restriction that she must only use rectangular shapes or the deal is off. In this way, the teacher is introducing constraints to the mathematical problem through the events of the story (Zazkis & Liljedahl, 2005). Students will soon discover there are many different areas that can be contained by this amount of rope under this constraint. For example, a $200\text{ m} \times 100\text{ m}$ rectangle has a perimeter of 600 m, and an area of 20000 m^2, whereas a long narrow rectangle with dimensions $1\text{ m} \times 299\text{ m}$ also has a perimeter of 600 m, but only an area of 299 m^2 (Dunham, 1994). The students will eventually see that at a square will encompass the maximum area.

The teacher might want students to generalize their findings using a simple proof: Consider a fixed perimeter arranged in the shape of a square having side x, with area x^2. Now if we were to shorten the height of this square by a units, we would have to lengthen its base by a units in order to maintain the perimeter. The area of the resulting rectangle is given by $(x-a)(x+a)$. When expanded, we have x^2-a^2, which students might recognize as the difference of squares—a beautiful connection between algebra and geometry the teacher might want to explore at this time. From this last expression, we can clearly see that the resulting area will always be less than x^2 (the area of the original square) (Dunham, 1994).

Again staying with the story, the Princess, using her charms, convinces the chieftain to let her use triangles, and then other regular polygons, increasing the number of sides with each attempt. Students can explore how these different configurations have an effect on the enclosed area—eventually discovering that a circle is the best shape to maximize the area. Again, students can generalize their findings: Consider a square with perimeter x and a circle with circumference x. Then the length of the side of the square is $\frac{x}{4}$, with area $\left(\frac{x}{4}\right)^2=\frac{x^2}{16}$. The radius of the circle is given by $r=\frac{C}{2\pi}=\frac{x}{2\pi}$, since the circumference is x. The area of the circle is then $\pi\cdot\left(\frac{x}{2\pi}\right)^2=\pi\cdot\frac{x^2}{4\pi^2}=\frac{x^2}{4\pi}$. Given that $16>4\pi$, the area of the circle, $\frac{x^2}{4\pi}$, is greater than the area of the square $\frac{x^2}{16}$ (Pappas, 1998).

Evidently, Dido arranged the bull's hide in the shape of a semi-circle (one side running along the coast) (Dunham, 1994). The ideas encountered in Dido's problem can lead to an exploration of other related concepts. “High school students are ready to look for connections among the many mathematical ideas they are encountering” (NCTM Standards, 2000, p.64). For instance, the method of maximizing the volume of a rectangular prism is suggested by the method for maximizing the area of a rectangle. In the same way, the volume of a sphere is connected to the area of a circle.

> As students develop a view of mathematics as a connected and integrated whole, they will have less of a tendency to view mathematical skills and concepts separately. If conceptual understandings are linked to procedures, students will not perceive mathematics as an arbitrary set of rules (NCTM Standards, 2000, p.64).

3.5.2 Kathy’s Wardrobe

This time, let me “reverse the direction of the aesthetic flow so that it originates in the mathematics” (Gadanidis & Hoogland, 2004, p.4). Consider the following abstract representation of a problem in set theory a teacher might give students (Zazkis & Liljedahl, 2005):

$$\mathrm{A} = \{x, y, z\},\ \mathrm{B} = \{1, 2, 3, 4\},\ \ |\mathrm{A} \times \mathrm{B}|?$$

Let me recast this using some familiar natural language:

If set A has 3 elements and set B has 4 elements, how many elements are in the Cartesian product $\mathrm{A} \times \mathrm{B}$?

Now consider the same problem as a word problem :

Kathy has 3 skirts and 4 tops. How many possible outfits can she wear?

This word problem situates the mathematics in a familiar context (choosing what to wear is an important consideration for any teenager), and uses familiar natural language (“skirts”, “tops”, “how many”, “outfits”, “wear”) (Koedinger et al., 1999, 2004). As I have already shown in section 3.2, grounding the problem representation in this way might make it more meaningful to students, but to truly engage their imaginations, some further elaboration might be in order (Zazkis & Liljedahl, 2005):

> Kathy woke up late for school today. She had no time for a shower since the school bus would be at the corner in 5 minutes. But she had to wear something nice considering it was the day before the dance, and she might see Jim…[the reader sees where this is going]. She looked in her dresser to find it was empty save for 3 skirts and 4 tops. There was a black skirt, a red skirt, and a green skirt. The tops were white, pink, yellow, and blue. Maybe she should wear the black skirt and yellow top? Or maybe the red skirt and white top? How many different outfits could she wear to school today?

Again, the story is intertwined with the mathematical content. Students might use an “exhaustive counting” method to arrive at a solution, but then the deeper mathematical ideas may not be discovered. The teacher could extend the story by adding more choices to Kathy’s wardrobe. She might have 35 hats in her drawer, which would make counting difficult. Eventually students will see they could simply multiply the number of elements in each set to arrive at a solution. In this way, the students learn to generalize mathematical concepts from specific cases—an activity that requires a great deal of imagination (Dialogue, 2005). The teacher could add constraints through the story, such as adding pants to the drawer, which would further engage students’ imaginations with the mathematics. Kathy cannot wear both a skirt and pants and so students would have to consider how the solution changes. At the end of this exercise, the teacher could lead students in a discussion of the “fundamental theorem of counting” (Zazkis & Liljedahl, 2005). It is evident from the example above that the telling of a story weighs heavily as to how much a story intertwines with a subject as opposed to simply introducing one (Zazkis & Liljedahl, 2005).

3.5.3 Pure Mathematical Narrative

Perhaps the highest degree to which a story can intertwine with the mathematics is in the form of what Mazur (2005) calls "pure mathematical narrative". In this type of story, the mathematics itself is the story, and the drama lies wholly in the way mathematical ideas unfold. According to Egan (1985), if a story is to keep our attention, there must be some type of conflict or tension present that gives rise to a pattern of suspense. There must be some precipitating event that breaches "canonical" behaviour (Bruner, 1991). It should force the listener to ask, "What happens next?" In this type of story, however, the conflict is not between two mathematicians, or two schools of thought, but it occurs within the realm of ideas (Senechal, 2003). Some might call this "intellectual curiosity" (Mazur, 2005, p.7). *It is a Story*, by Barry Mazur (2005), is a story about elliptical curves that illustrates this type of narrative. Some of the most recent advances in mathematics have been tied to the theory of elliptical curves—Fermat's Last Theorem being the most famous. But for the last 20 years, mathematicians have puzzled over the question of "How many rational points do elliptical curves have?" Is it a finite amount, or an infinite amount (Tomlin, 2005)? The conflict arises when you consider the probability that a given elliptical curve has infinitely many rational points. If you consider the large amount of data that has been collected, the probability is approximately $\frac{2}{3}$, whereas most mathematicians believe it to be closer to $\frac{1}{2}$, based on theoretical research (Senechal, 2003). Therein is the drama—whether the data is correct, or the theory? This is the same type of drama found in the incommensurability of the length of the diagonal of a square with the length of one of its sides to the Pythagoreans, or the *indivisible line* to *Xenocrates* (Mazur, 2005). The answer to the elliptical curve problem is yet to be found, but it wasn't until the problem was expressed in narrative terms (an unfolding of ideas in time) that it became clearer (Tomlin, 2005).

While this form of mathematical narrative is engaging to the mathematician for its conundrum and its purity, it leaves most non-mathematicians somewhat less enthused. It appears that a necessary prerequisite for enjoying this kind of narrative is a high level of mathematical sophistication. Most people find mathematical ideas boring and largely inaccessible. "We cannot engage them in a drama of abstraction" (Senechal, 2003, p8). Or maybe we can? Egan (1985) contends that, contrary to many of the current theories, children have a certain proficiency at abstract thought. Combined with the fact that children have a proclivity for narrative, can stories then be used to frame abstract mathematical ideas in a way that engages students' imaginations? So far all the characters in our stories have been human (Archimedes, Guass, Descartes, Princess Dido, Kathy). According to Bazzini (2001), in order to access these higher levels of thought, we often need some other more indirect instrument. Zazkis and Liljedahl (2003) suggest that one way to bring abstract mathematical ideas to life is to cast mathematical objects as characters in our stories. Giving inanimate objects, animals, or concepts human qualities is a familiar technique used in a wide variety of children's literature. *Thomas the Tugboat*, *The Little Engine that Could*, and *Peter Rabbit* are just a few of the many examples. This approach takes into account the way young children perceive the world around them, and their affinity for fantasy play (Egan, 2005). We should however, consider whether students at the secondary level will appreciate talking rabbits or insecure locomotives. Although the tendency to humanize inanimate objects diminishes as children approach secondary school age, Zipes (1982) contends that in later adolescence and adulthood, it regains its fervour—a way to sort of recapture their childhood. Part of this demographic is of interest to us and so I will give the idea of mathematical objects as characters more attention in section 5.2.3 of this thesis. Nevertheless, this literary device has already worked its way into a wide variety of mathematics literature. *The Greedy Triangle*, *The Silly Story of Goldie Locks and the Three Squares*, and *The Dot and the Line* all have familiar geometric shapes as their main characters (Bintz & Delano Moore, 2002). Stories at the secondary level include: *Flatland: A Romance of Many Dimensions*; *The Parrot's Theorem* and *Zero* by Denis Guedj; and *Class Reunion* by Colin Adams (Senechal, 2005). According to the NCTM Standards (2000, p.360) "different representations support different ways of thinking about and manipulating mathematical objects. An object can be better understood when viewed through multiple lenses."

3.5.4 Numberville

Consider how the properties of even and odd numbers can be imaginatively engaging when using this technique:

> Numberville was a sleepy little suburb that lay just over the hill from Mathopolis. Numberville was a small town like any other small town. It had an ice cream shop, barber shop, a grocery store, a Mayor that knew everyone by name, two police officers, one police car, and three stop signs. The train tracks ran right through the centre of town and, I guess, that's where the trouble began. On one side of the tracks lived the Evens and on the other side lived the Odds. And they hated each other! The Odds would claim that the Evens thought they were so much better because of their special characteristics. Life was just easier for them, as for the past 20 years the mayor, as well as both policemen had always been Evens. The Evens, likewise couldn't stand the Odds and their uneasy character. The Evens knew that the Odds were just jealous (Zazkis & Liljedahl, 2003, Chapter 2, p.17-18).

From here, the story could unfold in a number of ways depending on what properties of even and odd numbers are to be explored. For instance, the property that all even numbers are divisible by 2 and odds are not could be elaborated in the story. At the secondary level, students can explore these ideas first by using specific cases and then moving to more general terms. The general expressions students create could be thought of as the "laws" of Numberville. Evens are easily identified on the streets because they are of the form $2k$, where k is any integer. Odds on the other hand, are of the form $2j+1$, where j is also an integer. Perhaps Evens only married Evens and Odds only married Odds. If we imagine "marriage" to mean addition, what does the marriage of two Evens look like? $2m+2n=2(m+n)$. This is still an Even. What about two Odds? $(2x+1)+(2y+1)=2(x+y)+2$. This is an Even! So perhaps when Odds marry, they appear as an Even? What would happen if an Even dared to marry an Odd? $2k+(2j+1)=2(k+j)+1$. The result is an Odd and is probably one reason why the Evens have laws in place against intermarriage! What do the children of married couples look like? If we take "procreating" to mean multiplication, do the children of Odds stay Odds? What do the children of an Even and an Odd look like? This could be another *Westside Story* or *The Outsiders* in the making—but in purely mathematical terms! Convoluted as some of these potential storylines might sound, they offer a unique and imaginative way for students to explore abstract mathematical ideas (Zazkis & Liljedahl, 2005).

3.6 Stories to Explain a Concept

Another type of story is a story that explains a rule or algorithm (Zazkis & Liljedahl, 2005). Mathematical content is mostly perceived as a set of logical facts, algorithms, and skills that students need to learn. Many of these facts, algorithms, and rules have been constructed through centuries of mathematical endeavors and carry with them a part of the rich culture of mathematics we want students to embrace (Schiro, 2004). Disappointingly, direct instruction often promotes rote mechanistic memorization of those facts, algorithms and skills (i.e. invert and multiply, do no divide by zero, negative times a negative equals a positive) (Zazkis & Liljedahl, 2003; Schiro, 2004). This seems counterintuitive to establishing mathematical meaning for students. Schiro (2004) argues, however, that the teaching of facts, algorithms, and rules can be a powerful learning activity that connects students' skills and understandings to real mathematical meaning. "It is a delicate balancing act that in which the teaching of understandings, skills, meanings, and values must be carefully coordinated, sequenced, and synchronized" (p.97). According to Egan (2004), story and story structures can provide the framework for giving meaning to these facts, algorithms, and rules. Even facts, algorithms, and rules need the development of narrative tools in order to be used effectively. We incorporate new skills and facts into our conceptual frameworks more readily according to their significance and emotional meaning rather than chronologically or sequentially. We organize and remember facts and rules more easily in narratives, than from logically ordered lists (Egan, 2004). Few people will remember the precipitation on a particular day five years ago—unless perhaps, it was the day their house flooded, or the day they were splashed by a car while walking along the sidewalk. Likewise, few adults will remember the relationship between the perimeter and area of a rectangle, until it becomes part of a plot involving treachery and seduction, and few students will care if a hexagon has 5 sides or 6, unless it means the difference between being a member of the middle class or the upper classes of society. Stories can better capture the complexity of experience than a list of facts, rules, and procedures (Gadanidis and Hoogland, 2004). More importantly, they provide a familiar way for students to connect the seemingly disjointed bits of information into a meaningful network of concepts (Roach & Wandersee, 1995). According to Bruner (1991), "knowledge and skill are domain-specific and consequently are uneven in their accretion. Principles and procedures in one domain do not automatically transfer to another" (p.2). It is through stories—and connectedly in the realm of the imagination—that students can reconcile these incongruencies.

3.6.1 The Twelve Diamonds

According to Gadanidis and Hoogland (2002), the goal is to transform a difficult topic, such as "division by zero", and give it meaning using students' "aesthetic sensitivities such as symmetry, repetition, rhythm, and pattern" (p.5). Instead of the typical method of telling students they cannot divide by zero and showing instances of when to follow this rule, division by zero can be introduced and explained using stories (Gadanidis and Hoogland, 2004, p.6). Consider the following story I call, *The Twelve Diamonds*:

> A king's will orders that his 12 diamonds be given to his 6 living offspring. The question for the executor of the will is "how many diamonds does each heir receive?" Clearly, each heir receives 2 diamonds, since $12 \div 6 = 2$. Now let's say there are instead 4 offspring. In this case, each heir receives 3 diamonds because $12 \div 4 = 3$. Similarly, if there are only 2 offspring, then each heir receives 6 diamonds. Now, consider the case if there is only one offspring. Then all 12 diamonds will go to this one heir since $12 \div 1 = 12$. Imagine now that the one remaining offspring dies. Now there are zero heirs left. The situation is now undefined since there are 12 diamonds but the question of "how many diamonds will each

heir receive" makes no sense. Therefore division by zero is undefined (Zazkis & Liljedahl, 2005).

One of the powerful features of narrative that works to our advantage is, what Bruner (1991) calls, "*hermeneutic composability*." That is, the meanings one gathers from a text must be seen in light of the constituent parts, and visa versa. In other words, a story has its own internal system of validation—otherwise it is not a story. When we read or listen to a story, we expect that story to cohere and comply with its own internal logic. When an idea is placed within a story, the events of that story must, as a whole, necessarily support that idea (Gadanidis and Hoogland, 2002). "The parts of a narrative serve as functions of the narrative structure as a whole" (Bruner, 1991, p.8). In the example above, we gather from each subsequent division statement, that division by zero is undefined. It is only when we see each "event" in light of the whole story, that we draw the true meaning from the story. To crystallize this idea, consider what would happen if in the last step, the one remaining heir does not die, but instead, one of the diamonds is lost. Our question then becomes $11 \div 1 = 11$, which seems confusing in establishing the overall meaning of the story. According to Bruner (1991), "great storytellers have the artifices of narrative reality construction so well mastered, that their telling pre-empts momentarily the possibility of any but a single interpretation" (p.9).

3.7 Stories to Introduce an Activity

Another type of story is a story that introduces an activity (Zazkis & Liljedahl, 2005). Activities can play a vital role in helping a story establish meaning. Murphy (1999) points out that children often understand difficult mathematical concepts presented within the context of a story only after they engage in activities that support those ideas. According to Schiro (2004), there are three different levels at which students are asked to engage with the mathematics they encounter in stories. At the most basic level, students observe what characters in the story do with the mathematics they encounter, but are not given any explanation as to how the mathematics is done. At the next level, students listen in to a character or narrator's thoughts while mathematics is being done, so that they can see how to do the mathematics. However, students do not do any mathematics themselves. At the highest and most effective level of Schiro's hierarchy, students are required to do the mathematics in the story in order to fully understand both the story and the mathematics within.

Stories often prompt activities that emerge from within the story and that support connections to students' own lives. Sherrie Martinie (Martinie & Bay-Williams, 2003) describes how she used the story, *Wilma Unlimited: How Wilma Became the World's Fastest Woman* to teach eighth grade students the concept of rates. After reading the book, students investigated the rate of travel as they hopped, walked, and ran various distances just like the character in the book. They later graphed the results and discussed their findings as a class. Activities often lead to discussions about mathematical concepts. When children share their work with each other and help each other, they become involved, not only in the individual construction of meaning, but a social construction of meaning as well. This connects them to "a larger and more varied knowledge base, while at the same time adds connectedness and comradeship to the ongoing web of intrigue that forms about their lives" (Schiro, 2004, p.62).

Sara Moore (2002) describes how she used the book *A Cloak for the Dreamer* to provide a context for her unit in geometry. The story involves a tailor and his three sons who were given the task of each creating a cloak using different tiling shapes for the Archduke. One son used rectangles, another used triangles, while the youngest used circles. The problem arose in the design of the youngest son's cloak in that it contained "holes". What alterations could be made to this pattern in order to remove the holes? This story illustrates that the successful completion of a cloak depends on the shape one uses to tile it. At the secondary level, students might use a similar approach to further explore more advanced ways to tile a plane. David Buhl (2003) describes how his students used the legend of Paul Bunyan[3] to engage in three activities that involved clay modelling and connecting the concepts of length, area, and volume to scale factor and proportion. When Doris Lawson (as cited in Schiro, 2004), tells her grade four students *The Wizard's Tale* to help them understand the multi-digit addition algorithm, she has them interact with the characters in the story through various activities. The main characters in the story, Gandalf and Tinkerbell, cannot solve the mathematical problems they encounter on their own and necessarily require the help of Lawson's students. They use magic pens and stone slates; they push base ten blocks around using a magic bulldozer; and they sing songs and make sound effects to communicate their solutions to the characters in the story (Schiro, 2005).

While activities involving "magic pens" and "singing fairies" almost certainly do not belong in the high school setting (I will discuss this later in section 4.4), they do infer something about the interconnectedness of stories and activities. Chris Shore (1998), has developed a five-lesson unit entitled, "Princess Dido: The Geometry of an Ox Skin" in order to explore the relationship between area, perimeter, measurement, and metric

[3] Paul Bunyan was a legendary lumberjack from the forests of the Northern United States. He boasted the size of ten men and was accompanied by an equally extraordinary blue ox.

conversion. The unit begins with students listening to *The Story of Princess Dido*, and then attempting a similar problem using a scale drawing of a "rat's hide." In the next several lessons, students use a bed sheet (the approximate shape of a bull's hide) to continue their exploration of the now familiar isoperimetric problem. Students take the bed sheet and cut it into one long 5 cm wide strip that they can arrange on the school field. Before actually cutting the bed sheet up, students are encouraged to guess how much of the field the strip will cover, and plan out their strategy using graph paper, paying particular attention to scale. It might be pertinent to note here the value of extending an activity over several days. Schiro (2004) suggests that when problems are situated in a lengthier story, students will be more likely to dream about, discuss, imagine, share ideas about, and re-play in their minds the events of the story, what will happen next, and how the activities and related problems emerging from within the story can be solved.

Writing stories is another type of activity teachers might want to incorporate into their lessons. Children are natural storytellers. Long after a story has been told, children can be heard reciting verses, acting out scenes, and manipulating storylines (Teaching Storytelling, 2007). Articulating experiences in stories and sharing responses to those stories, "allow feelings, emotions, and instincts to be confirmed, refined, or adjusted" (Hoogland, 1998, p.87). Not only do student-generated stories reveal what emotional concerns dominate children's thought lives, they also reveal what students know about a topic and whether they can communicate that knowledge effectively (Halpern & Halpern, 2006). Cara Halpern (Halpern & Halpern, 2006) used writing activities to assess her ninth grade students' understanding of polygons. After listening to *Sir Cumference and* *the First Round Table* and *The Greedy Triangle,* students were assigned a project in which they were to write a story about polygons. They could modify a story or fairytale they knew, or create one of their own. One student wrote a play entitled *Goodnight Polygon* in which a mother asks her child if she knows what type of "polygon" their house is. The dialogue that ensues reveals that the student has some misconceptions and incomplete knowledge about two and three-dimensional objects and the terminology used to describe them.

Stories can be used in the classroom in various ways to highlight, introduce, or even explain the mathematical content we wish our students to understand. The degree to which students actually understand the mathematics embedded within a story largely depends on the type of story chosen, as well as the level at which students engage with the mathematical content they encounter (Zazkis & Liljedahl, 2005; Schiro, 2004). Presenting a problem using familiar natural language and recognizable contexts may not be enough. Students must be given a chance to, not only observe what characters in the story do with the mathematics, but also interact with the mathematics themselves and respond to elements in the story that capture their imaginations (Schiro, 2004).

4 SHAPING STORIES TO MAXIMIZE IMAGINATIVE ENAGEMENT

In this chapter, I focus on how stories can be shaped to maximize their effect on students' imaginative engagement with the content. I consider several factors that must be considered when shaping stories, including certain other cognitive tools that emerge with the development of language, literacy, and theoretical thinking (Egan, 1997). Additionally, I look at the contexts and the delivery of stories and how an attention to these can increase a story's ability to engage students' imaginations. The first of the cognitive tools is binary opposites. I will argue that organizing stories within a binary structure helps students to initially grasp the mathematical ideas presented. Additionally, binary opposites play a significant role in creating conflict within a story and help keep the story moving along (Egan, 1986). I will also present several examples of how binary opposites can be woven into any storyline in order to enhance imaginative engagement with the mathematical content. The next of the cognitive tools I will look at is metaphor (Egan, 1997, 2005). Like binary opposites, metaphor is basic to human cognition and language. I argue that in the same way metaphor has been used to shed light on scientific processes, it can also be used in stories to teach and reinforce mathematical ideas. I will further argue that using metaphor by giving mathematical objects human qualities is an effective way to help students see mathematical ideas in a new and unique way (Zazkis & Liljedahl, 2005). Another cognitive tool I will examine is knowledge and human meaning (Egan, 1997, 2005; Zazkis & Liljedahl, 2005). Seeing a mathematical concept in terms of the stories and the human emotions involved in its discovery helps to make it more meaningful. In this regard, how much poetic license should we take in telling our stories—should we keep the bounds between fiction and non-fiction intact? I argue that taking time to develop the most sensational attributes of the characters and settings in a story can enhance its emotional impact. In terms of the contexts of stories, I will also look at whether fantasy or real-life contexts are more effective in engaging the imaginations of students. Rhyme, rhythm, and pattern is another of the cognitive tools that must be considered when shaping stories. The search for patterns seems to be a fundamental characteristic of human nature. I argue that incorporating patterns into stories can make them more inviting as well as highlight important mathematical content. Closely related to this is the cognitive tool of jokes and humour that relies primarily on incongruities in our conceptual patterns. I suggest infusing stories with humorous elements can actually accentuate the mathematical ideas we want students to understand. Finally, I will examine how the delivery of stories impacts their effectiveness—in particular, I will compare the orally told story with the read story.

4.1 Binary Opposites

One of the primary cognitive tools that students have available to them is that of binary opposites (Egan, 2005). It is a characteristic of human thinking that comes with the development of language (Egan, 2005) and that is necessary for the development of our "mythic understanding" (Egan, 1997). In order to initially grasp a concept, we first divide it into opposites. Our minds tend to discriminate new information into stark opposites such as hot/cold, good/bad, high/low, earth/sky, courage/cowardice, and so on (Egan, 2005). We see this "elementary differentiation" particularly in the way children perceive the world around them (Egan, 1997). They describe temperature in terms of hot or cold—“hot” meaning hotter than their body temperature, and “cold” meaning colder than their body temperature. In terms of size, they seem limited to words like “big” or “small” (Zazkis & Liljedahl, 2003; Egan, 1997). Vygotsky (as cited in Egan, 1997, p.38) noted the use of elementary differentiation in children performing simple tasks: “Association by contrast, rather than by similarity, guides the child in compiling a collection.” This process of dividing things into binary opposites seems to be the initial way the mind organizes and orders new concepts. It helps us gain access to new material in a clear and comprehensible way. Once these opposites are established and their principles are understood, we can then use them to ascribe meaning to the various degrees of discriminations in between (Egan, 1997). Children eventually learn that they can use words like “warm” and “cool” to describe temperature or “tiny” and “enormous” to describe size. They soon realize that not every stranger is bad, and that there can be various degrees of both bad and good qualities in a person (Zazkis & Liljedahl, 2005). But this type of elaboration must take place only after binary structuring, or the topic might seem confusing to students, even inaccessible, since they have nothing secure to grasp on to (Egan, 2005). The process of seeing characters and events in terms of binary opposites is not exclusive to children. When we watch the news, we naturally position people or events into opposing categories. That is, we search for facts that help us fit the event or person into this binary structure (Egan, 1989). The media recognizes our propensity for binary opposites and often presents stories in a way that will orient our emotions to the content—binary opposites provide that orientation. One of the keys to holding the viewer’s attention is capitalizing on the inherent binary opposites in a story (Zazkis & Liljedahl, 2005).

Considering that binary structuring is a basic feature of our thinking, it is no surprise that most children's stories almost always rest on such underlying binary sets as security/danger, good/evil, courage/cowardess, beauty/hideousness, love/hate, happy/sad, poor/rich, health/sickness, permitted/forbidden, cleverness/stupidity, familiar/strange, and so on (Egan, 1997, 2005; Zazkis & Liljedahl, 2005). Consider what lies below the surface of many of the classic fairy tales. *Hansel and Gretel*, delineates between the binary opposites of security and danger; *Cinderella* pits poverty against wealth, vanity against modesty, and innocence against corruption; and *Jack and the Beanstalk*, courage against cowardice, cleverness against stupidity or familiar against strange (Egan, 2005). Pitting binary opposites against one another is also at the heart of popular teen and adult fiction (Zazkis & Liljedahl, 2005). In *The Da Vinci Code* (Brown, 2006), we see the binary sets of truth/lies, knowledge/ignorance, and permitted/forbidden embodied in the conflict between *Opus Dei* and the *Priory of Zion*. Horror novels, like those written by Stephen King or Wes Craven, capitalize on the binary opposites of known/unknown, courage/cowardice, and safety/danger. *Harry Potter* and *Lord of the Rings* rely on the use of such binary oppositions as good/evil, ingenuity/banality, and security/danger.

Binary opposites play a key role in developing the plot within a story (Zazkis & Liljedahl, 2005). Settings, characters, and descriptive language are meaningless devices if there is no conflict or crisis to engage the listener (Mazur, 2005). This is what Bruner (1991) calls, "canonicity and breach." The presence of binary opposites provides the opportunity for a "breach" in what is considered right or normal in a culturally defined situation. Without a breach, there is no story. In *Cinderella*, the presence of an evil stepmother provides the means by which a conflict can occur. The further introduction of characters, such as the evil stepsisters, and events, such as the prince's ball, must necessarily further establish the "evilness" of the stepmother and the "innocence" of Cinderella. In other words, the binary opposites that underlie this story serve as criteria for the selection of content. Connectedly, these binary opposites provide the main lines by which the story moves forward (Egan, 1986).

If thoughtfully crafted, the stories we tell can be more imaginatively engaging and meaningful. I suggest that using binary opposites in plot-lines will accomplish this very task (Zazkis & Liljedahl, 2005). As Rose (1966) so elegantly puts it, "all opposites call for reconciliation in the life of imagination" (p.347). I consider here several examples of how this can be done.

4.1.1 Archimedes Revisited

Recall Archimedes' "Eureka" story mentioned earlier. The way it was presented did little to encourage the reader to ask, "What happens next?" Cast in the light of binary opposites, however, this story has the potential to engage the imaginations of students and connectedly draw their emotions in a number of ways (Zazkis & Liljedahl, 2005). For instance, I could cast Archimedes as the town eccentric, who is so intoxicated by his studies, he fails to attend to his personal hygiene. Plutarch writes (Hawking, 2005):

> Often times Archimedes' servants got him against his will to the baths, to wash and anoint him, and yet being there, he would ever be drawing out of the geometrical figures, even in the very embers of the chimney. And while they were anointing of him with oils and sweet savors, with his fingers he drew lines upon his naked body, so far was he taken from himself, and brought into ecstasy or trance, with the delight he had in the study of geometry (p.120-121).

Such an approach pits Archimedes, the eccentric mathematician, against the rational townspeople. I can further develop the plot of our story by accentuating these binary opposites. For example, I could introduce a family that lives above Archimedes. They are so distressed by the stench from Archimedes' room, they plead with the town council to have Archimedes evicted—right at a crucial moment in his investigation of the king's crown! Students may, however, be divided as to what pole they naturally assemble to—certainly body odor is a major concern for students at this stage in their physical and social development. Alternatively, I could cast Archimedes against the tyranny of the king of Syracuse (Zazkis & Liljedahl, 2005). This would require a great deal of embellishment, however, as it is a common held belief that Archimedes was on intimate terms with King Hieron II and that he may even have been related to him (Hawking, 2005). Nevertheless, for the purposes of establishing a conflict, I will position these two personalities within a binary structure. To establish the king as a tyrant, I can elaborate his hunger for power and his thirst for war. At one point, he challenged Archimedes to construct a weapon that would hurl large objects at attacking ships (Hawking, 2005). The king knew war was imminent, as he had recently broken trade agreements with some neighbouring powers. I could say that the king even commanded Archimedes to make these weapons or else Archimedes would be banished—the act of a desperate king! Under much duress, Archimedes managed to fulfill the king's request. Plutarch describes the effct of Archimedes' devices (Hawking, 2005):

> When Archimedes began to ply his engines, he at once shot against the land forces all sorts of missiles, and immense masses of stone that came down with incredible noise and violence, against which no man could stand; for they knocked down those upon whom they fell in heaps, breaking all their ranks and files. Meanwhile huge poles were thrust out from the walls over the ships, and sunk some by great weights which they

> let down from on high upon them; others they lifted up into the air by an iron hand or beak like a crane's beak and, when they had drawn them up by the prow, and set them on end upon the poop, they plunged them to the bottom of the sea; or else the ships, drawn by engines within, and whirled about, were dashed against steep rocks that stood jutting out under the walls, with great destruction of the soldiers that were aboard them. A ship was frequently lifted up to a great height in the air (a dreadful thing to behold), and was rolled to and fro, and kept swinging, until the mariners were all thrown out, when at length it was dashed against the rocks, or let fall (p.120).

In this light, we see Archimedes as the humble genius coerced into making weapons of destruction for a greedy tyrant. I will continue to thicken the plot. The king is so proud of his victory over the Roman armies, that he commissions a goldsmith to make him an elaborate gold crown commemorating the event. Of course, King Hieron is trusting of no one and soon suspects the goldsmith of stealing some of the gold and replacing it with an equal amount of silver (Guzman, 2000). He commands his most trusted advisors to come up with a way to determine whether the crown has been compromised. After several days, however, they return empty-handed. The king flies into a rage and has them immediately executed! A few days later, Archimedes returns home to find all the wise men in the land are dead, and now the task of finding a solution once again falls on him. It is under this trepidation that Archimedes ponders the problem of the gold crown at the bathing pools (Zazkis & Liljedahl, 2005). From this example, we see how carefully constructed binary opposites can provide the impetus for students to engage with the content (Zazkis & Liljedahl, 2005).

4.1.2 Princess Dido Revisited

Consider again *The Story of Princess Dido* (Parada, 2007; Mortal Women of the Trojan War, 2006). An attention to binary opposites can make this story come alive in the minds of students. Why did Dido leave her homeland in the first place? To make it into the annals of Greek mythology, some unearthly conflict must have taken place. Just before his death, the King of Tyre left his two children, Dido and Pygmalion, as joint heirs to the throne. However, having two rulers was unacceptable to the people and so they chose Pygmalion as their sole ruler (one can accentuate the gender issues here as a way to engage students). This was a grave mistake, however, for Pygmalion was an evil man. He was corrupt, greedy, and he demanded the absolute obedience of his subjects. He had heard that his father secretly left all his gold with his uncle, Acerbas—who also happened to be married to Dido (his own niece)—and so he had him killed in order to seize the gold. But warned of Pygmalion's treachery by her dead husband's ghost, Dido took the gold in the night, and fled with a group of loyal followers. I can, at this point, contrast the heroic qualities of Dido with the villainous nature of Pygmalion. Dido was a caring and compassionate leader willing to sacrifice all for the well being of her people. According to Egan (1997), students at this stage in their cognitive development, are fascinated by heroes. As part of their emergent "romantic understanding," our students are interested in the transcendent qualities that allow heroes to venture beyond conventional constraints. They associate with heroes because it allows them to explore their own heroic qualities and the limits of the world around them. "Another feature of human qualities is that they are inevitably emotional in some degree. Heroes engage our emotion" (Egan, 2005, p.89). It is interesting to note here that the archetypal hero in Western culture has traditionally been the powerful male usually performing violent acts. As we have already seen, Princess Dido possesses all the qualities associated with heroes—sanctity, compassion, wit, ingenuity, selflessness, elegance, patience—but without the violence.

All of this can be considered a build up to the rich mathematical problem that would be presented to the princess along the Northern coast of Africa. The real drama of the encounter between Dido and Iarbus is one worth telling as it draws on the binary opposites of ingenuity/incompetence, security/danger, and dominant/subordinate to pose a very pertinent mathematical problem relevant to our curriculum (Egan, 1997). The need for a large tract of land was essential if Dido was to establish a city powerful enough to defend against the eventual attack of her evil brother. So students have some incentive as to why they might engage the mathematical problem presented to Dido. The local ruler, Iarbus, was not interested in giving land to the newcomers and so his offer to give them only “what a bull’s hide could cover” was a devious way of getting rid of them without eliciting much offense. This pits the good-hearted Dido against the crafty Iarbus. But Princess Dido was no fool as we have already seen. Digging deeper into Greek mythology, we find Dido had the favour of the Gods on her side as well! If the reader recalls, Dido was at first limited to rectangular shapes, but after some coercion, Iarbus abandoned this constraint.[4] He was so taken by her cunning, that he desired her for his wife. The rest of the story is for the reader to discover, but the point of this loose retelling of the myth is to show how binary opposites can play a vital role in increasing a story’s imaginative attractiveness.

[4] There was no such constraint in the original version of the story. This was added to my version of the story for instructional purposes.

4.1.3 Hippasos and the Pythagoreans

Almost every mathematical concept at the secondary level can come alive in the minds of students through the use of binary opposites. Let me now consider how binary opposites can be used in an exploration of irrational numbers. According to the NCTM Standards (2000):

> High school students should develop an understanding of the system of real numbers. They should understand that given an origin and a unit of measure, every point on a line corresponds to a real number and vice versa. They should understand that irrational numbers can only be approximated by fractions or by terminating or repeating decimals. They should understand the difference between rational and irrational numbers. Their understanding of irrational numbers needs to extend beyond π and $\sqrt{2}$ (p.291).

How can rational and irrational numbers become meaningful to students? Can the differences between these two sets be connected to any manner of intrigue or suspense? An investigation of the topic normally begins with students converting decimals to fractions and then finishes with students recognizing that certain decimals cannot be written as fractions—a somewhat dry introduction to the idea of rational and irrational numbers. If students only knew what outrageous personalities and circumstances—steeped in stark binary opposites—lie just below the surface of this exciting topic.

The discovery that the square root of two is irrational was not only one of the most important developments in mathematics, but it was also the first real “crisis” in the foundations of mathematics (Crew, 2007). As previously discussed, the drama here lies in the way mathematical ideas unfolded. It is a conflict between theory and data (Mazur, 2005). In order to engage the imaginations of students, I will develop our story along this binary structure.

The “theory” side of this binary set is represented by the Pythagoreans, a mysterious sect, who believed in the sanctity of numbers and their power over reality. Egan (1997) contends that students at this stage of their cognitive development are fascinated with the extremes of experience and the limits of reality (see section 4.4 for a further discussion of this). Accentuating the bizarre and extreme aspects of the Pythagoreans will capture students’ interests and engage their imaginations. The Pythagoreans were a secret organization devoted to a pursuit of knowledge and a lifestyle that included many odd rituals, including the transmigration of souls into other species, which accounted for their strict vegetarianism (although they had an aversion to beans since their leader likened them to testicles), and the worship of Apollo (Egan, 1997). They believed disembodied aliens existed on other planets, and the further one travelled away from the Earth, the purer these aliens became. In fact, if one reached a certain level of purity, an alien might choose to inhabit that person (many of Pythagoras’ followers believed their leader to be inhabited by an alien who came from the Sun) (Pickover, 2005). But how does one reach this level of purity? To the Pythagoreans, the whole numbers from 1 to 10 were divine, pure, and free from material corruption. They were alive, possessing the ability to communicate telepathically with those who sought after them. Meditating on these numbers and their properties was like praying and such a practice could lead to a higher plane of existence (Pickover, 2005). Now this is probably enough information to establish the extreme nature of the Pythagoreans in the minds of students. What about the mathematics? The Pythagoreans were devoted to the notion that "everything is number" and that everything could be measured with whole numbers or a ratio of whole numbers (Harris, 2002). The mathematics here is not abstract, but concrete. For instance, consider a right triangle with sides built on the ratio 3:4:5. The Pythagoreans would represent each side with a series of dots, called "monads" (Hulik, 1993). If a square is built on each side of the triangle it becomes clear that the number of dots that make up the square on the hypotenuse is equal to the sum of the dots of the squares on the remaining two sides. In this way, the Pythagoreans demonstrated that reality, at its most fundamental level, is mathematical (Hulik, 1993).

The "data" side of this conflict is embodied by Hippasos of Metapontum, an eager young Pythagorean who was more concerned with the pursuit of truth and less interested in blindly following the dogmas of an old and outdated sect (students will likely identify with this notion of blind acceptance, particularly, in the way they typically learn mathematics). It was this rogue, Pythagorean that came upon the discovery that the side and diagonal of a square are incommensurable (Benson, 2003). This was disastrous to the very foundations of the Pythagorean way of life as it meant that not everything could actually be understood in terms of whole numbers (Grant, 2003)! Hippasos actually held to the belief that "fire" and not "number" is the first principle in the universe (Culik, 1993). How could the diagonal of a square—a square being an example of symmetrical beauty—be incommensurable with its sides? This was doubly damaging in that the Pythagoreans had a penchant for mathematical symmetry (Harris, 2002). Pappus says about this discovery:

> [The Pythagoreans] sought to express their conviction that firstly, it is better to conceal (or veil) every surd, or irrational, or inconceivable in the universe, and, secondly, that the soul which by error or heedlessness discovers or reveals anything of this nature which is in it or in this world,

> wanders [thereafter] hither and thither on the sea of non-identity (i.e., lacking all similarity of quality or accident), immersed in the stream of the coming-to-be and the passing away, where there is no standard of measurement (Mazur, 2005, p.7).

The Pythagoreans attempted to conceal the discovery of “irrational numbers” made by Hippasos and legend has it that they threw him overboard while at sea. If carefully constructed along these binary oppositions, students can be drawn into an engaging exploration of the irrationality of numbers (see section 6.3.2.1 for opportunities of student engagement).

4.2 Metaphor

Another cognitive tool that emerges during the development of language is metaphor (Egan, 1997). Metaphor is pervasive in both language and thought (Gentner, Bowdle, Wolff, & Boronat, 2001). Levi-Strauss (as cited in Egan, 1997) suggests that metaphor “is not a later embellishment of language, but is one of its fundamental modes—a primary form of discursive thought” (p.54). It is one of the foundations of all human mental activity (Egan, 2005). Simply put, metaphor is the capacity that allows people to see one thing in terms of something else (Egan, 2005). We do not have to look very far to find examples of metaphor (Egan, 2005): “I was walking on eggshells”; “This job is hell”; “It was no walk in the park”; “She got back on her feet after the divorce”; “Don’t dig up the past”; “His eyes were burning coals” (Gentner et al., 2001); and the like. From these examples, we see that metaphorical value can be attached to any object or idea (Hailey, 2007). We frequently make these types of associations to emphasize and give richer meaning to the ideas being expressed in a way, simple literal phrases or sentences cannot (Egan, 2005). In this regard, metaphor is more than just a tool for communication, but it is a way to establish meaning (Schwarz, 1988). It is one of our "cognitive grappling tools" that enables us to see the world in multiple ways and to "engage with the world more flexibly" (Egan, 1997, p.58). "A metaphor, in short, tells us something new about reality" (Schwarz, 1988, p.32).

Writers of fiction and prose recognize our propensity for metaphor and regularly make use of it to define human experience (Hoogland, 1998). We value those writers, like Shakespeare or C.S. Lewis, who can stir our emotions and make our reading come alive through new and carefully constructed metaphors (Egan, 2005). John Bainville, in *Beauty, Charm and Strangeness: Science as Metaphor*, suggests "the trick that art performs is to transform the ordinary into the extraordinary and back again in the twinkling of a metaphor" (as cited in Pickover, 2001, p.265). But is the use of metaphor just an attention-getting decoration characteristic of the arts? Schwarz (1988) contends that metaphor is more than just a wielding of language reserved for the writer or poet, but it is a powerful tool that can be used to help students better grasp new concepts within any field of study. Consider the role metaphor has played in the learning of science: It was not until the invention of the pump that the function of the heart became clear (Egan, 1986); The computer gave psychology a better model—as opposed to the telephone switchboard—for understanding how the brain processes information (Egan, 1986); Einstein has attributed his insights into relativity to his ability to see the cosmos as a particle approaching the speed of light (Egan, 2005); The Earth's atmosphere has been better understood when seen in terms of an invisible shield (Cameron, 2002); Electric current is often studied in terms of water flow (Cameron, 2002).

Metaphor “establishes a new relationship between heterogeneous ideas in a way that adds something to, or throws new light on the thing talked about” (Egan, 1997, p.53). It is a kind of “triangulation” process—where “two old ideas or things are connected by some function, unrecognized until the metaphor makes the connection obvious” and then “old knowledge leads to new insight” (Schwarz, 1988, p.32). In this way, it enables one to quickly transfer “chunks of experience from well-known to less well-known contexts” (Schwarz, 1988, p.32). The use of metaphor therefore, involves an expansion of understanding into new domains—which is certainly of chief concern in education (Egan, 1997). However, in order for students to truly understand the new domain, they must correctly interpret the metaphor. Misinterpretation would undoubtedly hinder learning (Cameron, 2002). It would be pertinent then, to explore the process involved in interpreting a metaphor and the potential pitfalls.

4.2.1 The Shield Metaphor

Consider the following example: “The atmosphere is an invisible shield of air surrounding the Earth.” The conceptual domains underlying the words “atmosphere” and “shield” are distinct, and mapping attributes from one domain to the other creates a metaphor (Cameron, 2002). The words “atmosphere”, “invisible”, and “air” signify the *topic* domain—the less well-known context, while the word “shield” signifies the *vehicle* domain of the metaphor—the well-known context. In order to successfully interpret a metaphor, we must bridge the gap between these two domains, so that the topic is understood in light of the vehicle (Cameron, 2002). This is inherently problematic, as the interpretation of metaphor is neither a single invariant process, nor is it independent of how familiar one is with the particular metaphorical uses of language (Cameron, 2002).

Some of the conceptual difficulties students might have in interpreting a metaphor have been identified.

First, students may not realize that what they hear or read is metaphorical and needs interpretation. More explicit markers of metaphor, such as simile, or direct teacher mediation of the metaphor can easily resolve this problem. Restating the above metaphor as “The atmosphere is *like* an invisible shield of air surrounding the Earth” or “The atmosphere can be compared to an invisible shield of air surrounding the Earth” will not only alert students that interpretation is required, but will also help them to identify the topic and vehicle domains (Cameron, 2002).

Second, students are affected by how much they know about the conceptual domains of the metaphor when attempting to interpret the metaphor. Here, vehicle knowledge seems to be of greatest concern—a student who knows little about “shields” will have difficulty mapping their attributes to corresponding attributes of “the atmosphere”. Choosing a vehicle that students are fairly familiar with or further vehicle development can easily solve this. Cameron (2002) also found vehicle contextualization—some aspect of the vehicle term is related to something outside of the text and more familiar to students—can help students better understand the attributes of the vehicle domain that need interpretation. For instance, including the additional statement “Roman soldiers used shields to ward off enemy arrows” helps to define what type of “shield” we are concerned with and what aspects of shields are of particular interest—their ability to stop something from penetrating through. Cameron (2002) however, has noted that even when vehicle domain knowledge is in place, inaccurate or incomplete topic domain knowledge can also contribute to misinterpretation. Introducing the vehicle domain after topic domain development can help to minimize this problem. This enables the target domain to "override" those cases where the mappings from vehicle to target do not support the intended interpretation (Bazzini, 2001).

Finally, even when students realize that metaphorical interpretation is necessary and they know enough about the vehicle and topic domains, they may still have problems selecting what aspects of the two domains connect to each other. Mapping attributes of the vehicle to attributes of the topic requires a "leap of the imagination" and therefore affords the possibility of "missing the target" (Cameron, 2002, p.681). A "leap of the imagination" is defined as the process of taking some abstract concept and positing in the mind, considering it, transforming and manipulating it like a real object, and finally realizing it in some tangible way (Tomas, 1958). Part of the reason students might miss the target depends largely on the vehicle domain selected. A "shield", while addressing the protective nature of the atmosphere, is limited in its ability to help students understand how the atmosphere performs this function. In this regard, metaphors seem constrained to mapping only simple attributes as opposed to relational properties. Research by Gentner and colleagues (2001) however, suggests that this is not necessarily the case, particularly when higher-order relations, such as causal relations, are present in the metaphor. Moreover, "metaphorical mappings are not isolated, but occur in complex systems and in complex ways" (Bazzini, 2001, p.261). That is, people seldom map random attributes from vehicle to topic, but will instead project inferences that complete the system of relations—in other words, a local instance of metaphor will almost always invoke a domain-level mapping between domains (Gentner et al., 2001). For instance, it would be natural after topic and domain development have occurred to, not only map the attribute that the atmosphere protects life on Earth to the attribute that a shield protects the person behind it, but also to map the way the atmosphere reflects certain harmful wavelengths of light to the way a shield deflects arrows. Sometimes the entire structure of a vehicle domain can be "projected" onto a target domain. In this way, the vehicle domain 'adds' structure to the target domain (Lakeoff & Nunez as cited in Bazzini, 2001, p.261). Another point we must consider is how the "shield" metaphor fails to address other functions of the atmosphere, such as how it keeps useful gases from escaping. Cameron (2002) suggests that this can be remedied by explicitly stating which attributes of the vehicle and the topic domains are to be connected. In most cases, simply paying careful attention to vehicle and topic term adjectives is enough. For example, the adjective "invisible" in the Topic domain adds nothing to the learning potential of the "shield" metaphor and may even create confusion or unintended interpretation (Cameron, 2002).

So far we have seen that metaphor is basic to human cognition and language, and that if used correctly, can be a powerful teaching tool. According to Egan (2005), the ability to recognize and generate metaphors seems to peak around age four and thereafter goes into steady decline, only making a slight spurt again during puberty. If we are to keep this cognitive tool alive in the minds of our students, then we must look for opportunities to draw attention to it in our teaching. Robert Hanson, Harvey Silver, and Richard Strong (as cited in Schwarz, 1988), suggest there are two ways teachers can use metaphor:

> The first is to create something new for the purpose of making the familiar strange; i.e., to help students see old problems, ideas, or products in a new, more creative light. The second strategy is the making of the strange

> familiar, i.e., designed to make new, unfamiliar ideas more meaningful (p.32).

Both of these applications of metaphor can be used within the context of stories, for the purpose of teaching mathematics. Although it may seem like an odd pairing, the connection between metaphor and mathematics is reasonably transparent. Bazzini contends (2001) that, “metaphors are an essential part of mathematical thought, not just auxiliary mechanisms for visualization or ease of understanding” (p.261). Like metaphor, mathematics has evolved “from our need to describe the world in which we live” (Koehler, 1982, p.81). The natural numbers themselves boast of metaphor in the way they represent groups or collections of objects (Court, 1964). Instructors of mathematics may not realize the extent to which they already incorporate metaphorical thought into their lessons and communication. Consider the following examples:

- The notion of ‘set’ is often perceived in terms of a ‘container’ (Bazzini, 2001).
- The familiar balance is useful in helping students understand how to solve equations (Bazzini, 2001).
- A mathematical function is often viewed as a machine that receives inputs and produces outputs (Bazzini, 2001).
- A proof is sometimes seen as a story (Senechal, 2007).

4.2.2 Metaphor in Fiction

Some would say that all of mathematics is metaphoric in nature, and as such, is a source for creating interesting themes and images in fiction and prose (Koehler, 1982). It would be pertinent then to examine some of the stories that incorporate both mathematics and metaphor and consider the interplay between them. According to Koehler (1982), writers often look to geometry, probability, or statistics, since they are closely related to our physical world, but almost any field of mathematics can be a source of metaphorical activity.

The French poet and novelist, Samuel Beckett, relied on the metaphoric power of mathematics to express his contempt for the way language is limited in its ability to describe all aspects of human experience. For this, he includes in several of his works, various references to the Pythagoreans and the failure of their system to recognize irrational numbers. In fact, Beckett's many references to Hippasos suggest he saw himself playing a similar role among the literary elite of his day. His interest in mathematics as a metaphor no doubt grew from the mathematical debates that occurred during his time that eventually culminated in Godel's Undecidability Theorem (Culik, 1993).

Several references to calculus are made in Thomas Pynchon's *Gravity's Rainbow*, a story set during the Second World War. The following excerpt reveals a mapping from the process of integration to the character transformations taking place within the story (Koehler, 1982):

> She tried to explain to him about the level you reach, with both feet in, when you lose your fear, you lose it all, you've penetrated the moment, slipping perfectly into grooves, metal-gray but soft as latex…as Δt approaching zero, eternally approaching, the slices of time growing thinner and thinner, a succession of rooms each with walls more silver, transparent, as the pure light of the zero comes nearer… (p.85).

In Dostoyevsky's *The Brother Karamazov* (as cited in Lipsey & Pasternack, 2006), the main character Ivan, uses aspects of non-Euclidean geometry as a metaphor to explain how he believes in the existence of God but doubts His hand in creation:

> Even if the parallel lines converge and I actually witness it, I shall witness it and say they have converged, but all the same I shall not accept it (p.2).

Russian novelist Leo Tolstoy's famous quote, "A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction," resembles the "virtue" ratio $M\colon A$ in Jane Austen's *Emma*, where A is perfect virtue and M is attained virtue. The main character Emma, aspires to perfection—that is, when M increases until $M{=}A$ (Pickover, 2005, p.117; Lipsey & Pasternack, 2006).

Symmetry is the vehicle domain for the metaphor in Jorge Luis Borges' detective story, *Death and the Compass* (as cited in Lipsey & Pasternack, 2006):

> Lonrot [the main character] circled the house as he had the villa. He examined everything: beneath the level of the terrace he saw a narrow Venetian blind. He pushed it; a few marble steps descended to a vault. Lonrot, who had now perceived the architect's preferences, guessed that at the opposite wall there would be another stairway. He found it, ascended, raised his hands and opened the trap door. A brilliant light led him to a window. He opened it: a yellow, rounded moon defined two silent fountains in the melancholy garden. Lonrot explored the house. Through anterooms and galleries he passed to duplicate patios, and time after time to the same patio. He ascended the dusty stairs to circular antchambers: he was multiplied infinitely in opposing mirrors; he grew tired of opening or half-opening windows which revealed outside the same desolate garden from various heights and various angles… (p.87-88).

Many stories, like the ones mentioned here, accurately and consistently throughout incorporate mathematical ideas and concepts. In most cases however, the mathematics is the vehicle domain of the metaphor and human activity is the topic domain. In other words, the mathematics is only there to enrich the reader's understanding about some aspect of human experience—other than mathematics. This, however, does not necessarily have to be the case—the story can serve to help the reader better understand the mathematics it contains. It may simply be a matter of switching the vehicle and topic domains (Koehler, 1982). In this way, metaphor can be used to shape stories in order to highlight the mathematical content.

4.2.3 Flatland

One of the most famous examples of the use of metaphor in a mathematical narrative is Edwin Abbott's *Flatland: A Romance in Many Dimensions.* Written over a century ago, *Flatland* makes use of metaphor in two very powerful ways. The first and most obvious is the way in which the characters in Flatland are animate geometric figures endowed with human qualities. *Flatland's* narrator, Mr. "A. Square," is the typical anthropomorphic object trying to make sense of the two-dimensional world he lives in (Senechal, 2006). In *Flatland*, social class is dependent on the number of sides one possesses: At the very base of society are women who are simply lines. Women are required by law to sway side to side as they walk so as not to stab and kill someone with their pointy ends (Peterson, 1998, p.83); Isosceles triangles are the working class men; Equilateral triangles make up the professional class; Squares and hexagons make up the business class, to which A. Square belongs; Noblemen are regular polygons with six or more sides; and at the very top of society are the priests, comprised of regular polygons whose sides are so numerous they are almost indistinguishable from circles. Each successive generation gains an additional side. For instance, A. Square's father was an equilateral triangle and his son is a pentagon (Peterson, 1988). According to Dewdney

(2001, p.1261), Abbott has discovered a "marvelous door" to the world of mathematics. *Flatland* provokes the reader into noticing properties and relationships among shapes in a unique and unfamiliar way—it asks, "Can you put yourself in a Flatlander's shoes" (Dewdney, 2001, p.1261)? This is very different from the typical way students learn about geometric figures at the secondary level. Nicol and Crespo (2005) found, in their research with sixth through eighth graders, that the imaginative setting of *Flatland* and the underlying themes present elevated student engagement with the mathematical ideas: The way women were represented as lines elicited emotional responses that encouraged discussions about the properties of lines and triangles; Students considered the class of "priest" and asked questions such as "How many sides are necessary for a regular polygon to be classified as a circle?"; They asked questions about similarity and how Flatlanders recognize each other. In this regard, human emotions and activity are part of the vehicle domain in the metaphor for understanding the properties of geometric shapes.

The other way metaphor is utilized in *Flatland* becomes apparent when A. Square unexpectedly encounters a being from a higher dimension—a sphere. The sphere's existence confirms some of the ideas A. Square has been wrestling with and to which he has been secretly teaching his son concerning the possibility of a third or even higher dimensions. His ideas are not welcomed by the social elite and eventually end in A. Square's imprisonment (Baum, 1984). According to Dewdney (2001), this is the central metaphor of *Flatland*—the ability to appreciate higher, as well as lower, dimensions. Abbott's views concerning the fourth and higher dimensions, paralleling those of his main character, were not well received by the mathematical establishment in Britain during his time. In a culture dominated by three-dimensional Euclidean geometry, the established mathematicians of Abbott's day believed that such concepts would shake the very foundations of mathematics (Peterson, 1988). But as many in the field know today, *Flatland* showed, metaphorically, a novel way to approach something that is not fully understood—something in which its meaning may never fully be grasped (Peterson, 1988). It was *Flatland* that inspired Brown University's Thomas Banhoff to construct a computerized model of the fourth dimension (Baum, 1984). Mathematicians have since been using metaphor and analogy to venture beyond the areas of what they hold to be concrete. Mazur explains in a dialogue with Peter Pesic (Dialogue, 2005):

> The ubiquitous activity of generalizing, which is one of the staples of mathematical and scientific progress, is a way of analogizing. We start with a structure or concept we feel at home with (say, multiplication of ordinary numbers), and we see a broader realm for which the same or at least an analogous structure or concept may possibly make sense (say, think of transformations as a kind of multiplication operation). We make ourselves at home with this more general concept, initially at least, by depending heavily on the analogy it has with the more familiar, less general concept (p.127).

According to Mazur (Dialogue, 2005), we really are just "squashing and slicing" these more complex spaces in order to fit our limited senses. But this squashing and slicing requires a great deal of imagination and intuition. Metaphor is a powerful cognitive tool that can be used to transform a typical lesson into an imaginatively engaging storytelling activity. Jarrett Wilke (2006) describes how he used the process of driving from town to town using a road map to guide him, as a metaphor to teach his ninth grade students the steps to factoring polynomials. The first stop on the road is the town of "Common Factorville." If there is a common factor in each of the terms of the polynomial, it must first be removed. The road does not stop there. Students must "keep driving." They will soon arrive at a fork in the road in which they must choose which path to take. If there are two terms, they must go to the town of "A Difference of Squares"; if there are three terms, they must take the path that goes to "Trinomialville." The story takes more twists and turns depending on the types of polynomials the students encounter. Wilke (2006) asserts that presenting a concept in this way can be more enjoyable for students and will live long in their memories. As Gadanidis and Hoogland point out, whenever an idea is "regenerative"—that is, it lives fruitfully in future experiences—the imagination is at work (2002).

4.3 Knowledge and Human Meaning

Another cognitive tool that students have available to them is knowledge and human meaning (Egan, 2005). We tend to teach mathematics as a set of rules, algorithms and processes independent of its human origin. "All knowledge is human knowledge, discovered or invented as a result of some human emotion, and seeing knowledge through the emotions that were involved with its past creation or current use helps us grasp its deeper human meaning" (Egan, 2005, p.80). Casting a problem in the light of how it was first encountered and the meaning it had for the discoverers has the potential to engage students' imaginations. Ernest Renan wrote, "The simplest schoolboy is now familiar with facts for which Archimedes would have sacrificed his life" (Woodard, 2005). Consider, for example, the familiar relationship $a^2 + b^2 = c^2$, where *a* and *b* are the sides of a right triangle and *c* is the hypotenuse. Few instructors will connect this relationship to its geometric meaning—the sum of the areas of the squares constructed on the sides of the triangle is equal to the area of the square constructed on the hypotenuse (Zazkis & Liljedahl, 2005). Fewer instructors however, will explore this relationship as a product of human necessity—to construct right angles. Ancient engineers and carpenters would tie knots in ropes at equal distances apart in order to construct triangles. They noticed that only certain configurations would result in right-angle triangles. Stories that introduce "Pythagorean triples" in this way will, not only engage students' imaginations, but will also establish its deeper human meaning (Zazkis & Liljedahl, 2005).

4.3.1 Historical Narrative or Historical Fiction?

From section 3.4, we saw how "origin stories" are typically used to introduce mathematical concepts. But what if the characters in our stories lead dull or uneventful lives? How much poetic license should we take when we introduce a mathematical idea in the context of the lives of those who discovered it? Should teachers explore all the dramatic possibilities available to them when incorporating the stories of mathematicians into their lessons (Senechal, 2005)? Even our example above illustrating ancient construction techniques may be a gratuitous detour into fiction. The Metropolitan Museum in New York has a sample of a rope with knots; several modern paintings depict ancient Egyptian surveyors using ropes in this way (Stein, 1996). However, there is no evidence that suggests this method was ever used. The assertion originates with a "guess" made by Moritz Cantor in his *Lectures on the History of Mathematics*, published in 1907. He states:

> Let us *postulate without any foundation* [italics mine] that the Egyptians were aware that when three edges of lengths 3, 4, and 5 are joined to form a triangle, the two shorter edges enclose a right angle. Now let us *postulate* [italics mine] that they divided a rope of length 12 by means of knots into pieces of lengths 3, 4, and 5. It is clear that this rope, when stretched around three stakes, creates a right angle (Stein, 1996, p.38).

Cantor makes several other assumptions throughout the rest of this text. As with many of the stories about mathematicians, with time, these assumptions often turn into historical fact (Stein, 1996).

4.3.1.1 Archimedes Revisited Yet Again

Archimedes' "Eureka" story also appears to be a pleasant blending of history and fiction. The account is a familiar one and is worth incorporating into any introduction to the laws of buoyancy and/or volume properties. The story grabs students' attention and engages their imaginations. However, how true is this story? Archimedes made no reference to the event in any of his writings. It was the Roman architect, Vitruvius, who first mentioned it some 200 years after the time of Archimedes (Stein, 1996). As Stein (1996) suggests, the account sounds more like an embellishment long after the fact. With the many exciting discoveries Archimedes made, it seems highly unlikely he would run around naked every time he came upon a new idea.[5] Nevertheless, the reader of this account is compelled to feel an elevated level of joy at Archimedes' discovery—something they may not have felt in the absence of such embellishments.

4.3.1.2 Hippasos and the Pythagoreans Revisited

As another example of this brash approach to history, let me return to the discovery of irrational numbers. Is it true that the Pythagoreans were involved in a "cover-up" of the discovery that the square root of two is irrational? How likely is it that Hippasos, a Pythagorean, was killed for divulging a mathematical truth that would threaten the Pythagorean's world-view? Probably the earliest modern source of this

[5] Even the account by *Plutarch*, describing Archimedes lack of personal hygiene, was written some 300 years after his death (Hawking, 2005).

account comes from Morris Kline's, *Mathematical Thought from Ancient to Modern Times* (as cited in Crew, 2007):

> The discovery of incommensurable ratios is attributed to Hippasus of Metapontum (5th cent. B.C). The Pythagoreans were supposed to have been at sea at the time and to have thrown Hippasus overboard for having produced an element in the universe which denied the Pythagorean doctrine that all phenomena in the universe can be reduced to whole numbers or their ratios (p.1).

There is very little evidence from ancient sources, however, that support Kline's account. The first reference to this event is seen in the scholium to *Euclid Book X* after a lengthy discussion about irrationals and the divisibility of geometric magnitudes (as cited in Crew, 2007):

> There is a legend that the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck (p.4).

Another ancient account is from the tract of Iamblichus, *On the Pythagorean Life* (as cited in Crew, 2007):

> It is related of Hippasus that he was a Pythagorean, and that, owing to his being the first to publish and describe the sphere from the twelve pentagons, he perished at sea for his impiety, but he received credit for the discovery, though really it all belonged to HIM (for in this way they refer to Pythagoras, and they do not call him by name (p.4).

From these two accounts it is clear that the Pythagoreans did not like their secrets being revealed. But which secret did Hippasos reveal? Was it the irrationality of the square root of two or the construction of the dodecahedron? It is not clear from these texts (Crew, 2007). Furthermore, it is unlikely that a group that refused to sacrifice animals and believed in the transmigration of souls into other species—and on that account refused to eat meat—would murder another human being for divulging its secrets

(Pickover, 2005; Crew, 2007). As Benson (2003) suggests, the drowning of Hippasos was probably "a calumny promulgated by outsiders suspicious of this secret brotherhood" (p.20). As for the modern version of the legend, it is probably not a coincidence that Kline's book was published around the same time cover-ups and conspiracy theories dominated popular media culture (i.e. JFK assassination, Watergate) (Crew, 2007). Grant (2003) argues that the discovery was not as eventful to the Pythagoreans as many would have us believe. He contends that the cult was more concerned with adhering to a lifestyle promulgated by its founder, and less concerned with any particular mathematical dogma. The Pythagorean notion that "all is number" has been misconstrued from its probable original meaning, that "all can be explained in terms of numbers." Even this notion of number is more associated with Aristotle than with the Pythagoreans (it has long been known that Aristotle did not expound the ideas of others with any concern for accuracy) (Grant, 2003). Much of what we know about Pythagoras and the Pythagoreans has been passed down from obscure and fragmented texts, whose authenticity is somewhat questionable. "Probably more sheer nonsense has been written about Pythagoras and his followers than about any other figure(s) in all the annals of mathematics" (Grant, 2003, p.343).

4.3.1.3 Partition

Now consider the staging of Ira Hauptman's play, *Partition* in 2003. It is the story of how two very different mathematicians, G.H. Hardy and Srinivasa Ramanujan, collaborate together on the theory of mathematical partitions (Oliver, 2005). The play is about Ramanujan's search for a proof to Fermat's Last Theorem and incorporates a small amount of graduate level mathematics, such as Poincare and modular forms. At first glance, it would appear the play caters only to an erudite few. However, by focusing more on the invisible partitions that existed between the two men—personality, culture, and mathematical style, the playwright was able to, not only draw a wider audience, but was also able to engage the audience in a type of mathematics it would normally not be interested in (Senechal, 2005). Oliver (2005) suggests, what makes the mathematical ideas palatable for the average person is the mutual passion for mathematics both men share in spite of their stark differences:

> The non-mathematical part of our audience (people like me) must find their way into the play by understanding that a passion for mathematics is as consuming and as motivating as any other passion whether it is for another art, or a cause, or (indeed) an idol—human or otherwise (Oliver, 2005, p.9).

Ramanujan lays around waiting for inspiration from the goddess Namagiri who writes formulas on his tongue after consulting with dead mathematicians, while Hardy, who is portrayed as an exaggerated eccentric, takes a more classicist approach (Senechal, 2005; Oliver, 2005). Some mathematicians have pointed out that these characters are nothing like the real Ramanujan and Hardy (Senechal, 2005). It is a well-known fact that neither mathematician ever worked on Fermat's Last Theorem. As Corry (2006) suggests, most mathematicians today would have an easier time accepting Ramanujan's divine inspiration.

Some of the most heated discussions during the meeting at Mykonos involved the way narratives often compromise historical accuracy (Tomlin, 2005). This is, of course, a concern that transcends the field of mathematics. Take *Dr. Atomic,* an opera by John Adams and Peter Sellars, that depicts the events leading up to the development of the atomic bomb: Robert Oppenheimer is portrayed as a tormented soul sacrificing his spiritual values for power; Edward Teller, a conniving genius; and General Groves a bigheaded loud-mouth full of empty threats (Ross, 2005). How much did the characters in the opera resemble their real-life counterparts? Should this account simply be deemed a lie, or does it say something about the nature of humanity (Senechal, 2005)? Is the mathematics teacher's duty to simply convey mathematical content accurately, or to highlight various aspects of humanity to which mathematics speaks? As one statistician and author of a book on mathematics and magic, states, "There's no reward for expository stuff, and a bias against it. If we are to communicate we have to lie. If we don't, we're dead boring" (Tomlin, 2005, p.623). Perhaps what he is suggesting we do when telling our stories is along the lines of Roman Jakobson's famous definition of the artist's task, "to make the ordinary strange" (Bruner, 1991, p.13). According to Corry (2006), there is a delicate balance in the triangular relationship between mathematics, the history of mathematics, and what he calls "mathematics in fiction". While real characters and historical settings may be included as part of the plot, they are more like archetypes that represent universal people or situations. There is nothing about the genre that says authors must stay close to what they believe to be historical truth. Bruner (1991) suggests this "referentiality" property of narrative allows, 'truth' to be "judged by its verisimilitude rather than its verifiability" (p.13). There must be, what Samuel Taylor Coleridge first called a "suspension of disbelief" on the part of our listeners (as cited in Corry, 2006, Chapter 2, p.1). This suggests, that while our students may not get a complete view of a topic, they will certainly get an engaging one (Egan, 2001).

Introducing the concept of buoyancy in terms of Archimedes' indiscretions, allows the reader to catch a glimpse of the excitement he may have felt during that historic moment at the baths; Developing the notion of irrational numbers through a tale involving murder and intrigue, gives the listener a sense of the extent of the Pythagoreans' treachery; Placing Ramanujan and Hardy's otherwise mundane colloquy against a backdrop of divine visions and revelations, gives the mathematical content a type of value even a non-mathematician may relish. Students will see the concept in terms of the emotions and motivations involved in its discovery—regardless of how far-fetched some aspects of the account may seem—and the seemingly irrelevant facts and rules in their textbooks will begin to take on real meaning.

4.4 Real-life or Fantasy Contexts

It was believed that the best way to contextualize mathematics was to place it in objective reality, by associating it with those things that might arise in the everyday lives of students (Schiro, 2004). Stories should dwell around experiences and activities that students are most familiar with (Colgan, 1965). This idea is based on previous notions that children's learning progresses from the concrete to the abstract, from the local "known" experiences to the 'unfamiliar" (Egan, 1986). Current research shows, however, that simply placing mathematics in a child's objective reality does not necessarily mean it will correspond to the subjective reality in their mind. "Simply doing so will not make it any more meaningful to a child than an unopened book sitting in the same room with the child" (Schiro, 2004, p.63). Mathematics must be contextualized in a way that will catch their attention, stir their emotions, come alive to them, and encourage them to engage in its problems. Not all experience makes for a good story. According to Schank (as cited in Gadanidis and Hoogland, 2002), "Good stories are not always about things we experience day to day. They are often about things that are not routine-like and involve some element of surprise" (p.3). It is more about what the listener finds interesting (Gadanidis and Hoogland, 2002). Consider the "footprint problem" given to a group of middle school students (Nicol & Crespo, 2005). Students were asked to search for vandals who left only footprints as potential clues. Using only the footprints, they had to calculate the suspect's height and then generalize their methods as a sort of "toolkit" for police. Clearly there is much about this story that is unrelated to the everyday lives of students. The problem could easily have been set up as a contest at an amusement park where students had to guess the height of someone based on their shoe size. Nicol and Crespo (2005) suggest "the mystery, intrigue and role play of the detective story is more compelling and imaginatively engaging precisely because it is not an ordinary task for students" (p.241).

Schiro (2004) contends that fantasy stories have the ability to capture the interests of students, engage their imaginations, and establish mathematical meaning like no other type of narrative. Children in the early stages of development prefer the magical and the fantastic. We see this in the way they enjoy fairy tales, which are often their initial acquaintance with ideas and vocabulary. If a child encounters a word or experience that is unfamiliar to them, their natural curiosity leads them to inquire as to the meaning of those words or experiences (Colgan, 1965). We see children use their imagination in play pretending to be star-warriors, pirates, and monsters (Egan, 1986). In a fantasy setting, children can explore pressing themes at a safe distance. Such distancing makes it safe to explore concepts of fear, security, or loneliness, or even manipulate adult and child power relations—something that is not easily carried out in the objective world (Hoogland, 1998). This propensity for fantasy settings does not go away as children enter adolescence. Stories, like *Lord of the Rings* and *Harry Potter,* take children far from their own objective reality to distant unknown worlds. As one seventeenth-century poet put it, fantasy "digs without spade, sails without ship, flies without wings… fights without bloodshed, in a moment striding from the center to the circumference of the world, by a kind of omnipotency creating and annhiliating things in an instant" (as cited in Rose, 1966, p.348).

According to Schiro (2004) and Cory (2005), fantasy contexts ask the listener to suspend beliefs about the objective world and how it really works. This is a powerful intellectual activity since students must intentionally make distinctions and draw connections between what is real and what is imagined (Hoogland, 1998). This is not an easy task as the lines between what is real and what is imagined is often not so obvious. Bruner (1991) contends, that the real difficulty is not necessarily making those distinctions, but sustaining them. Given hermeneutic composability, we tend to interpret events in light of the whole story. For example, "a story that requires a betrayal as one of its constituent functions can convert an ordinarily mundane event into something that seems like a betrayal" (Bruner, 1991, p.14). Simply stated, we read into a story what we think should be there. If negotiated carefully, students will become more aware of the interplay between the imagined events of the story and the events in the story of their own lives (Hoogland, 1998). They picture themselves in the role of characters or as helping characters in the story solve the problems they encounter. Thus, the story and the embedded mathematics becomes a part of both their subjective and objective reality. And in this way, the mathematics becomes meaningful and consequently will impact their views of what is important to them (Schiro, 2004). Egan (2005) contends however, that students at the secondary level need some semblance of reality in the stories they hear. Unlike the previous period of their cognitive development, where they simply accept magic fairy dust or beanstalks that can support giants, students are now interested in reality and need explanations for why things are the way they are. Does this mean that a story must completely comply with reality for students to enjoy it? The popularity of such movies like, *Lord of the Rings* or *Star Wars*, suggest otherwise. But students do require some accommodations to reality. They will accept that Superman can fly and has super strength only because he comes from a solar system where the sun is red. They may not accept the magic in *Cinderella*, but they will accept the "force" of *Star Wars* because it has been harnessed by the efforts of the Zen-like, Jedi warriors, through centuries of discipline (Egan, 1989). In both examples, they "make accommodations with reality that students require, presenting them with a recognizable and justifiable world—though unfamiliar" (Egan, 2005, p.84). It is the unfamiliar part that gets their attention, but it is the accommodations to reality that hold their interests. At this stage in their development, students are fascinated with the extremes of experience and the limits of reality—like the type of subject matter one encounters in the *Guinness Book of World Records* (Egan, 1997). "Students begin to lose their ready engagement with giants who were a mile high and midgets no bigger than your thumbnail. They turn intellectually to discover who was really the biggest and smallest person who ever lived" (Egan, 1997, p.86). The simple binary structures of their mythic understanding concede to reality. They have advanced into what Egan (1997) calls a "romantic understanding." It is a compromise with reality. Whenever they find themselves in a new environment, students will seek to find the limits of that environment and its most outstanding features. It is these elements that will attract their attention and engage their imaginations. Moreover, it will help them gain a better understanding of their own everyday world as they delineate between these extremes—and therein establish meaning (Egan, 2005).

4.4.1 The Story of Tan

Schiro (2004) gives an excellent comparison of how a mathematical problem can be more meaningful when placed in the context of a fantasy situation. Consider the following two cases using the familiar *tangram* puzzle:

- Case 1: The teacher gives each student the seven pieces of the *tangram* and tells them to construct a square. The teacher further explains, that upon completion, they can go for recess (Schiro, 2004).

- Case 2: This time the teacher first tells the students the following story (Schiro, 2004):

> Tan was a poor potter in ancient China. He was so poor that he had only one set of clothes, could afford only one meal a day, and could not afford the dowry he needed to marry the woman he loved. However, Tan was well known for his ability to make beautiful floor tiles. One day, the Emperor of China asked Tan to make him a beautiful square floor tile. Tan did so while thinking about how his fame and fortune would be secured if the Emperor liked his tile. If the Emperor liked his tile, then he might be able to afford more than one shirt, more than one meal a day, and the dowry to marry the woman he loved. On his way to deliver his square tile to the Emperor, the tile fell from Tan's hands, landed on the ground, and broke into seven pieces. Tan was very distressed, for it seemed as though his fame and fortune had just slipped through his fingers. He looked at the broken tile pieces and noticed that they had broken into triangles, squares, and parallelograms. Tan wondered if the tile could be reassembled. He sat down on the ground and tried to put the seven pieces into a square. He was delighted when he could. He decided to bring the Emperor his broken square tile and tell him

> that it was a special square tile puzzle, for he had heard that the Emperor liked puzzles. On the way to the Emperor, Tan wondered if he could make the seven pieces into something other than a square. He wondered if the tile pieces could be assembled into a triangle, a rectangle, or a parallelogram. He wondered if the seven pieces could be made into a fish, a cat, or a boat. When he showed the Emperor of China his tile puzzle, the Emperor was delighted with it because he loved puzzles. The Emperor told Tan that he would pay him for each new recognizable shape that Tan could make out of the seven-piece puzzle. The Emperor named the puzzle a tangram in honor of Tan, and the Emperor said that he would create a book called Tangram Puzzles that contained each recognizable shape that Tan could make out of the seven tangram pieces. Tan's fame and fortune would be made if he could create many new puzzle shapes for the Emperor (and others) to solve. (p.60-61).

After the story is complete, the teacher can present students with the same problems that Tan encountered. In this way, Tan's problems become the students' problems, and his success is linked to their success. With each new puzzle they solve, the teacher can remind students of the reward Tan would receive (Schiro, 2004).

In Case 1, students will likely become interested in solving this problem since it involves the use of manipuatives, as well as their going to recess and having "real fun" hinges on its completion (Schiro, 2004). However, the problem is an isolated task that is not connected to any "ongoing web of intrigue" that forms the student's intellectual life (Schiro, 2004, p.61).

In Case 2, the problem is situated in a fantasy context, where the events of the story are quite different from the students' everyday lives. However, there are real dramatic elements in the story that will occupy their thoughts and ignite their emotions. What will happen to Tan if he cannot make enough geometric shapes? How can they help Tan make these shapes? Will Tan be able to escape poverty and marry the woman he loves (Schiro, 2004)? This last point is critical in understanding the differences between the two cases. In the first case, the students are simply trying to construct a square in order to receive a reward that has no mathematical meaning, whereas in the second case, the students do mathematical work in the context of a fantasy story where they are helping Tan get out of his predicament. The mathematics is connected to the students' fantasy lives through the story, when they imagine themselves in Tan's situation (Schiro, 2004). Because the outcome of the story depends on the decisions students make, the story is tied to their emotions, and consequently, plays a part in establishing mathematical meaning (Schiro, 2004).

Doris Lawson, when telling her fourth grade class, *The Wizard's Tale,* a fantasy story designed to help students learn the multi-digit algorithm, found some promising results. Over the five-day period, her students regularly "moved" in and out of the fantasy world and the real world in order to solve the mathematics problems embedded in the story. This was seen in the way they played with each other and in the vocabulary they were using (Schiro, 2004).

4.4.2 Gulliver's Travels

De Bock and colleagues (2003), contend that fantasy contexts for mathematics do not always produce the desired impact on students' performance. In fact, they assert that it could actually impede learning. In a follow-up study of the "linear illusion" problem, De Bock et al. (2003) tested to see if the poor performance of students in previous studies was due in part to the inauthenticity of the problem context. A large group of 13-16 year old students answered test questions again involving the enlargement and reduction of geometric figures, but this time, half of the students answered the questions situated in a more engaging and attractive context. Preceding the tests, this second group of students watched a ten-minute video collage of scenes from the movie, *Gulliver's Travels*. The following announcement preceded the test (De Bock et al., 2003):

> In the world of the Lilliputians, all lengths are 12 times as small in our world, the world of Gulliver. A tree, being 12 m high in our world, would thus be only 1 m high in the world of the Lilliputians. A road being 12 km long in our world, would be only 1 km long in the Lilliputian world. And, of course, the Lilliputians themselves are also 12 times as short as Gulliver (p.447).

The results of the study showed that the even though the problems were placed in a fantasy context, the students who watched the video performed significantly worse than students in the other group. It would appear that placing the problem in a potentially engaging context does not necessarily have a positive effect on student performance. However, De Bock et al. (2003), offer several possible explanations for these results:

- First, the way the context was established appears problematic. It does not take into account that students learn in multiple ways (Schiro, 2004). Studies by Streefland and Treffers (as cited in De Bok et al., 2003) show that when students are actively engaged in several lessons around the context with various types of activities (e.g. drawing a giant's footprint or making a giant's handkerchief), the context becomes more engaging—this stands in stark contrast to simply watching a ten-minute video and writing a test.

- Second, the way in which the problems were presented to the students that did not watch the video, were more like the ones they normally encounter at school (De Boc, 2003). In a typical mathematics lesson, students are presented with abstract, decontextualized facts and algorithms (Schiro, 2004). By adding the non-routine element of the fantasy context, the students may have become unclear as to what they were expected to do and the rules they should follow. In general, mathematics education at the secondary level "does not envision itself as a place where fantasy coexists with reality," nor does it usually support 'dreaming' and 'let's pretend' except for a few years at the lower grades" (Hoogland, 1998, p.85).

- A third explanation for the unexpected results may be related to the way students perceive media and the effect it has on their willingness to engage a problem. According to Salomon (as cited in De Bok et al., 2003), students believe video to be a less difficult medium than writing, and therefore will put less effort into understanding information transmitted in this way. If we extend this idea further, problems presented in story form may be perceived as less difficult than the regular abstract theoretical number problems.

- Lastly, the problem context may unintentionally lead students away from the deeper mathematical structure of the problem. A study by Boaler (as cited in De Bock et al., 2003) revealed that when students encountered mathematical problems in the context of fashion, the girls in the group attained significantly lower grades than the boys, even though the girls were notably more interested in the activities. Boaler contends that the girls' underachievement had to do with their greater emotional engagement with the problem context instead of the mathematics embedded in it. The story elements are intended to serve the mathematical ideas presented as opposed to lead students away from them.

Given that high school students are drawn to the dramatic, the improbable, and to the extremes of human experience, a failure to develop the context of a story along these lines suggests the story might only cater to an inspired few. However, with careful planning, and just the right balance of reality and fantasy, mathematical concepts can be taught in a way that sparks students' curiosities and stirs their emotions.

4.5 Rhyme, Rhythm, and Pattern

> "Creativity is not the finding of a thing, but the making something out of it after it is formed" –James Russell Lowell (as cited in Perrella, 2006, p.116).

Another cognitive tool that students have available to them is a sense of rhyme, rhythm, and pattern (Zazkis & Liljedahl, 2005). In oral cultures, important cultural information was often passed down through patterned sound usually structured within a story. Drum beats, vibrating strings, or songs and chants, all aided in the memorization of the stories and myths that were important to those cultures (Egan, 2005). A great deal of research shows that children have a propensity for rhyme, rhythm and patterns. Nursery rhymes, storybooks, television ads, and the sing-a-longs found in children's shows like Sesame Street, all attest to their appeal (Moyer, 2000; Opies, 1959, 1969, 1985; Paley, 1981, 1984, 1990; Knapps, 1976, as cited in Zazkis & Liljedahl, 2005). Egan (2005) contends that these "small-scale" linguistic patterns really just echo the more general emotional patterns that encompass our lives—patterns of hope and despair, of fear and release, of youth and age, and so on. "We might be wise to recognize that rhyme and rhythm pervade our language and lives" (Egan, 2005, p.20). Elaboration of these linguistic rhythms and patterns to match the more general patterns of our lives results in, what we call, stories (Zazkis & Liljedahl, 2005). According to Gadanidis and Hoogland (2002, p.1), we are all predisposed to our "sense of rhythm and fit, balance, motion, and symmetry." We are fascinated by the waves crashing on the sand, the brush strokes in a painting, or the vibrations of the strings on a guitar (Egan, 2005). We continually look for patterns in what may often seem chaotic (Zazkis & Liljedahl, 2003). Indeed, this may account for the recent popularity of both chaos and string theory. Our students are developing their "philosophic understanding" wherein they are particularly interested in establishing overriding ideas and schemas about the world. They look for patterns in history to derive some general belief about how the world works and what role they play in it (Egan, 1997). "One feature of the imagination, it seems fair to say, is the ready recognition of pattern in the world" (Zazkis & Liljedahl, 2005, chapter 1, p.18).

It has been argued extensively that mathematics itself is essentially "the classification and study of all possible patterns" (Hoogland, 2002, p.406). It is perhaps the one consistent strand that ties together all the seemingly unrelated fields within the discipline (Devlin, 1997; Moyer, 2000). G.H. Hardy wrote, "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas" (Huckstep, 2002, p.406). Colgan (1965, p.143), on the other hand, argues that, "the essence of creativity and the imagination is that no one really evolves new ideas. They merely evolve new combinations, groupings, or associations of ideas...." The ability to recognize, interpret, and create—or re-create—patterns plays a key role in a child's cognitive development (Moyer, 2000). For example, when children recognize growing patterns—numbers, symbols, letters, or objects that become increasingly larger as the patterns progress—they tend to make generalizations, which, not only exercises their imaginations, but cultivates algebraic thinking (Moyer, 2000). Attention to rhyme, rhythm, and patterns can therefore play a significant role in the learning of mathematical concepts. However, Zazkis and Liljedahl (2005) argue that their use should entail more than just the simple rote memorization of rules (i.e. SOHCAHTOA, BEDMAS, or FOIL), as is the way they are normally incorporated in mathematics instruction. Patterns can be used to introduce concepts or activities, or explain mathematical ideas that learners find difficult (Zazkis & Liljedahl, 2005). When these patterns are situated within the context of a story, the potential to engage the imagination multiplies. Egan (1986) argues, good stories have a certain rhythm that keeps the story moving forward. Bruner (1991) suggests good narratives are necessarily structured according to certain underlying patterns. In addition to those previously mentioned (hermeneutic composability, canonicity and breach, referentiality) a story must have the attributes of "narrative diachronicity" and "narrative accrual." The former suggests every story is made up of a unique pattern of events occurring over time. Whether by means of such literary conventions as flashbacks, flash-forwards, temporal synecdoche and the like, some indication of temporality is essential to narrative construction. The latter feature, narrative accrual, implies a collection of stories "builds" a culture or history—according to what the writers have in mind. The events and happenings are eventually converted into a more or less coherent whole—a sort of "arithmetic growth" of ideas (Bruner, 1991). As with *The Twelve Diamonds*, juxtaposing mathematical content alongside the events of a story can assist in making clear the existing mathematical patterns (Hoogland, 1998).

4.5.1 Anno's Mysterious Multiplying Jar, Antaeus, and One Grain of Rice

An example of a pattern that introduces a concept is *Anno's Mysterious Multiplying Jar* (Anno, 1983). The story is about a jar that contains a sea. On the sea, there is 1 island, on the island there are 2 countries, within each country there are 3 mountains, on each mountain there are 4 kingdoms, within each kingdom there are 5 villages, in each village there are 6 houses, in each house there are 7 rooms, in each room there are 8 cupboards, in each cupboard there 9 boxes, and within each box there are 10 jars (Zazkis & Liljedahl, 2005). The question remains, "What is the total amount of jars in all the boxes?" A beautiful pattern emerges as students are introduced to the concept of factorials (Zazkis & Liljedahl, 2003, chapter 1, p.22):

$2 \times 1 = 2! = 2$ the number of countries

$3 \times 2 \times 1 = 3! = 6$ the number of mountains

$4 \times 3 \times 2 \times 1 = 4! = 24$ the number of kingdoms

Consider this next story as way to introduce students to the concept of exponential growth. From Greek mythology we have the story of how Heracles (Hercules) defeated Antaeus, the powerful son of Gaea (Lipke, 1996). As we have already seen, many of these ancient legends have hidden deep within, nuggets of mathematical wisdom (Gandz, 1936). Each time Antaeus touched the Earth, his strength doubled. This made it difficult for Heracles to defeat him. It was only until Antaeus was lured away from the Earth that Heracles was able to pick him up and strangle him (Lipke, 1996). What if Antaeus' strength tripled or doubled—or even halved—each time he touched the earth? The mathematical explorations generated by these patterns are almost limitless. In *One Grain of Rice* (Demi, 1997), a young Indian girl, Rani, saves her village from starvation through her mathematical prowess. The greedy raja decides to hoard all of the rice in the land during a time of great famine. One day he hears how this little girl performs a good deed and decides to give her whatever she asks for. She responds:

> If it pleases Your Highness, you may reward me in this way. Today, you will give me a single grain of rice. Then, each day for thirty days you will give me double the rice you gave me the day before. Thus, tomorrow you will give me two grains of rice, the next day four grains of rice, and so on for thirty days (Demi, 1997, p.12).

Not realizing the magnitude of rice involved, the raja agrees to her request. Needless to say, the villagers received their rice and were saved from starvation. Students can easily see the relationship between the events of the story and the patterns present:

$2^0 = 1$ doubled 0 times, 1 grain of rice

$2^1 = 2$ doubled 1 time, 2 grains of rice

$2^2 = 4$ doubled 2 times, 4 grains of rice

$2^3 = 8$ doubled 3 times, 8 grains of rice

Clearly one can see a one-to-one mapping of elements in the story to elements in the mathematical patterns. Following these patterns can be an enjoyable activity for students, and at the same time, it can stimulate a sense of exploration and discovery. Students might be compelled to ask, "What if the rice was tripled each time—how does the pattern change?" In both the traditional approach and narrative approach, students may be exposed to the same concepts and ideas, such as division by zero, factorials, or exponential growth, yet the potential to engage their imaginations differs significantly. Stories, like *The Twelve Diamonds, Anno's Mysterious Multiplying Jar*, and *One Grain of Rice*, create "a patterning environment in a way that grabs their attention and that facilitates the experience of mathematical surprise, discovery, and insight" (Gadanidis & Hoogland, 2002, p.6).

4.5.2 Mathematically-Structured Story

Mathematical patterns may also be seen in the structure of the story itself (Lipsey and Pasternuk, 2002). Students are already familiar with rhyme, rhythm, and patterns that occur in poems, limericks, and various other literary forms. Sonnets and Haikus require the writer to work within the parameters of certain numerical constraints.[6] However, to what ends do these imposed structures and their intended emotional responses serve? Patterns are essential to understanding mathematics and any exercise in exploring them is beneficial. “Finding ways to eloquently express these thoughts in rhythm, rhyme, measured syllables, or with specific initial consonants is a complex problem” (Rule & Kagan, 2003). But the primary purpose of a sonnet is to engage the reader in feelings of lust, unrequited love, or heartbreak, while a Haiku ultimately draws attention to the nature of humanity. In both cases, however, it is for a literary end. What about for a mathematical end? Gowney (as cited in Lipsey and Pasternuk, 2002. p.2) created a poem that is structurally based on the prime factorization of the natural numbers (1, 2, 3, 2x2...). The first four lines of his poem are as follows:

1	cold
2	winds howl
3	geese go south
2 x 2	nights long tea steeps

[6] A Shakespearean Sonnet consists of fourteen lines of iambic pentameter. The "iambic" part means that the rhythm goes from an unstressed syllable to a stressed one, and the "pentameter" part means that this pattern repeats five times. Haiku is comprised of three non-rhyming lines, the first having five syllables, the second seven, and the third five.

An analysis or writing activity involving such narrative forms can engage the imaginations of students in a way that a traditional lesson in prime factorization may not.

4.6 Jokes and Humour

Another cognitive tool I consider when shaping stories is that of jokes and humour (Egan, 2005; Zazkis & Liljedahl, 2005). If one spends even a small amount of time with high school students, they will soon discover that their lives are filled with humour (Egan, 2005). Teasing, bantering, and idle chitchat, although routine activities, are often the testing grounds for developing this powerful cognitive tool. For teachers, the careful interjection of humour into a lesson or discussion can turn a seemingly mundane event into a memorable one. Unfortunately, humour in the mathematics classroom usually appears as nothing more than a simple "hook" to get students' attention or as a refreshment before returning to the "more serious" task of learning (Egan, 2005; Moreall, 1989). Yet jokes and humour can play a fundamental role in the learning of mathematical ideas. Egan (2005) suggests jokes and humour can encourage students to become more aware of the language they use. Instead of seeing it as some unconscious activity, they will begin to see language as an object that can be manipulated. In terms of mathematics, "jokes [and humour] can point to the subtleties of mathematical language and to the subtleties of intention and the implicit underlying assumptions" (Zazkis & Liljedahl, 2005, Chapter 1, p.34). As an example, consider this excerpt from *Math Curse* (Smith & Scieszka, 1995) in which a young girl wakes up to find she sees everything in terms of mathematical problems:

On Tuesday I start having problems… I take the milk out for my cereal and wonder:

1. How many quarts in a gallon?

2. How many pints in a quart?

3. How many inches in a foot?

4. How many feet in a yard?

5. How many yards in a neighbourhood? How many inches in a pint? How many feet in my shoes?

I don't even bother to take out the cereal. I don't want to know how many flakes in a bowl (p.1-6).

The first few points seem like typical questions one might encounter in a grade eight mathematics class. But by point 5, things get strange. The words "yard" and "feet" are no longer used in the way we expect them to be used—as units of measurement—and the association of inches with pints is obviously misdirected. A similar approach can be seen in the following joke (Runde, 2008):

One day, Jesus said to his disciples: "The Kingdom of Heaven is like 3x squared plus 8x minus 9." A man who had just joined the disciples looked very confused and asked Peter: "What, on Earth, does he mean by that?" Peter replied: "Don't worry - it's just another one of his parabolas" (p.1).

Here, the word "parabola" is associated with the word "parable." One cannot help but to go back and verify the description of the parabola. In both examples, our conceptual patterns have been violated—what we expect to occur does not, and we are surprised. As a result, we find these situations humorous while at the same time we become more alert to the meaning of each term and the contexts they are used in (Morreall, 1989).

As another example, consider the following joke that might be used to critique the stereotype notion that women cannot do well at advanced mathematics (Renteln & Dundes, 2005).

> Two mathematicians are in a bar. The first one says to the second that the average person knows very little about basic math. The second one disagrees, and claims that most people can cope with a reasonable amount of math. The first mathematician goes off to the washroom, and in his absence the second calls over the waitress. He tells her that in a few minutes, after his friend has returned, he will call her over and ask her a question; all she has to do is answer, "One third x cubed." She agrees, and goes off mumbling to herself. The first guy returns and the second proposes a bet to prove his point. He says he will ask the blonde waitress an integral, and the first laughingly agrees. The second man calls over the waitress and asks, "What is the integral of x squared?" The waitress says, "One third x cubed." Then, while walking away, she turns back and says, "Plus a constant!" (p.33).

Once again, something surprising happens that clashes with the mental framework into which it is received—a waitress who presumably knows very little about advanced mathematics actually points out an omission made by someone who is an expert in the field (Morreall, 1989).

In each of the examples above, there is a discrepancy between what is expected and what actually unfolds. In the first two examples, it pertains to our use of mathematical language, while in the last example it has to do with our notions of mathematical ability. According to Willmann (1940), this is the essence of humour—it is the result of "some sort of contradiction or incongruity" (p.72). We could say the incongruous idea or event is radically different from what we are familiar with—the conceptual patterns we tend to adhere to (Wilmann, 1940). In this regard, we might even say that the cognitive tool of jokes and humour is really just a sub-category of the cognitive tool of rhyme, rhythm, and pattern. We find something funny because it deviates from the patterns we have come to accept. It is similar to the way we find pleasure in the twist at the end of a movie or exhilaration at that moment of "aha" after a string of disappointments.

It is not surprising then that we remember more readily those elements of a story that are humorous—those moments in the story that are out of place, that clash with our conceptual schemata. "We like trains of thought more when they have jolts, reversals, and even 'wild-goose chases,' than when they proceed in a smooth progression to a conclusion" (Morreall, 1989, p.254). The idea of disrupting the logical patterns that surround us however, certainly goes against the rational thinking of many mathematicians and mathematics educators—we need only glance back to our section on rhyme, rhythm, and pattern to substantiate this point. Indeed, the basis of scientific inquiry is primarily concerned with eliminating any forms of dissonance, so to introduce an incongruity seems irrational. But as Morreall suggests (1989), such anomalies tend to foster imagination and encourage us to think more flexibly about a topic. We begin to see a familiar topic in a new way. By looking at something in so obviously the wrong context or category, we actually "reinforce the category it pretends to disrupt" (Egan, 2005, p.25).

Any story can be infused with humorous elements (Zazkis & Liljedahl, 2005). Consider again Archimedes' *Eureka* story. What events or ideas might our students associate with this historical figure after listening to our version of the legend? I suspect they would remember his incessant body odour, or his friends dragging him to the baths. They would undoubtedly recall his running through the streets naked screaming "Eureka!" All of these are surprising—they are incongruous with what we know to be responsible adult behaviours. Do these humorous aspects of the story take away from the scientific discovery Archimedes made or do they actually accentuate the steps leading up to it? How else might we highlight the principle of buoyancy Archimedes discovered using the cognitive tool of jokes and humour? We might suggest Archimedes had a large spherical shape growth on his belly that had veins and boils all over it and that pulsated with each beat of his heart. Perhaps it was when Archimedes saw this growth submerge in the water that he made his discovery? What about *The Story of Princess Dido*? What really happened in Tyre that caused the riff between Princess Dido and her brother? Could it have anything to do with the fact that the young and beautiful princess married her old and decrepit uncle? We could easily tie this into the situation with the local chieftan, Iarbas, and his desire to marry Dido. Perhaps Dido came up with her circle configuration after noticing Iarbas' big round belly as he ran in slow motion through the glistening Mediterranean Sea? Any elaboration of these peculiar circumstances would make for an unforgettable storytelling event.

4.7 Orally Told Story or the Read Story

One last point concerning shaping stories is, whether it is more effective to use mathematics literature in the form of children's trade books, or is it better to deliver stories orally from within the conscience of the storyteller. This may seem inconsequential, as it is generally believed that orally told stories are simply the unwritten counterparts of written or literary narratives. In this regard, both forms share the common features of narrative devices (Georges, 1969). Until now, I have made no attempt through the many examples in this thesis to make this distinction clear. There is a myriad of children's literature available that can enhance the learning of mathematical concepts and spark students' imaginations. Studies of literature-based approaches have shown that using a book or poem creates a non-threatening atmosphere which students are already familiar with and that reduces anxiety (Martinie & Bay-Williams, 2003).

Zazkis and Liljedahl (2005) contend, however, that the telling of the story is as important as the story itself, and that the delivery plays a significant role in the shaping of the story. We will consider two ways in which the orally told story is different from the read story.

First, the orally delivered story is different from the read story in that the oral storyteller is free from any text (Schiro, 2004). According to The Committee on Storytelling (Teaching Storytelling, 2007), stories have a greater impact when told from memory, as opposed to reading a story straight from a book. This allows more flexibility in the selection of content. This means the storyteller can craft the story according to their own unique needs, that is, what learning outcomes they wish their students to meet (Schiro, 2000). Many of the trade books teachers try to use have contrived story lines that attempt to derive some obscure mathematical concept from it in a "fun way." Sinclair (as cited in Gadanidis and Hoogland, 2002) cautions that "a pernicious consequence of appealing to students' love of something else in the hopes of increasing their interest levels in mathematics is that it endorses the belief that mathematics itself is an aesthetically sterile domain" (p.4). Another problem with using trade books is that many are written with a certain grade level in mind and to teach a certain skill (Whittin, 2002). The wrong book could inadvertently hinder learning. Moreover, teachers often thwart valuable mathematical explorations that emerge from within the context of a particular story in order to stay with the text. Some of the best mathematical discourses with students are those that arise spontaneously (Butterworth & Cicero, 2001).

Second, the orally told story is different from the read story in that the storyteller has the added ability to tailor their stories to the unique interactions that occur between themselves and their audience (Schiro, 2000). Georges (1969) contends that every storytelling event is unique. The message of a story is inseparable from all aspects of the storytelling event—primarily, the storyteller and the audience. The reader of a story is limited to what is written on the page to draw any emotional meaning, whereas the storyteller has the unique ability to adapt their story based on the interactions they have with their audience and the background knowledge they possess, in order to create emotional meaning (Bruner, 1991). For instance, a good storyteller might elaborate certain parts of the story they wish to emphasize or downplay parts that will fail to move the story forward (Schiro, 2000). This also suggests a storytelling event is not just a linguistic phenomenon, but entails certain other codes such as paralinguisitic and kinesic codes (Georges, 1969). When telling *The Twelve Diamonds*, the teacher might speak in a majestic manner and stand upright when describing the king, and crouch over and speak in an "evil" voice when revealing the intentions of the wicked sons. According to The Committee on Storytelling (Teaching Storytelling, 2007), when stories are told orally and from memory, a certain dynamic develops between the storyteller and the audience—an intimacy follows that facilitates an exchange of ideas and emotion, and is a factor in the way the story unfolds. This means the audience is not simply a receptacle for information, but is engaged in an ongoing dialogue with the storyteller (Bruner, 1991). In T*he Story of Princess Dido*, the students might react to the way Iarbus wanted to marry the princess. In response to their reaction, the teacher might elaborate this point further (i.e. Iarbus is an old, fat, hairy man; he has many wives; he killed his last wife, etc).

Simply presenting students with a story does not guarantee that their imaginations will be engaged with the mathematical content embedded within, neither does it ensure that learning will take place. A story must be carefully crafted taking into consideration certain cognitive tools students possess, as well as paying particular attention to the context and the manner in which the story is delivered. In the chapters that follow, I will explore specific cases of how carefully crafted stories can be used in the classroom to teach mathematical ideas.

5 RESEARCH QUESTIONS

5.1 The Imagination

Central to the arguments presented thus far is the idea that the imagination is crucial to learning. Much of my understanding about the prominent role of the imagination in education has developed through a study of the works of Egan (1986, 1997, 2005), Zazkis and Liljedahl (2005), and the Imaginative Education Research Group (IERG). Indeed, through the course of my research, I have become somewhat fixated on the belief that entreating the imagination should be as big a pursuit as the successful transference of the very information we deem important. Unless endearing the imagination becomes of paramount importance in mathematics education, there is little reason to believe the facts, algorithms, and rules students encounter will move beyond the stature of irrelevant, disconnected, and meaningless bits of information. In a general sense then, this is the pervading theme that underlies all of the ideas presented in the literature review, as well as the basis for the questions that I present here.

5.2 Stories and the Imagination

It should be clear by now that there are several cognitive tools available to students that can enhance their imaginative engagement with the curriculum content (Egan, 1997, 2005). Among them is the cognitive tool of story, of which plays an especially significant role (Zazkis & Liljedahl, 2005). Using Zazkis and Liljedahl's classification of stories (2005) and the theoretical frameworks of Egan (1997, 2005) and Zazkis and Liljedahl (2005), as described in chapters 3 and 6 of this thesis respectively, I have shown how stories can be utilized and shaped to interact with the curriculum content in a way that makes the mathematics come alive in the imaginations of students. In a specific sense then, it is this cognitive tool of story that interests us here and that we will attempt to delineate in the following pages of this thesis.

5.3 Stories and Mathematics

Two more points must be made before I present the two research questions I will be addressing hereafter. That stories and mathematics tend not to be considered compatriots in the educational process is significant. The obstacles to blending these two seemingly unrelated domains, in a practical sense, are readily available—time, curriculum goals, and resources are but a few. In terms of perception, one can spend just a few moments talking with students, colleagues, or even mathematicians about this odd pairing and you will almost certainly be met with some solicitous trepidation. I must concede however, that this is not necessarily the case at all levels of mathematical instruction. A great deal of research and resources are available that support the use of stories with children at both the elementary and middle school levels. What about at the secondary level? It may be surprising to the reader to discover that many of the stories presented up to now were gleaned from the addendum-like droppings of discourses celebrating the uses of stories with a much younger audience. Nevertheless, while a more concerted effort has been made in recent years to integrate mathematics and stories at the secondary level (i.e. the meeting at Mykonos), very little data has been generated to study the results of such endeavours (Senechal, 2007).

Lastly, I must point out that, unlike the trend to make mathematics more meaningful by highlighting its practical applications, I am mostly interested in using stories within a purely fictional setting. That is, while stories about measuring the angle of a roof for construction purposes or calculating the speed of a plane travelling between two cities may be of interest to students, I am primarily concerned with stories about pirate ships, ogres, Samurai warriors, and other imaginary-like characters and settings, and how effective they are at engaging the imaginations of secondary school students. I am certainly not suggesting we ignore stories about the angle of a roof or the speed of a plane. On the contrary, a portion of the research I present here accommodates this type of approach. What I am suggesting is that alternatively, we consider stories about measuring the angle of attack of aliens arriving from Uranus descending upon your bedroom, or the speed of a hi-jacked plane travelling from Vancouver to Toronto with a bomb on it that is about to go off, and you are the passengers' only hope (ironically, I am writing this section while on a flight from Vancouver to Singapore). Consequently, I have formulated two research questions that primarily address these issues and the answers to which will add to the body of literature available.

5.4 Research Question One

In what ways can stories be used to engage the imaginations of students with the secondary mathematics curriculum?

It is clear that stories have the potential to engage the imaginations of students at the secondary level. What is not clear is how these stories can be integrated into our instructional practices. I am interested in finding the practical ways stories can be used in the classroom for the purposes of establishing mathematical meaning. I hope to understand through my research whether or not using stories in the high school context is a valid means of instruction. Can stories take the place of traditional modes of mathematics instruction or should they be used only to compliment or enrich these other forms? Can stories provide students with the same learning opportunities as these traditional forms? Do they offer a new or alternate perspective to approaching mathematical ideas? Should we even expect students to sit and listen to our stories or is that a practice reserved for the younger grades? Is it more effective to deliver stories orally or to present it in printed form? Using Zazkis and Liljedahl's classification of stories (2005), we might ask what types of stories are most effective in eliciting an emotional response from students? Does writing stories help students better comprehend the mathematics they encounter? What other types of activities and assignments can we use in conjunction with stories? Through my research, I hope to answer all of these questions as well as provide mathematics instructors and researchers with an array of examples of the various story types and how to use them in the classroom setting.

5.5 Research Question Two

What are the characteristics of the stories that are most effective at engaging the imaginations of students with the mathematical content?

Where the first question speaks to the practical ways stories can be integrated into classroom practice, this question really addresses the stories themselves and the qualities that make them effective at eliciting an emotional response from students. I hope to gain some insight as to the nature of good stories and their particular effectiveness at the secondary level. It became clear to me through my own teaching practice that some stories work better than others in the secondary school environment. What are the qualities of these stories? How can we shape these stories to maximize their potential for imaginative engagement with the curriculum content? From chapter 4 of the literature review, we know that a story can be shaped by paying particular attention to various elements in the story, such as its context and the manner in which the story is delivered. Additionally, a story can be shaped by taking into consideration certain cognitive tools students possess. Which cognitive tools work best and what are the effects of using these cognitive tools on students' thinking? To what degree do students associate the stories they listen to with the mathematical content we want them to learn? Do they think of the mathematics in terms of the stories we tell? Do stories help students better communicate their understanding of mathematical ideas?

In the chapters that follow, I will address these questions through a series of pointed studies. I do not however, deal with each research question in isolation and the reader is left to determine which information is relevant to which particular question. Nevertheless, the data generated from each study will allow us to make some inferences and draw some conclusions about the use of stories at the secondary level, the sum of which will add to the collective body of knowledge already available.

6 METHODOLOGY: PARTICIPANTS, SETTING, DATA, AND THEORETICAL FRAMEWORKS

In this chapter, I will explain the methodology for my research. First I will look at the participants involved in the research and where the research took place. Next, I will look at the type of data collected in each of the studies and the theoretical frameworks used in both the treatment of stories and in the analysis of the data collected.

6.1 Participants and Setting

The participants in this research were students enrolled in Mathematics 8 Honours (MATH8H), Mathematics 9 Honours (MATH9H), Mathematics 9 (MATH9), and Principles of Mathematics 10 (PMATH10) at a secondary school in the greater Vancouver metropolitan area. At the time of this study, there were 141 students (28 in MATH8H, 30 in MATH9H, 23 in MATH9, and 60 in PMATH10) enrolled in these courses of which I was the instructor. Participation in this research was completely optional and students wishing to participate submitted a consent form signed by their parents (see section 17.2 in the Appendix). Of the 141 students, 105 students chose to participate in this research: 20 in MATH8H (10 girls and 10 boys), 23 in MATH9H (10 girls and 13 boys), 15 in MATH9 (7 girls and 8 boys), and 43 in PMATH10 (30 girls and 13 boys). I will describe the school and each of the courses in some detail:

6.1.1 Location of the Studies

The studies took place at a public school in an average to lower socio-economic area of the lower mainland of British Columbia. It is the oldest of the secondary schools (grades eight through twelve) in the district with a rich history dating back more than fifty years in the community. It is often referred to as "the flagship of the district." Situated within minutes of the downtown core, it is also considered the only "inner-city" secondary school in the district. A large portion of its approximately 1500 students comes from the apartments and townhouse complexes nearby. Additionally, many students are drawn to the school for its renowned basketball, music, gifted, and French immersion programs. Running on a 5 x 8 linear system, students attend each 75-minute period two to three times per week.

6.1.2 Mathematics 8 Honours

Mathematics 8 Honours is one of two grade eight mathematics courses offered at the school (the other being Mathematics 8). This is the highest level of mathematics in grade eight and students enrolled in this course usually intend to go to university or college after graduation. These students are selected solely on the basis of the scores they receive on a comprehensive test written during the last week of grade seven. Approximately 75 of the top mathematics students in the district write this test in hopes of winning one of the 30 spots in this course. All of the students enrolled in this course have strong mathematics skills and would likely say that mathematics is their favourite subject. The course itself is considered "enrichment," which means the curriculum is the same as the Mathematics 8 course (MATH8), but with additional units or lessons. In most cases these units or lessons consist of discovery activities and assignments that expound on topics normally given little attention in the regular grade eight curriculum. These additional assignments often run concurrently with those assigned in MATH8. Additionally, these students have been exposed to a wide variety of instructional and assessment techniques including: writing activities, discussions, debates, computer explorations, individual and group presentations, take-home tests, journaling, role-play activities, and storytelling events. Students must maintain an average of 80% or higher if they wish to remain in the honours program.

6.1.3 Mathematics 9 Honours

Mathematics 9 Honours is one of two grade nine mathematics courses offered at the school (the other being MATH9). This is the highest level of mathematics in grade nine. Students enrolled in this course are usually the same students who have successfully completed MATH8H with an 80% or higher average. Additionally, one or two students may have opted to enrol in this course after both receiving 90% or higher in MATH8 the previous year and writing a comprehensive test. Like the students in MATH8H, most of the students enrolled in this course have strong mathematics skills and would likely consider mathematics their favourite subject. The course is setup in the same way as MATH8H, with the same variety of instructional and assessment techniques, but students are not required to meet any minimum standard percentage as there is no honours mathematics course offered in grade ten.

6.1.4 Mathematics 9

As mentioned above, Mathematics 9 is the other grade nine mathematics course offered at the school. This is the standard mathematics course grade nine students are required to enrol in (unless they are enrolled in MATH9H). The makeup of the particular class involved in this study, however, consisted of several students with relatively weak mathematics skills. Many of the students in this class would probably have said mathematics is the subject they like the least. This may account for the relatively low percentage of students from this group willing to participate in the research. Like the honours students, these students have also been exposed to a wide variety of instructional and assessment techniques.

6.1.5 Principles of Mathematics 10

Principles of Mathematics 10 is one of three grade ten mathematics courses offered at the school (the others being Essentials of Mathematics 10 and Applications of Mathematics 10). PMATH10 is the top level of mathematics in grade ten. Unlike the students in MATH8H, MATH9, and MATH9H, these students have had minimal exposure to alternate instructional and assessment techniques. The curriculum itself consists of a large amount of new and difficult material and culminates with a large-scale high stakes external assessment, and so a stand-and-deliver approach is unofficially prescribed and expected both by the mathematics department and by students—this is largely driven by a "get through the curriculum" mentality. In comparison to many of the other subjects in grade ten, this course is concerned with mostly abstract symbolic language and is mostly perceived as lifeless and uninteresting. As a result, many of the students enrolled in this course would likely say mathematics is the subject they like the least. Furthermore, many of the students who might do exceptionally well in subjects like English or History, achieve average marks in PMATH10.

6.2 Data

The data for this study consists of student work produced as a result of each of the nine studies outlined in chapters seven through sixteen. Additionally, quiz and/or test answers corresponding to particular story assignments were also included as data in this research. Because the use of stories in various forms had become so much a part of my instructional practice over the last few years, the data collected in these studies were generated as a result of the normal ongoings of the mathematics classroom. In this regard, while all of the students did not participate in the research, they all participated in the activities and assignments associated with the research. In some cases, the reader may wonder why certain quiz and test questions were considered as part of "the normal ongoings of the mathematics classroom," as they appear to only serve the purposes of this research. However, from section 3.4.1 in the literature review, we know that the more connections students can make to a mathematical idea—whether through grounded familiar language or abstract symbolic equations—the more opportunities they have for establishing mathematical meaning. Skemp (as cited in Zazkis & Leikin, 2007) even argues that the formation of a concept requires several examples of where some common element is observed. In this respect, writing about a mathematical concept may be equally valuable as solving an abstract mathematical equation in terms of establishing mathematical meaning since both activities require the learner to explore common elements of the concept—albeit in vastly different ways. It is no coincidence then that in each of the following studies, students are required to respond to the stories they hear through writing or other creative activities. Additionally, as a research tool, the stories students write can reveal flaws in their perceptions and the other conceptual difficulties they might have when approaching mathematical ideas (Zazkis and Leikin (2007). The data collected from students in each of the nine studies is outlined below:

The Legend of Chikara Ni Bai

- Student-written stories about exponential growth

The Pythagorean Problem

- Student-written scripts about the discovery of irrational numbers
- Students' answers to two test questions concerning:

    1) The connection between The Pythagorean Theorem and $\sqrt{2}$;

    2) The relationship between the circumference and diameter in a circle.

The Four 4s

- Students' solutions to *The Four 4s* problem homework assignment.
- Students' answers to one question about *The Four 4s* problem.

Skull Island

- Student-created maps, stories/themes, and clues.

The Skippers

- Student-written stories about uniform motion/rate of work problems.

Kelloggs Needs Your Help

- Student-created cereal boxes
- Students' answers to one test question about the connection between *The Story of Princess Dido* and the *Kelloggs Needs Your Help* cereal box assignment.

The Wicked Sons

- Students' answers to one test question about division by zero.

Mathematically-Structured Story

- Students' answers to one *test* question about The Fundamental Theorem of Arithmetic.
- Student-written stories about prime factorization.
- Students' answers to one *quiz* question about The Fundamental Theorem of Arithmetic.

The Story of Princess $\sqrt{3}$

- Student-written stories about converting a mixed radical to a mixed radical in simplest form.

6.3 Theoretical Framework

The theoretical framework for this research is based on the works of Rina Zazkis and Peter Liljedahl (2003) and Kieran Egan (1986, 1997, 2005). Central to the arguments presented in this thesis is the idea that the imagination is essential to learning.

If we are to engage students with the material of the mathematics curriculum, then we must engage their imaginations—and connectedly their emotions (Egan, 2005; Zazkis & Liljedahl, 2003). The typical way mathematics is taught at the secondary level does not take into account the important role imagination plays in helping children understand new concepts and establish meaning (Egan, 1997, 2005).

6.3.1 Egan's Theoretical Framework

Students acquire, at certain stages, specific "cognitive tools" that can be used to effectively engage their imaginations (Egan, 1997, 2005; Zazkis & Liljedahl, 2003). Understanding what these mental devices are and what types of understanding students possess is important if we are to engage them with the material of the mathematics curriculum. According to Egan (2005), the primary cognitive tools that come with the development of oral language are: story; metaphor; binary opposites; rhyme, rhythm, and pattern; jokes and humour; mental imagery; gossip; play; and mystery. When students acquire these tools, they will develop what Egan (1997) calls, a "mythic understanding." Children, around the age of seven or eight, begin to develop a new set of cognitive tools as they become literate. These are: the sense of reality; the extremes of experience and the limits of reality; association with heroes; the sense of wonder; knowledge and human meaning; and a narrative understanding (Egan, 2005). The acquisition of these tools leads to the development of Egan's "romantic understanding" (Egan, 1997). This type of understanding is characterized by some concessions to reality and a need to make sense of the world in mostly human terms (Tyler, 2006). The cognitive tools that come with the development of theoretic thinking are: the sense of abstract reality; the sense of agency; and a grasp of general ideas and their anomalies (Egan, 2005). When students acquire these tools, around the time of adolescence, they will develop what Egan (1997) calls, a "philosophic understanding." Each of these understandings makes a distinctive contribution to a student's understanding. This accrual of understanding does not, however, occur "naturally." Rather, the process occurs only when the array of corresponding cognitive tools are used appropriately (Tyler, 2006).

6.3.2 Zazkis and Liljedahl's Theoretical Framework

While Egan focuses on using all of these cognitive tools to engage the imaginations of students and develop their various understandings, Zazkis and Liljedahl (2005) concentrate primarily on the cognitive tool of story—with a particular focus on teaching mathematical content. In their framework, story is the most powerful of the cognitive tools and the rest are there to support its use. For example, the cognitive tool of binary opposites is most effective, when used within the context of a story to create conflict and to keep the story moving; the cognitive tool of metaphor is particularly engaging when it is seen in the form of mathematical objects taking on human qualities; attention to rhyme, rhythm, and pattern often makes a story more memorable and underscores the inherent mathematical content. Using stories for the purpose of teaching mathematics is both an efficient and relatively simple means for teachers to present mathematical ideas in a way that is imaginatively engaging. Zazkis and Liljedahl (2005) categorize stories according to their function in the classroom and their potential for imaginative engagement: Stories can be used to ask a question; accompany a topic; introduce an idea; intertwine with a topic; explain a concept; or introduce an activity. These categories are neither exhaustive, nor are they completely disjoint—a story may fall into one or several of these categories. Both of the frameworks described above start with the assumption that you should address a particular curriculum item, and then design instruction accordingly using the appropriate cognitive tools. This process is summarized in the following planning framework developed by Zazkis and Liljedahl (2005, Chapter 3, p.2-17):

6.3.2.1 Planning Framework

1. Identify the target

The target is the specific part of the curriculum content we wish to address. It could be a topic such as irrational numbers, or it could be related to a specific formula or strategy, like using the Pythagorean theorem to find the missing side in a right triangle. The target is often referred to as the "intended learning outcome."

2. Identify the problem (mathematical problem or problematic issue).

After we have identified the target, we must ask: What aspects of the target are problematic? What is the mathematical problem or activity that has the most potential for imaginative engagement with the target? The problematic issue might be irrational numbers and why they cannot be written as numbers with terminating or repeating decimals.

Identifying cognitive tools: Exploring what cognitive tools are appropriate to present the problem in a way that engages students' imaginations. Identifying the main tool and supporting tools to be used.

We will use a story as our main cognitive tool and then consider what other cognitive tools will support its use. For instance, we might choose the story of Hippasos' murder to explore the nature of irrational numbers. To create a conflict, we might use the supporting cognitive tool of binary opposites and pit the "heavenly-minded" Pythagoreans with the "down-to-earth" Hippasos (see section 5.1.3 for a more complete discussion of this topic).

3. Organizing presentation of the problem using cognitive tools identified in (2). Considering possible students' engagement.

A story with secret organizations, religious cults, murder, deceit, and cover-ups has the makings of a best-selling Dan Brown novel! It would take little effort to introduce the problem in the form of a story accentuating its inherent binary structuring. Perhaps setting up Hippasos as in charge of building the temple for Apollo and then asking students to design it using repeated applications of the Pythagorean theorem. At first students will have to construct floor plans using right triangles based on Pythagorean triples. But then comes the dilemma—Apollo wants a beautiful square pavilion…with a wall running diagonally from two of the corners (okay it sounds like a terrible design, but I am sure the reader can come up with something better). Here is the catch: Each wall must be a whole number or a ratio of whole numbers—it is the order of the gods!

4. Extending or varying the initial problem situation (optional).

Varying or extending the problem often leads students to a richer understanding of the original problem. Perhaps students might be challenged by Apollo to come up with a general formula for generating Pythagorean triples. They might further explore the relationship between the circumference and diameter of a circle. Can a circle ever have both a rational circumference and diameter?

5. Conclusion, closure.

There are many ways to conclude this mathematical activity. Students might be invited to finish the story or write a play that describes the outcome of the conflict—maybe with a twist as to who actually murdered Hippasos! Perhaps students might be put into two groups and debate each side of the conflict or conduct a mock trial that determines Hippasos' fate. (Remember our discussion of poetic license?).

6.3.2.2 Treatment and Analysis

Because of its particular focus on the cognitive tool of story to establish mathematical meaning, the two main components of this research are based primarily on the theoretical framework of Zazkis and Liljedahl (2005). These are as follows:

a. Treatment of Stories

In chapters seven through fifteen, I present the reader with nine stories/story assignments. Each of the stories/story assignments laid out in these chapters were designed and administered according to the planning framework described in section 6.3.2.1. In each case, the story/story assignment was used as an instructional strategy to introduce, develop, or reinforce curriculum content. A detailed discussion of these processes and the reasoning behind each one is found at the beginning of each chapter.

b. Analysis of Student Work

Collected data was analyzed according to the previously mentioned framework and organized according to content. In chapters seven through fifteen, I present evidence of imaginative engagement paying particular attention to any use of the cognitive tools described in this framework; as well as those described in the larger more broad framework described by Egan (1997, 2005). I also identify several other instances of students' imaginations at work as described in the literature review. Some of these are listed below:

- They incorporate new ideas, whether finding new ways to solve an old problem or new ways to communicate ideas (Gadanidis and Hoogland, 2002).
- They anticipate upcoming events and look for alternative plots and solutions to problems (Gadanidis and Hoogland, 2002).
- Their ideas are "regenerative"—that is, they live fruitfully in future experiences (Gadanidis and Hoogland, 2002).
- They make generalizations or "squash and slice" difficult concepts in order to make them understandable (Dialogue, 2005).
- They struggle to visualize new concepts or extend their grasp of something that is beyond common limits (Dialogue, 2005).
- They form mental images, alter these images in countless ways, and make distinctions and draw connections between domains. (Rose, 1996, p.347).
- Their thinking becomes more flexible, creative, and energized (Egan, 2005).
- They are emotionally involved with the material (Egan, 2005).

Clearly, students engaging in some sort of writing or creative exercise is essential in order to identify the imagination at work.

In each of the following nine chapters, I describe the treatment of a particular story/story assignment and then give an analysis of the data collected. A number of themes emerged as a result of this process. These themes provide the substance of the findings of this thesis. A discussion of these themes and the conclusions one can draw from these themes will be left to the final chapter of this thesis. The reader should note however, that almost immediately after beginning my research I soon realized that to collect and analyze this amount of data was a formidable a task. To some degree, what I present in the following nine chapters is a brief but trenchant look at what can happen when stories are integrated into the secondary mathematics curriculum. Nevertheless, I include the treatment of each story/story assignment, the data analysis, and in some cases a brief conclusion in order to reveal the extent to which stories became a regular part of my instructional practice and to give mathematics instructors a starting point for their own more generous explorations. The table below provides an outline of each of the nine studies that follow:

Table 6.1 Outline of Studies

Study	Story	Target	Cognitive Tools	Data
I	The Legend of Chikara Ni Bai	Exponential Growth	Rhyme, rhythm, and Pattern; Jokes and Humour	Student-written stories
II	The Pythagorean Problem	Irrational Numbers	Binary Opposites; Knowledge and Human Meaning; Jokes and Humour	Student-written scripts; Students' answers to two test questions

III	The Four 4s	Order of Operations	Binary Opposites	Student solutions of order of operations problem; Students' answers to one test question
IV	Skull Island	Graphing Linear Equations	Binary Opposites; Knowledge and Human Meaning; Jokes and Humour	Student-created maps; Student-written stories/themes; Student-created clues
V	The Skippers	Word Problems	Binary Opposites; Jokes and Humour	Student-written stories
VI	Kelloggs Needs Your Help	Volume and Surface Area	Binary Opposites	Student-created cereal boxes; Students' answers to one test question
VII	The Wicked Sons	Division by Zero	Bianry Opposites; Rhyme, Rhythm, and Pattern; Jokes and Humour	Students' answers to one test question
VIII	Mathematically-structured Story	Prime Factorization	Rhyme, Rhythm, and Pattern	Students' answers to one test question; Student-written stories; Students' answers to one quiz question
IX	The Story of Princess $\sqrt{3}$	Radicals	Metaphor	Student-written stories

7 STUDY I: THE LEGEND OF CHIKARA NI BAI

7.1 Treatment

7.1.1 Target and Problematic Issue

The Legend of Chikara Ni Bai is a story to explain the concept of exponential growth. According to the planning framework developed by Zazkis and Liljedahl (2005) outlined in section 6.3.2.1, the concept of exponential growth is the target—the specific part of the curriculum I wish to address. The problematic issue becomes apparent only after students begin to develop the exponent laws and use them with problem sets. Students tend to rely more on the rote memorization of these laws to solve exponent problems, instead of seeing them as a product of the rich mathematical patterns inherent in the concept. Identifying and following these patterns will better facilitate the learning of more difficult concepts such as zero and negative exponents. In this regard, my secondary vehicles for engaging students with the mathematics are the cognitive tools of rhyme, rhythm, and pattern and jokes and humour.

7.1.2 Cognitive Tools

An appropriate context to explore the patterned nature of exponential growth is found in the ancient Japanese art of sword-making. This approach is largely inspired by the work of Theonni Pappas, in *More Joy of Mathematics* (1991). According to Pappas, the construction of a Japanese sword was an elaborate and invariable process dominated by pattern. The sword-maker would heat a piece of steel until it was at welding temperature, fold the steel over, weld it, and then forge it to its original dimensions. This process was repeated 22 times, which produced over 4 million layers of steel (Pappas, 1991). This explains why Samurai swords are so light, yet can cut through even the densest of metals. The real value of this tradition to the learning of mathematics, however, is the explicit patterns that emerge in the process of which students can discover, follow, and predict, and thereby enlarge their grasp of the concept of exponents and exponential growth:

$2^0 = 1$,	0 folds,	1 layer of steel
$2^1 = 2$,	1 fold,	2 layers of steel
$2^2 = 4$,	2 folds,	4 layers of steel
…		
$2^{22} = 4{,}194{,}302$,	22 folds,	4,194,302 layers of steel

According to Pappas (1991), "the art of swordmaking in Japan was an old and revered profession which was secretly passed from father to son or master to pupil. The swordsmith followed specific religious observances and traditions, and wore a ceremonial costume" (Pappas, 1991, p.168). We could easily ignore this aspect of the craft and get right into examining the mathematical patterns. However, by connecting the mystical elements of Japanese sword-making to the mathematical patterns, the story takes on a new dimension—leaving the confines of the natural world and entering the realm of the fantastic. From section 4.4, we know that this combination of reality and fantasy, may lead to imaginative activity. We might say that the fantasy elements of the story are in a one-to-one correspondence with the real elements in the mathematical pattern. In order to further engage students' imaginations with these mathematical patterns, the supporting cognitive tool of binary opposites was also incorporated into the story (Zazkis & Liljedahl, 2005; Egan, 2005). On one side, we have the legendary Samurai warrior, Chikara Ni Bai, whose purity is founded on a lifetime of service to the emperor and his village, and made obvious by the love he has for his son. Opposing our hero is Akumo No Niisan, Ni Bai's mortal enemy, and one-time apprentice, who had been lured to the dark side during a moment of weakness (yes, everything eventually comes back to *Star Wars*). His profane nature is made explicit by his unmerciful acts of violence to Ni Bai's village and family. In our story, the listener follows Ni Bai in his quest to avenge the death of his only son. To further highlight the binary structuring in the story, Japanese teacher, Aaron McKimmon, carefully devised names for characters and places in the story that clearly identify their position in the conflict:

Chikara Ni Bai (the great warrior)–"Strength Times 2"

Akuma No Niisan (the evil warlord) –"Devil's Older Brother"

Jigokubi Hanzo (the sword-maker) –"Hellfire Hanzo"

Tengoku No Tani (Ni Bai's village) –"Heaven's Valley"

Ja Aku No Donzoku (No Niisan's village) –"Evil Abyss"

Tetsu No Tamashi (the magical sword) –"Iron Spirit"

The cognitive tool of jokes and humour played a key role in this story. Incongruities, like Chikara Ni Bai's heroic demeanour superimposed against the backdrop of the silly antics of characters such as Chacho the Village Idiot or Jigokubi Hanzo, served to strengthen the binary structure of the story and thereby heighten the importance of the sword's construction. A head on a stick, a mortal enemy resembling a character from Star Wars, a misfortune at the grocery store, the fires of Hell, screams of uncertain intent, and the like, invited students to look at exponential growth in a new and unfamiliar way.

7.1.3 Presentation of Problem and Conclusion

According to our planning framework (Zazkis & Liljedahl, 2005), the presentation of the problem was organized according to the cognitive tools already identified. Our primary cognitive tool is story, and so the problem was embedded within *The Legend of Chikara Ni Bai*. I chose to deliver the story orally from the front of the classroom.[7] The names of characters and places, along with their meanings, were written on the whiteboard as they were introduced in the story. In addition, as the story was told, I drew diagrams on the whiteboard illustrating the concepts as they unfolded and recorded any discoveries students made concerning the nature of exponential growth. From section 3.7, we know that an effective way to promote student engagement with the content is through writing stories. After students listened to the story, I gave them an assignment in which they had to write their own exponents stories. There were no restrictions as to the topics students could write about, nor any prescription for how exponents were to be incorporated into their stories.[8] Their story must be at least one paragraph long, and could include a question at the end, which would be posed to the class the following period.

[7] Unless otherwise indicated, all stories in this research were delivered to students in this format.
[8] Students were instructed to write "exponent" stories and not specifically "exponential growth" stories.

7.1.4 Participants

28 students in MATH8H, 30 students in MATH9H, and 23 students in MATH9 participated in this assignment, but only 18 students in MATH8H, 23 in MATH9H, and 15 in MATH9 chose to allow their work to serve as data. All of these students had prior experience with exponents.

7.1.5 The Story

> Steam rose from the ground as the great warrior rode onto the bloodied field. The smell of raw flesh and guts would cause anyone to curl over and vomit… but not Chikara Ni Bai. He had seen his fair share of death as a Samurai and leader of his people. He was finished fighting now. His only desire was to return home and hold his beloved son in his arms once again. He gathered the other warriors together and began the long journey back to *Tengoku No Tani*, which means "heaven's valley." Ni Bai was known throughout the land for his skill with the sword. Some say he received this gift as a child from the spirits, but Ni Bai knew it was his apprenticeship under his sensei, Gogun Yamaguchi[9], that had led him to discover the subtle art of sword-handling. Ni Bai thought about all the things he would do with his son...play…tell jokes…play stuff…give him money…all the things a father is supposed to do with an eleven year old boy. As they neared the village, Chacho the Village Idiot, and apprentice to Ni Bai, who had not yet learned the art of biting one's tongue, let out a blood-curdling yell [wait for one student to yell]. For there, across the meadow rose smoke. It was coming from their village. "My son!" yelled Ni Bai. The whole village had been destroyed. While the warriors, and Chacho, the Village Idiot, had been away, Akuma No Niisan, Ni Bai's one-time apprentice and now mortal enemy, had ravaged the village. Nothing was left…the women had been killed, the men had been carried off, and the animals had all been fixed. Ni Bai ran to his house in the center of the village. There, lying in the doorway was his wife. She had been stabbed. Blood was squirting everywhere. Ni Bai, however, only had one thing on his mind—his son. He ran into his son's room and at once, fell to his knees. There, on the cold wooden floor was his son's body—but no head. Ni Bai fell down beside his son—or at least most of his son—and let out a blood-curdling yell [wait for a student to let out a yell].

---

[9] The reader may note that Gogun Yamaguchi, whose name I do not know the meaning of, was the founding father of Gojo-ru Karate in Okinawa—the same brand of karate Daniel Laruso learns in *The Karate Kid*. Unfortunately, I washed out of martial arts shortly after receiving my yellow belt.

Next to his son's body was a piece of rice paper with the words "I have your son's head" written on it. Ni Bai had only one thought, revenge… revenge. But to defeat his mortal enemy, the great Chikara Ni Ba would need help. Not the help of man or beast. No, Ni Bai would need a sword…a sword of great power and speed. There was only one man who could create a sword of this distinction…Jigokubi Hanzo. Hanzo was famous for his sword-making skills. Many believed that the person who carries one of Hanzo's swords will never know defeat. Ni Bai set out immediately for Hanzo's village which was several moons away. When he arrived at Hanzo's home, the door swung open and the sword-maker fell down at the warrior's feet and let out a blood-curdling yell [wait for students to yell]. "Word has come to me that the great Chikara Ni Bai is to avenge the death of his son," Hanzo began. "I know what it is like to lose someone you love. I once lost my father. He was at the grocery store. One minute he was looking at the zucchinis and the next minute he was gone. What's worse, the zucchinis were gone too. Every week they call trying to get me to pay for those zucch…." Ni Bai interrupted, "I have come…" "Wait…I know what you have come for," said Hanzo. "You want a sword. One like no other that has come before it. A sword so deadly, that even the sight of it will bring fear to its enemies. A sword so strong it can cut through the most solid of metals and so light it can move through the air more swiftly than a greased chicken on a water slide." "You can make a sword of this nature?" asked Ni Bai. "I begin tonight," said Hanzo. That evening, Hanzo took one sheet of steel and placed it in the fire—a fire hotter than the very fires of hell. The steel began to turn red and then blue and then white. Hanzo let out a blood-curdling yell [wait for student] and then let his hammer fall upon the now obedient metal. He hammered it and hammered it putting his very soul into each swing. After some careful maneuvering he folded the metal over, and then beat it into submission. [How many layers are there now? How many folds?]. Again, he took the folded sheet of steel and placed it in the fire...and again he let out a blood-curdling yell [wait for student]. Once again he hammered the naughty piece of steel until it seemed like the life had left it. And again he folded the steel over. [How many layers are there now? How many folds? How many times do you think Hanzo did this process in total?] 21 times the sword-maker punished the steel in this way.[10] [Now how many folds? How many layers?]. After seven days and seven nights, Hanzo emerged from his workshop to present the warrior with the greatest sword he had ever forged. "I give you, 'Tetsu No Tamashi'," said Hanzo. [This means "Iron Spirit"]. Ni Bai took the sword and held it up to the sky. A shimmer of light worked its way up the sword. "Careful, Ni Bai…even looking at this sword can draw blood," said Hanzo. Ni Bai let out a cry of victory, "aaaaaaaaeeeeeiiiiooooooo." He jumped onto his horse and rode off to meet his enemy No Niisan. The

[10] This process is actually conducted 22 times in total (Pappas, 1991).

warrior knew this would be his greatest battle… Tetsu No Tamashi went before him…

7.2 Analysis

7.2.1 Attention to Patterns

In *The Legend of Chikara Ni Bai*, the abstract concept of exponential growth is juxtaposed alongside a concrete event in the story—the forging of a mystical sword—*Tetsu No Tamashi*. Because recognizing and re-creating patterns is an example of the imagination at work, I will focus my analysis on students' use of patterns. Although unsophisticated and sometimes mathematically incorrect, most of the stories students wrote tended to focus on this patterning nature of exponents. Consider for example Lee's story in which he explicitly illustrates the mathematical patterns:

> One day I was in my backyard when an alien spaceship came and beamed me up! One alien started poking me with a stick… every second after that, two aliens for each alien that was appeared and started poking me. How many aliens were poking me after one minute? 2^0 2^1 2^2 2^3…

A similar attention to pattern can be seen in Julia's, *Halloween Candy*,

> It was Halloween night and the children were pouring into my door yelling, "trick or treat, give me something good to eat". So I listened. The first kid came so I gave him 2 candies. Then 2 kids came, I gave them 4 candies in total. Then two kids came and I gave them 8 candies in total. So I wanted to know if I continued on the same streak and if another 2 kids came how much chocolate/candy I would have to give them?
>
> 1) 2^2
> 2) 2^3
> 3) $2^4 = 2 \times 2 \times 2 \times 2 = 2 \times 2\, [sic] = 4 \times 2 = 8 \times 2 = 16$
>
> So I would have to give them 16 candies in total and if I continued on the same streak all my candy would be gone quite quickly! I then decided to eat them all. Unfortunately, I was sick the next morning and was rushed to the hospital for halucinating.

Corey's exponent story includes a table to elaborate on the mathematical patterns:

> There is a man in "New York City" named Bill. Bill opened up a hot dog stand when he was younger. On his very first day of business, Bill sold only 5 hot dogs. On the next day, business got better [and] Bill sold 25 hot dogs. On day three, business got even better [and] Bill sold 125 hot dogs. On the fourth day, business was BOOMING! Bill sold a whopping 625 hot dogs. Bill is now a multy BILIIONAIRE being the creator of Microsoft. After all, his full name is Bill Gates.

Hot Dogs	Repeated Multiplication	Exponents
5	5	5^1
25	5×5	5^2
125	$5 \times 5 \times 5$	5^3
625	$5 \times 5 \times 5 \times 5$	5^4

Many of the stories students wrote included tables like in Corey's story, to help the reader see the patterns in exponential growth. In that way, they were using their imaginations to find new ways to communicate their ideas (Gadanidis & Hoogland, 2002). In some cases, the tables afforded students the opportunity to clarify what they had difficulty expressing in words. This is the case in Leah's story:

> One summer, Adam, Billy, Caroline, Dave and Elizabeth decided to make some money. They applied at the corner store and [they] all got the job. They talked to the owner and instead of just getting paid $5 an hour, each hour *they would get higher* [italics mine]. Adam worked one hour and got $5. Billy worked two hours and got $25 and by the time it got to Elizabeth who worked 5 hours, she got paid $3125.

1	5^1
2	5^2
3	5^3

4	5^4
5	5^5

7.2.1.1 Lack of Attention to Patterns

There were two cases however, where students used their imaginations to write the story, but did not engage their imaginations with the mathematical patterns. Consider Dennis' story for example:

> Bob was going to throw a small party. There would be 4^2 people at the party including Bob. His first attempt to cook for his guests was a disaster. First the burgers caught fire, igniting his apron and singeing his eyebrows. Now Bob wasn't that smart. Knowing that his guests would be arriving soon, he went to the drawer and got his scissors. He then cut pieces of the hair from his head, and glued them on his forehead where his eyebrows used to be. Now since he used some of his hair for new eyebrows, he had a big bald spot, Right in the middle of his head se he decided to wear his hat.
>
> Bob decided to try hotdogs. When the guests arrived, Bob told them that they were going to roast the hot dogs over a bon fire. When Bob was cooking his hotdog, it caught fire, so Bob tried to blow it out. When Bob blew, his hotdog flew into one of the guests' face, which made him bump into another guest and that guest hit another and eventually every one and their hotdogs were lying in a heap on the grass. Bob decided to order pizza. Bob knew that a large pizza had 3^2 slices and a medium had 2^3 slices. He estimates each guest would eat between 2 and 3 slices each. Bob knew he would need to order a minimum of 2 to the fifth power slices and less then 7^2 slices. Bob, who was not very good at math decided to order 3 medium pizzas and 3 large pizzas. He would eat the extra 3 slices for lunch the next day.

Karoline engages the concept of exponents in a similar way:

> One day Mark was going to the store to buy a new shower cap because his old was ripped. The store clerk told him that they were \$10 each. Mark spent \$10². He had \$200 with him so he spent ½ of his money. With the rest of the money he bought some goggles. He bought 2 pairs, they each cost \$2 he spent \$2². In the end he had spent a total of \$104

Consequently, this lack of attention to pattern corresponded to a lack of attention to the concept of exponential growth.

7.2.2 Change of Base

Perhaps, the most surprising result from this part of the study was the number of students who extended their concept of exponential growth to include bases other than 2. See Table 8.1.

Table 7.1 Bases Chosen by Students

Base	MATH8H	MATH9H / PMATH9
2	11	17
3	0	15
4	1	0
5	1	0
7	0	1
13	1	0
No specific base	4	4

Becky uses a story about trees to tackle the concept of exponential growth where the base is 3:

> All the trees in an elementary school's playground were cut down. It looked very bare out there so the school decided to plant a couple of trees in the playground. One of teachers went and bought some seeds for the kids to plant. After the trees were planted, the seeds started to grow. Everyday the trees got lots of sun, because it was during the summer, Fertilizer and some water which each kid will get to water the tree. Everyday the trees get fertilizer, sunlight and water it would triple in size and after 3 days the seed would start growing into a tree. After 10 days how tall will the tree be?

Tree	Tree getting bigger	Exp. form

3	1	3^1
3	3	3^2
9	27	3^3
27	81	3^4
81	243	3^5
243	729	3^6
729	2189	3^7
2189	6561	3^8
6561	19683	3^9
19683	59049	3^{10}

The tree will be 59049 mm tall after 10 days

Although Becky's reasoning seems unclear (i.e. the first few rows of the table), her story demonstrates a desire to extend her understanding of exponential growth.

Diana uses a base of 4 to determine the number of people at a party:

> A long time ago during the prehistoric age, Mr. Balakrishnan and three of his friends decided to have a party. Each person went out and found four friends. But it wasn't big enough for them. They wanted to have as many people as they could to come to their party. So again everybody went out and got four more friends to come. They did this 5 times until they finally thought they had enough people. Unfortunately, after all their hard work, an old woman named Mrs.Whitehead who lived on their street had called the cops and had their party shut down.
>
> How many people were at the party before it was shut down?
>
> 4 to the power of 5
>
> 4x4x4x4x4= 1024
>
> *Bonus: How long did it take Mrs.Whitehead to recover from the sight of Mr. Balakrishnan's afro-mullet.
>
> (Just Kidding)

Alec wrote a story using 7 for a base:

Generic Joe had a generic problem. He needed a generic cloth for generic face-washing time. He decided to visit the generically Abnormal Bob, the only known cloth-maker (not a generic cloth-maker). Abnormal Bob lived abnormally on the abnormal side of town, so Generic Joe needed to travel generically over there. He drove his generic car for a generic while until he reached the abnormal part of town. He stepped into Abnormal Bob's abnormal home. Abnormal Bob asked him about his generic problem. Abnormally, Abnormal Bob had the generic materials to make the generic cloth. In order to make a generic cloth, the generic materials have to be twisted 7 times, then hardened with generic starch, and then softened with generic fabric softener. The generic cloth is done when Abnormal Bob has done this 11 times.

$7^{11} = 1{,}977{,}326{,}743$

There are 1,977,326,743 twists.

In *The Magical Gumball Machine*, Hanna goes as far as base 13:

T'was a warm evening in the middle of August and the children of the village were putting on their costumes for trick or treating. You might ask why they were trick or treating in August, but that shall forever remain a mystery.

Back to the story, Xavier and his four friends, Zolt, Heath, Lief and MPLZ3

Were approaching the house that had the MAGIC GUMBALL MACHINE!!! The owners of the house were in fact very lazy on Halloween so they installed a fantastic self-serve gumball machine into their front door. (9 easy payments of 99$)

Xavier was first to his gumball. He turns the knob… and (gasp here) 13 pukey green gumballs roll out.

Leif skips up to the fantastic gumball machine and receive 169 mustard yellow gumballs.

"AYE BANANAS!" says Zolt. Why do get so many?

"HOTDOGS" says MPLZ3.

“WHAT?” asks Zolt

“never mind”

Anywho how many gumballs should Zolt get if were to go next, and how many would MPLZ3 get after Zolt?

People	#of Gumballs	Exponents	Repeated Multiplication
Xavier	13	13	13
Lief	169	13^2	13x13
Zolt	2197	13^3	13x13x13
MPLZ3	28561	13 to the 4	13x13x13x13

In the end Xavier was very jealous of how much candy his friends.

Too bad Xavier ☹

From these last examples, we see that students’ thinking became more flexible and creative. Their familiarity with a base of 2 made it easy for them to vary the base to other bases—a significant initial step in the generalization process (Dialogue, 2005). Interestingly, not one student tried to use fractional bases or negative exponents, and only a handful of students included a zero exponent in their explanations. This suggests fractional bases, and zero and negative exponents are concepts students are reluctant to investigate without some external prompting.

7.2.3 Realistic and Fantasy Contexts

Another interesting result from the study was the subject of the stories students wrote. Of the 55 stories analyzed, 25 were set within a realistic context. Realistic contexts were defined as those contexts in which the settings, characters, and scenarios could exist within the students' day-to-day lives, regardless of how unlikely they might be. This definition ignores the extremely large numbers generated through exponential growth. Supporting the findings of Baranes and colleagues (as cited in Koedinger & Nathan, 2004), many of these stories tended to focus on topics related to money—something students are already familiar with. For example, Brian writes about mowing lawns to raise money:

> There was a kid named Sid who decided to mow people's lawns every month. So Sid asked a guy named Fly if he could mow his lawn every month. Fly said sure and that he would pay Sid double the salary of the previous month. Fly gave him 3 dollars the first month. Sid wondered how much his salary would be on the 12th month.

Month	Salary
1	$3
2	$9
3	$27
4	$81
5	$243
6	$729
…	…
10	$59049
11	$117147
12	$531441

He would get $531, 441.

3^{12} or $531441. Fly is crazy!

Cassidy writes a story about raising money to purchase cheerleading outfits:

> Our school together a cheerleading group. We needed to raise money for the uniforms. To raise money, we decided to do bottle drives every second

> weekend. The first time we did the bottle drive 3 people showed up. In all, that day we made $10.00. Two weeks later we got together again. This time 12 people showed up. The total for that was 10 times the amount from the first time. The third time we did a bottle drive the whole team cam. That day we made the most money yet. We had made 10 times the amount we had made at the second bottle drive. If we needed $1500 for the uniform did we make enough to pay for them? If not, what is the difference?

Many of these realistic stories however, involved subjects students were not necessarily familiar with, but that they were interested in, such as selling cars, or saving trees. Gemma writes about a secret dare club in her high school:

> The students of Birchwood High gathered in the lunchroom. As always, competition lingered in the air. It could nearly suffocate you and being with this 'group' –though only seven students—kept you on your toes all the time. Maria glanced around nervously, she had been drawn into this so-called group through her own will of being noticed, feeling the temptation of winning. Yes, this was the Birchwood High dare council. Only those driven by competition could complete one of these dares. For a while no one spoke, they only looked at one another, through threatening eyes. Every one looked now at the leader. He would name the dare, eventually. It could be anything, last week it got three people suspended.
>
> At last the leader spoke. Maria cast her head down in embarrassment. What had she gotten herself into?
>
> "A fool would name one single dare, therefore mine progresses throughout the five days of the week. Think carefully before quickly accepting the challenge. Consequences could be 'bigger' than suspension. A generous prize will be rewarded, of course" he added with a smirk. This leader wasn't the nicest student, but he was bright and knew how pride could draw people into the most foolish situations.
>
> Maria let her mind wander. "I wonder what sort of prize this might be, it could be anything. This leader is rich and if I did this dare I could be recognized, forever!"
>
> "Let me name the dare" the leader said. All eyes shot up once again to him.

"As you know, the Birchwood High cafeteria has a large quantity of delicious hamburgers. Mouth watering ones. Big juicy ones."

"Hamburgers?" thought Maria. "What sort of dare was that? I thought that I might have to do something at lest remotely daring. How easy!" but her thoughts were interrupted with the leader again.

"Starting on the first day of the week the victim will eat two hamburgers." he said. The others in the group started to laugh. "What kind of dare is this?" someone shouted out. "All we need to do is eat hamburgers?!"

The leader smiled, "There's a bit of a catch" he said, "Every day of the week the hamburgers double in numbers and you must eat every one, to the last crumb. Any takers? Or will I pick someone myself?"

Maria, before calculating the total consumption number, shot up her hand before anyone else could. Her mouth was watering and to complete the dare would bring her popularity in the school, or so he hoped.

The leader smiled. "Thank you Maria, if you complete the dare the prize is yours."

The next day Maria ate two huge juicy hamburgers. To her she tasted only her own greed for the prize and the recognition that came with it. Over the course of the next several days she found her diet hugely increasing. She was feeling heavier and more grotesque by the day. And when on the third day and eating eight burgers she questioned her decision.

By the end of the dare session, she had gained exactly what she had wanted. To be noticed, and now weighing the mass of many burgers she was indeed noticed. Her dream was achieved. She would be forever noticed and the dare had turned out to be very big indeed. After doubling the burgers every day she finished the week having eaten 62 hamburgers finishing the last day with 32.

The remaining 30 of the 55 stories analyzed were set within a mostly fantasy context. Fantasy contexts were defined as those contexts in which the settings, characters, and/or scenarios could not exist within students' day-to-day lives. This was again ignoring the extremely large numbers generated through exponential growth. Some of the storylines included: alien samurais defending Earth; a mad scientist adding height to a midget with the help of a stretching device; a knight fighting off a dragon whose heads keep multiplying; a boy who grows so big his head breaks through the roof. 15 of these stories incorporated only a minimal amount of fantasy elements. For instance, Andrew describes a trip he takes with his parents to Paris that includes a magical transporter device:

> One morning I woke up. Except this morning I woke up differently. I thought I don't want to go to school today. So I called a cab and left my house to go to the airport (don't you wish you could do that). When I was at the airport I though 'cool I'm at the airport and I'm not at school. Hahahahaah' So I got out of the cab but I didn't have any money to pay the driver. "Excuse me sir, you have to pay me" said the dim- witter driver. I said "I don't have any money I guess you'll have to drive me back." The guy said Okay. I said "but don't worry, I'll let you go without having to drive me." The driver said "oh thanks buddy I owe you one." Idiot. Then he drove away. I entered the place and went to get my ticket. In a matter of minutes, I was on my one-way flight to Paris, France. It took a while so I napped and snoozed and dozed and slumbered and slept the whole way there. When I got there, I went to a bread-making place and got a scrumptious French Baguette. My bread and I decided to go to the Eiffel Tower. So we did. It costs money to get up there (can you believe it) *so I decided to use my super teleporter thing-ga-ma-jig to get up there*. It was exactly .072 seconds. Trust me, I counted. I read a plaque up there it was in Japanese and it said that the builders used the math to build the tower. As I was reading, by mistake I dropped my bread and it hit some hobo. Stupid butter. Anyways, exactly 33,554,432 metal triangles were used to build it. For each triangle they used they had to double that the next layer on the way down (they actually built the tower from top to bottom). If I wasn't in Mr. B's class, I would not find any math in this method. But I am so I understand that the architects used the exponents instead of repeated multiplication. The builders ended up with a massive, giant, big, large, huge, enormously gigantic square based pyramid. Then those Frenchie's used their baguette knives to cut out the middle of the tower and shaped it so that it was aerodynamically designed. Anyways back to math. There were 26 layers of triangles in the whole tower. If the layers were started at 1 and went down, and the triangles doubled for each layer, it would equal to 33, 554, 432 triangles. If in the second layer there are 2 triangles, then in the third layer there would be 4. if you repeat this process until the twenty sixth layer, you should have 33,554,432 triangles. The equation would be 2 to the power of 25.

Brent describes what happens when you eat too many chips:

> This is a story about what happens when you eat chips when you're on a diet. First you'll say that your only going to have just one. Then you have one and its so good that you think, well maybe 2 more wont hurt. Then they're so good that you decide they're not that fattening and have 4 more. And they're soo good that you look at the package that says 100g sunflower oil and you think that that means they're good for you, so you have 8 more. And they're soooo good that you figure you've been working sooo hard that you deserve this and eat 16 more chips. She keeps making excuses *until she explodes* [italics mine] and her family sues the chip company for making their chips so good.

While the number of realistic contexts and fantasy contexts chosen by students were comparable, the results of this study mostly support the claims of Schiro (2005) that suggest students are more readily drawn to settings that are not necessarily within the realm of their day-to-day lives. Moreover, in accordance with Egan's work (1997, 2005), students at this stage in their cognitive development enjoy the fantastical, but require some concessions to reality in the stories they read or write.

The results of this first study suggest stories can be used in the high school setting to explain mathematical concepts. Based on the interactions that occurred during the storytelling event, it appears that delivering a story orally to high school students is still a valid means of communicating ideas. There were moments however, where the telling of the story seemed to drag on and by the time the mathematics emerged, some students appeared to have lost interest. In this regard, a story delivered orally for instructional purposes must be relatively brief and the mathematics must weigh heavily on the plot, otherwise it may be reduced to nothing more than sheer ornamentation. The degree to which students engaged with the mathematics in *The Legend of Chikara Ni Bai* might be inferred from the way students constructed their own stories focusing on the patterning nature of exponential growth. Although they did not do the actual mathematics found in the story—which according to Schiro (2004) is the highest level of engagement with the content—they did however, create their own stories that incorporated the same type of mathematical content.

The data from this study also suggests that stories need shaping if they are to be effective at engaging the imaginations of students with the mathematical content. In terms of cognitive tools, rhyme, rhythm, and pattern and jokes and humour played important roles in keeping the story moving along as well as accentuating the nature of exponential growth. A fascination with patterns can be seen in the way most students chose to write stories about "exponential growth," as opposed to simply writing "exponent" stories. Considering that many of the stories students wrote were mere variants of the one I told, it is evident that *The Legend of Chikara Ni Bai* had a profound effect on the type of story students chose to write. Whether describing the antics of curious aliens or reflecting upon a prehistoric party at my house, students took every opportunity to incorporate humorous elements into their stories. This was no doubt influenced by the comedic nature of *The Legend of Chikara Ni Bai*. The inclusion of characters like Chacho the Village Idiot combined with the amusing dialogue between Hanzo and Ni Bai provided students with the courage to explore exponential growth from an uncharacteristically humorous disposition. As we will see in chapter 13, this type of approach can be quite effective when comical elements in a story are directly tied to particular elements in a mathematical pattern.

Considering that most of the stories students wrote were situated within a realistic or fantasy context, we might guess that students are particularly drawn to stories of this nature. It may be pertinent then to develop stories along these lines. I might concede however, that many students may have simply reproduced the type of story I told at the outset of the study. In any case, introducing the concept of exponential growth in terms of a fantasy context may have encouraged students to write stories that take a similar approach—one that seems necessary when dealing with such incredibly large numbers as those found in exponential growth. In one sense, fantasy or semi-realistic type contexts may give students the room to explore mathematical ideas without having to worry about the constraints of the natural world.

8 STUDY II: THE PYTHAGOREAN PROBLEM

8.1 Treatment

8.1.1 Target and Problematic Issue

The Pythagorean Problem is a story to introduce the concept of irrational numbers and an activity to reinforce the ideas presented. Again, according to the planning framework developed by Zazkis and Liljedahl (2005) outlined in section 6.3.2.1, the concept of irrational numbers is the target. The problematic issue is the notion that irrational numbers cannot be written as numbers with terminating or repeating decimal representations. In high school, making the distinction between rational and irrational numbers is closely tied to the procedure of converting numbers from their terminating or repeating decimal representations to fractions. This is usually taught in grade ten when students first learn about operations with radicals. But, like many topics in mathematics education, the concept of irrational numbers is presented as disjoint from other mathematical ideas—independent of any meaningful notion in the student's real or imagined world. We want students to ask: Why is the concept of irrationality significant? Why does it matter if the decimal representation of a number repeats or terminates? The problematic issue is then: How can we bring meaning to the concept of irrational numbers in the minds of students? Can it be elevated above the obscurity of a simple rote mechanistic procedure?

8.1.2 Cognitive Tools

In order to bring meaning to the concept of irrational numbers, I again used the cognitive tool of story as our main vehicle for engaging the imaginations of students. The supporting cognitive tools of binary opposites, knowledge and human meaning, and jokes and humour (Zazkis & Liljedahl, 2005; Egan, 2005) were also used to draw students into an exploration of the idea of irrational numbers, where the archaic, ultra-fanatical, and predominantly theoretical Pythagoreans are pitted against the innocent, young, and truth-seeking Hippasos (Pickover, 1998). When situated within the familiar context of the Pythagorean theorem, the idea of irrational numbers becomes quite approachable. In fact, many topics at the secondary level are better understood against the backdrop of this famous theorem. A consistent theme that permeates almost every unit of study at the secondary level, this age-old relationship is at the foundation of such concepts as the distance of a line segment on a Cartesian graph, the cosine law employed in trigonometry, and numerous others. The goal then, is to contextualize the less familiar concept of irrational numbers within the more familiar Pythagorean Theorem, through the use of an imaginative story. Inspiration for this story assignment came primarily from Clifford Pickover (1998), who in his book, *A Passion for Mathematics*, spends a significant amount of time describing the strange behaviours and beliefs of the Pythagoreans and their contributions to the social and mathematical world. Connectedly, Pickover provides an excellent model for how one might combine mathematical ideas with anecdotal facts and personalities and present it in the form of an amusing mini-narrative—an invaluable skill in the mathematics classroom.

8.1.3 Presentation of Problem, Extending the Problem Situation, and Conclusion

Again, I had to consider how to organize the presentation of the problem using the cognitive tools already identified in order to maximize student engagement (Zazkis & Liljedahl, 2005). To introduce the problem, I led students in a discussion of the Pythagoreans and their strange beliefs and behaviours (see section 4.1.3 of this thesis). After a spirited discussion, I introduced students to Hippasos and attempted to dramatize the intellectual process he may have gone through in his discovery of irrational numbers. Using monads (or dots) to construct squares along the sides of a 3-4-5 right triangle, I showed students the Pythagorean relationship. I then modified the problem so that the sides of the right triangle each consisted of 4 dots. I asked students to determine how many dots should be placed along the hypotenuse and to illustrate their solution. I gave them a reasonable amount of time to calculate this familiar application of the Pythagorean Theorem. After they stated that they did not know what to do about the hypotenuse, I drew a diagram of a square with side length 1 unit and then asked students to calculate the length of the diagonal. In light of the Pythagorean's penchant for symmetry and whole numbers, this posed a serious problem that had to be reconciled. I informed students that this was the discovery Hippasos made that led to his murder. Again, in order to further engage students' imaginations with the mathematical content, I chose to introduce a writing and drama assignment that takes into account the cognitive tools of story and binary opposites. I placed students in groups and instructed them to write a script for a play about this pivotal event and act it out for the class the following week.

In regards to the mathematical content, the play had to explicitly demonstrate an *understanding* of the mathematical concepts involved—and not just recite the discovery. I encouraged students to research more about the discovery of irrational numbers and try to incorporate their findings into their productions. They had a total of two full class periods spread out over one week to prepare their scripts and practice their plays.

Two weeks after the performances, students were asked the following questions in a test:

> 1. We know that $\sqrt{2}$ is irrational. That is, it cannot be written as a whole number, or a ratio of two whole numbers. A) Describe how this relates to the Pythagorean Theorem. B) Why would this result bother the Pythagoreans?
>
> 2. π is also irrational. The circumference of a circle is given by the formula: $C = \pi \cdot d$, where d is the diameter of the circle. We can then rearrange this formula to get: $\pi = \frac{C}{d}$. Therefore, π is the ratio of the circumference of a circle to its diameter. What does this tell you about the nature of C and d in any circle?

Question 1 was designed to see if students understand the nature of irrational numbers and how they are connected to the Pythagorean theorem. From section 3.4.1, we know that the more one knows about the history of an idea, the more connections they can make to other concepts and therefore develop a richer, more meaningful understanding of that idea. The question also examines the degree to which the story became a part of students' understanding of the concept of irrational numbers.

Question 2 was designed to extend or vary the initial problem situation according to Zazkis and Liljedahl's framework (2005). Could students extend their understanding of irrational numbers into other domains, thereby enriching their understanding of the original problem? Could they take what they learned and apply it to something else?

8.1.4 Participants

30 students in MATH9H participated in this assignment, but only 23 students chose to allow their work to serve as data. These students were already familiar with the Pythagorean theorem and its basic applications.

8.1.5 The Story

See section 4.1.3 for an outline of the story.

8.2 Analysis

8.2.1 Binary Opposites, Knowledge and Human Meaning, and Jokes and Humour

From sections 4.1 and 4.3 of this thesis, I know that situating the concept of irrational numbers within a binary structure and in a historical context, can foster imaginative engagement with the material. And certainly the strange practices and beliefs of the Pythagoreans added a humorous slant to this story. Students' fascination with the extreme nature of the Pythagoreans and the heroic qualities of Hippasus was evident in the way the classroom became a circus of sword-fights, impromptu songs and chants, and a display of elaborate costumes and props, in the week leading up to the performances. More importantly, each group eventually engaged in an intensive investigation into the nature of $\sqrt{2}$. Undoubtedly, this was strongly motivated by the upcoming performances. The plays students wrote contained many interesting characters and some involved plotlines that further developed the inherent binary opposites. For example, in one story, Hippasus has to convince the "Number God" about his discovery of irrationals before confronting his Pythagorean brothers:

HIPPASUS: My Lord, I have found a flaw in the Pythagorean theory!

NUMBER GOD: Show me my son.

(Hippasus scribbles on a piece of paper)

But then if this is not true my son, then I do not exist.

(Hippasus passes out)

Interestingly, their opening scene has some striking similarities to *Partition*, mentioned in section 4.3.1.3 where Hippasus is likened to Ramanujan, the young Indian mathematician, who routinely carouses with the goddess Namigiri. As with most of the plays, Hippasus is portrayed as a hero fighting against the corruption of the Pythagoreans.

Another group produced a musical to accentuate the strange behaviour of the Pythagoreans. Here is the chorus from one of the many humorous songs they wrote and performed:

We will worship numbers and believe they are Gods,

Meditate and talk to their brains.

7, 8, and 9, and 10, and evens and odds,

We'll love them 'till none others remain.

Aliens intelligent, and can't be outdone,

I'm an alien 'cause I'm the smartest one.

Lastly we'll be vegetarians, and no beans,

No meat for us, just give us our greens.

Another group decided to set their play within the context of a news team reporting on the recovered body of Hippasos. Here is an excerpt from their script:

> ANCHOR: The gruesome discovery of a drowned body was reported today in the island of Samos. Our own Eugene L. Peterson Waddlesworth the Third has gone to find out more. Eugene?
>
> REPORTER: Yes, Kent?
>
> ANCHOR: Can you tell us anything of the police's discoveries so far?
>
> REPORTER: Well, Kent, several chains with metal balls were found wrapped around the body, as well as heavy rocks in the pockets and cement blocks on its feet, as if to make sure it would sink. The police, for some reason, are suspecting foul play.
>
> ANCHOR: (Dripping with sarcasm) How did they figure that?
>
> REPORTER: (absolutely serious) No idea, Kent.
>
> ANCHOR: O...kay...(startled by the ignorance) So...what else is going on down there?
>
> REPORTER: All the police members have declined requests for interviews...however, there are some locals near the crime site. Let me talk to one of them.
>
> (Goes over to a suspicious looking local, wearing shades and looking around)
>
> REPORTER: Excuse me, sir, but would you mind if I asked you some questions?
>
> LOCAL: Questions? What kind of questions?
>
> REPORTER: I see you're wearing a unique sweater! (reads off it)
>
> The - secret - society - of - Pythagoras. Does this mean, and I don't mean to jump to conclusions in any way, that you indeed belong to a secret society?

LOCAL: What? Who told you that?

REPORTER: Well, sir, it's on your sweater.

LOCAL: You've got nothing on me, you hear? NOTHING!

(Runs away twitching)

8.2.2 Level of Engagement

The plays students wrote displayed various degrees of engagement with the mathematical content. Where some students simply regurgitated the examples presented at the introduction to the assignment, others successfully intertwined the mathematics with the events of the story in their own unique way. Consider another excerpt from the above news report:

REPORTER: What's that, constable?

CONSTABLE: This? I think it's a litter box. (gives it to reporter)

REPORTER: There's some stuff on this.

CONSTABLE: Well, obviously.

REPORTER: No, no, there are little sticks! It looks like they are arranged into some shape.

CONSTABLE: Looks like half a square.

ANCHOR: You mean like that? (holds up a picture of half a square)

REPORTER: Yeah. The base has three sticks...

CONSTABLE: The height is four sticks and five sticks connecting them.

ANCHOR: So it's a right triangle made of sticks?

REPORTER: Yeah.

(Anchor arranges sticks into a right triangle)

REPORTER: And there are squares here, too.

ANCHOR: What do you mean?

CONSTABLE: There are three squares.

REPORTER: Each of their sides are one of the sides of the right triangle.

ANCHOR: (puts on more sticks) Like this?

CONSTABLE: Yeah.

ANCHOR: Hey, I think I've seen this before. This is actually something Pythagoras found out. This square (the one with side 4) has the area of 16. And this square (the one with side 3) has the area of nine, while this side (5) has the area of 25.

CONSTABLE: What's that got to do with anything?

ANCHOR: It's quite interesting. If you add the area of this square (16) and this square (9)...

REPORTER: It's the area of the other square!

CONSTABLE: Here's another one of those litter boxes.

REPORTER: The shape on this is a lot smaller than the other one.

ANCHOR: What does it look like?

REPORTER: Well, the shape is exactly the same as the other one, except its height and base is only one stick.

ANCHOR: (after arranging it) Wait, but how many sticks is the other side?

REPORTER: There are two sticks, but they don't fit in perfectly into that side.

(silence)

CONSTABLE: I don't think a whole number of sticks can fit in there.

REPORTER: Yeah.

ANCHOR: What? But what about the squares?

REPORTER: What about them?

ANCHOR: Well, if we can't get the exact value of this side, how can we have a square?

CONSTABLE: Erm.

Again, students were operating at Schiro's highest level of engagement (2004) as they described and worked through mathematical problems similar to those found in *The Pythagorean Problem.* Another group removed the problem from its familiar classroom context and placed it in a believable, yet highly imaginative situation:

PYTHAGORAS: Do you want to join our cult, and worship Numbers?

HIPPASUS: Yes, I'd be honoured!

PYTHAGORAS**:** Well you're in Hippopotamus.

P, F, & H: Numbers numbers numbers numbers numbers numbers numbers numbers**…**

NARRATOR: "So the group prays and studies for years until one day, while they are on a Greek warship, sailing in the formation of a square, one of the warships sinks!

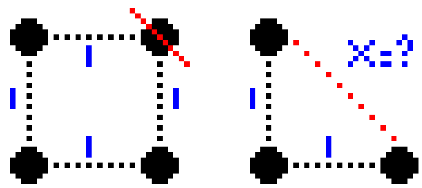
PYTHAGORAS: Hippopotamus, we need you to do a calculation for us.

FOLLOWER: We hang ropes of equal length in between each ship so that they stay in perfect formation. Since one of the ships has sunk; we need you to find out the length of rope required to connect the two now unattached ships.

Each rope is one unit, so we'll leave you to figure out how much rope we'll need.

HIPPASUS: One thing before you go, my name is Hippasus...

P & F: Right…



NARRATOR: So Hippasus meditates on the problem, but figures out that he can't get a *perfect* numeral, and he then figures out that numbers are not *perfect*... [italics mine]

HIPPASUS: Hey, guys, guess what, I found out that numbers are not *perfect*! The number that I got for the amount of rope that we need is not a *whole* number, but a decimal that goes on and on! [italics mine]

H&F: Oh no, what, no way, blasphemy, lies...

PYTHAGORAS: We can't tolerate this, he is speaking blasphemy against us and Apollo!

FOLLOWER: Let's throw him overboard!

In this case, simple isomorphic replacement brings the notion of irrationality to the forefront. Dots have been replaced with ships, and the lines between them with ropes. The problem of calculating the hypotenuse has now become a problem to find out how much rope is necessary to connect two of the remaining ships. Because the mathematics is so intertwined with the events of the story, an understanding of the mathematical content presupposes any grasp of the emerging plotline. Moreover, the stark binary opposites accentuate the mathematical content and compel the audience to ask, "What happens next?"

8.2.3 The Big Ideas

Another important point to consider is the way many of the stories students wrote revealed they still had an incomplete view of irational numbers. Notice the italics in the last five lines of the above story. We see that this group equates rational numbers solely to whole numbers and does not consider that a number whose decimal representation is both repeating and non-terminating might also be a rational number. However, even though these students could not easily identify or accurately describe irrational numbers, their use of words like “fit,” “perfect,” and “whole” suggest they are beginning to develop a larger more overriding view of the concept. In other words, they are starting to see some of the “big ideas” behind irrational numbers. Big ideas used in this sense are personal and individual to students and occur when they become aware of a concept that goes beyond their immediate familiarity. Often these big ideas occur prior to exposure to the details of the new concept and thus are often articulated without details and precise language. According to Cech (2007, p.1), “being able to look at the big picture is a source of power and knowledge,” and when unifying principles are presented first before instances of that principle, students can more easily integrate new information into their conceptual frameworks. We see this happening in social studies classes where the teacher tells students they are going to be studying a particular topic such as “Confederation.” “You give them an outline, and everything goes together. In math, we tend to treat each lesson as discrete” (Cech, 2007, p.1). The fact that not every real number can be represented as a simple fraction suggests to students that there exist sets and subsets of numbers or classes of objects with unique properties and that there may even be number systems outside of what they already know. With this big idea in mind, the introduction of such concepts as complex numbers or continued fractions in later grades will seem to make perfect sense. This notion of a 'big idea' is a theme that emerged out of much of my data, and thus will occur in several places in this thesis including the conclusion.

8.2.4 The Meaning of Irrational Numbers

From the previous excerpts of student work, we can deduce that *The Pythagorean Problem* played a significant role in establishing meaning to the idea of irrational numbers. However, the meanings students established were quite limited revolving around a single irrational number—$\sqrt{2}$. This is largely a result of the nature of the story I told as well as the focus of the first test question. In regards to the latter, most students attempted to describe the relationship between the Pythagorean Theorem and the irrationality of $\sqrt{2}$ in terms of a right triangle or in terms of the sides and diagonals of a square—much like the account I gave. For instance, Joanna describes it this way:

> The P. Theorem states that $a^2 + b^2 = c^2$. However, if *a* and *b* were equal to 1, their square would also be 1. So $a^2(1) + b^2(1) = 2$ [*sic*]. However, you cannot get *c* because $\sqrt{2}$ is irrational. The Pythagoreans would be disturbed as this meant that numbers were not perfect, nor did the theorem work every time.

Similar reasoning can be seen in Stacy's answer:

> The fact that $\sqrt{2}$ is irrational relates to the Pythagorean Theorem because when the base and altitude of a right triangle are 1, the hypotenuse is $\sqrt{2}$. This would bother the Pythagoreans because they believed everything could be expressed as a whole number or a ratio of whole numbers, but $\sqrt{2}$ cannot.

Alec gives this response:

> [Diagram of a right triangle with sides labeled *a* and *b* and hypotenuse labeled *c*]. $a^2 + b^2 = c^2$ if *a*=1, *b*=1, $1^2 + 1^2 = c^2$, $2 = c^2$. The Pythagoreans

believed that numbers were divine, perfect. When Hipposas discovered this he found a number that went on and on, and didn't make much sense in general. This made the Pythagoreans angry, because it went against their beliefs.

Jason offers a slightly different view:

[Diagram shows a square with side length of 1]. It does because of the square is two triangles put together if you take one away, it's long side is $\sqrt{2}$. They thought numbers were perfect so when this popped up and it squished their idea.

It would appear that these students are quite good at recalling events from the story. What is important, however, is that they perceive the concept of irrational numbers almost exclusively in terms of the story. This can be problematic however as we might also assume the story inadvertently hindered students from seeing irrational numbers in any other meaningful way. This may be why only a handful of students could reasonably answer the second test question which asks them to extend their notion of irrationality to one involving π —something not mentioned in the story. Their answers follow:

"*C* or *d* are irrational in any circle."

"They [*C* and *d*] are never rational at the same time."

"If $\frac{C}{d}$ equals an irrational number, *C* and *d* has no LCM or other common factors. And could this perhaps mean that *C* (and *d*) is also irrational?

"This tells us that *C* and *d* are not whole numbers (i.e. they are decimals)."

"$C/d=\pi$, which is the ratio of a circle's circumference to it's diameter rather than a solid measurement."

However, as with the scripts, the answers students gave to the first test question reveal once again they were entertaining one of the big ideas behind irrational numbers—that is, the notion of sets and subsets of numbers or classes of objects—even though they continued to have difficulty understanding the particulars. For example, Trevor writes:

> The Pythagoreans found that the square drawn around the legs of a right triangle equalled the square drawn around the hypotenuse. They found that if the legs were equal, then they couldn't get a *whole number*. They thought that numbers were *perfect* and this upset their religion [italics mine].

Although Noreen's algebra is flawed in the following example, we see that by her use of the term "fit," she is also beginning to grasp a larger, more overriding concept involving irrational numbers—and again, this is directly linked to the story:

> $a^2 + b^2 = c^2$, if $a = 1$ and $b = 1$, then $c = 2$.
> So if $c = 2$ and we're trying to figure out side a and b it will be $-\sqrt{2} =$ a +b, but $\sqrt{2}$ is irrational therefore it doesn't fit, throwing the equation off.

Clearly the story of *The Pythagorean Problem* impacted both Trevor's and Noreen's conceptualizions of irrational numbers, albeit in a very limited way. Like most of the students in this study, their descriptions suggest they still have an incomplete or inaccurate view of the concept.

Like Noreen, many students made simple procedural or algebraic errors in their explanations. For example, some students substituted in the wrong values for the variables in the Pythagorean Theorem, as is the case with Corey:

> [Diagram of a right triangle with sides labeled *a* and *b* and hypotenuse labeled *c*]. $a^2 + b^2 = c^2$. However, when $a = 1$ and $b = 1$, $1^2 + 1^2 = 2^2$ [italics mine]. Then $c = \sqrt{2}$.

Frank gives this answer:

> The Pytag theory is $a^2 + b^2 = c^2$ for a right triangle. And if the base and altitude were 1, *there hypotenus would be 2*, if so $\sqrt{2}$ is the length of the hypotenus [italics mine]. But the answer to $\sqrt{2}$ is IRRATIONAL! This bothered them because the triangle was a perfect shape and their god were the numbers but if they are irrational… DUN DUN DUN DUN.

> Here, Lauren is unsure about how to use the square root function:

> Because if $a^2 + b^2 = 2$, you wouldn't be able to square root it.

> The right triangle seems to have limited Ryan's view of the irrationality of $\sqrt{2}$:

> The longest side of the Pythagorean triangle is irrational. This result won't let the longest side be rational. The Pythagoreans were wrong.

We can see from these last few examples that the story is closely linked to the students' conceptualizations of irrational numbers. Even in cases where students seemed unsure about the mathematics, they often resorted to descriptive answers citing elements from the story. Here are several examples:

> "The Pythagoreans believed that the square was perfect, but $\sqrt{2}$ proved that it was not. Their perfect shape was irrational."

> "It relates to the Pythagorean Theorem because they loved numbers and there was a flaw in their theory and changed everything. It bothered them because what they believed in was fake and was incorrect."

> "It relates to the Pythagorean Theorem because this is what they discovered while drawing a triangle using dots. This result bothered the Pythagoreans because they believed numbers were god-like, this shook their perspectives and beliefs."

> "This relates to the Theorem because the Pythagoreans were surprised by this problem because it showed their perfect square wasn't perfect. The result bothered them because they believed the square was perfect."

> "Because in their geometrical chart thing it showed $\sqrt{2}$ with an exact answer. The Pythagoreans believed numbers to be perfect but when they realized a flaw it drove them mad."

The findings of this study, both from the analysis of the scripts students wrote and from the answers they gave to the two test questions, suggest the story of *The Pythagorean Problem* was closely linked to the meaning students assigned to the concept of irrational numbers. In most cases however, the meaning they assigned hardly extends beyond the bounds of the story, limited to the only irrational number they were exposed to. This brings up an important consideration when designing stories to use in the classroom: Whatever elements we choose to place in a story will be tied to the meanings students associate with the mathematical content. With this in mind, we might consider alluding to—within the events of the story—further applications of the mathematical content. In this way, the mathematics transcends the story. For instance, in *The Pythagorean Problem*, the story does not have to end with the death of Hippasus, but instead, it could end with the introduction of another young thinker who is pondering the nature of π. Leaving the story "hanging" in this way might imply there are other instances of where irrational numbers appear.

Another relevant finding from this study is the particular role the story played in developing certain big ideas around the concept of irrational numbers. This was apparent when analysing the language students used to describe the concept. The use of terms like "fit," "perfect," and "whole" suggests students were beginning to see the concept of irrational numbers in terms of sets and subsets or classes of objects—a larger theme they are not too familiar with and that has the potential to tie together many of the distinct topics covered in high school mathematics. Using stories to first develop the big ideas before engaging with the particulars may be an effective way to introduce new concepts.

9 STUDY III: THE FOUR 4S

9.1 Treatment

9.1.1 Target and Problematic Issue

The Four 4s is a story to introduce the concept of order of operations with rational numbers and to introduce an activity. According to the framework outlined in section 6.3.2.1, the target, or the specific part of the curriculum I wish to address, is order of operations. In grade eight, the concept of order of operations is extended to rational numbers, wherein the problematic issue arises. Students tend to have greater difficulty performing operations on fractions and negative numbers than on whole numbers. Combining this with the introduction of some less familiar operations (i.e. square roots and powers), the task of ordering operations can be a formidable one.

9.1.2 Cognitive Tools

In order to present the problem in a way that engages students' imaginations, once again, the cognitive tool of story was utilized. Furthermore, the supporting cognitive tool of binary opposites was also used to make the story come alive in the imaginations of students. Here, the mathematics student becomes the main character in the story, contending against the devil in a battle of mathematical wit. To escape from the pits of hell, the student is challenged to perform several calculations acknowledging order of operations. Inspiration for this approach came from two mathematical problems I encountered that are also situated within the context of a story. The first is from an article by Raymund Smullyan (1985) called, *Satan, Cantor, and Infinity*, in which Smullyan presents, through narrative, a problem in advanced set theory and mathematical logic. Satan tells Beelzebub about his ingenious plan to torment one of hell's inhabitants:

> "Well, tomorrow I am expecting a new victim and I will arrange matters so he can never go free!" "What will you do?" asked Beelzubub. "I have written down the name of a set of positive whole numbers," said Satan. "Each day he will be allowed to name one and only one set and if he ever names my set, he can go free. But he will never go free!" said Satan shrieking with delight. "Why?" asked Beelzubub. "Well, just look at what I have written!" said Satan. 'The set of all numbers *n*, such that *n* does not belong to the set named on the *n*th day' (p.118).

Fortunately, the victim was a prize student of George Cantor and was quite knowledgeable in set theory and mathematical logic. The second source of inspiration is from Enzensberger's, *The Number Devil* (1997). In this story, the main character Robert, a middle-school age boy, meets the "Number Devil" who presents him with various mathematical problems. While the mathematics in the story is quite simple, probably geared to preteen age students, it does draw attention to the patterned nature of mathematics. Extending these patterns and looking for general principles would be an excellent exercise in any course at the secondary level.

9.1.3 Presentation of Problem and Conclusion

Once again, the presentation of the problem was organized according to the cognitive tools already identified. However, the approach for this story and activity was quite different from previous ones in that the story itself outlined the writing activity students were to participate in. In fact, the assignment and criteria were delivered as the instructions given by the devil to the main character in the story. This is very similar to the approach Doris Lawson takes in *The Wizard's Tale*, where characters in the story require the help of students to solve mathematical problems they encounter (Schiro, 2005). As the assignment and criteria unfolded in the story, I wrote them down on the whiteboard and clarified any aspects of the assignment students addressed—ever remaining in the character of the devil when answering their questions. I further instructed students to find all the solutions—keeping all the correct mathematical rules for order of operations—which would be due the following class. This assignment was given as a way to reinforce the concept of order of operations students encountered at the beginning of the year.

In order to see the extent to which students associated the story with the mathematical problem, two weeks later, I asked students the following question that was placed the end of an integer review test:

Describe the "Four 4s" problem.

I was curious to see if students used terms or phrases from the story to describe the mathematical problem I gave them. Did they simply state the mathematical problem, or did they associate elements of the story with the mathematical problem? Were the two domains inseparable or fundamentally distinct?

9.1.4 Participants

28 students in MATH8H participated in this assignment, but only 20 students chose to allow their work to serve as data.

9.1.5 The Story

> It was quite unfortunate that I went to sleep last night. Perhaps it was that oversized bowl of chilli I devoured at dinner, or maybe even the burrito I snuck down to have as a late night snack? Whatever it was, within a few

nods I would soon leave this world of fine Mexican dining and find myself wandering within the caverns of the Stygian Deep. Of course this was only a nightmare and I am not really in Hell… I mean…I couldn't be… I have always followed the laws… I have devoted myself to live by the theorems… brackets before exponents, invert and multiply, refusing to divide by zero, and so forth. Did I really know what I was doing? Was my faith so blind that I only knew how to perform certain mathematical operations without really knowing why? I would soon find out the answers to these questions, for there in front of me stood the devil himself. Although he was somewhat like I always pictured him to be, he surprisingly looked a lot more like my grade eight math teacher, Mr. Schnickeldwart… cardigan sweater bursting at the seams, tie with a wrinkly white shirt poking out underneath, and pleated black pants with its fly obviously malfunctioning. He at once handed me an envelope that had written on it the most evil four-digit number you could imagine. No, not 666! This has only three digits—although this number is also quite evil. No, this envelope had the number 4444. It might not seem evil to you, but to me this number would later come to mean the difference between an eternity of monotony and boredom or exponential bliss. I opened the envelope and saw at the top of the page the title "The Seven Deadly Functions". They were listed as follows: "$+, -, \times, \div, \sqrt{\ }, !, \text{exponent}$." Okay, now I was scared, considering I had not paid attention to Mr. Schnickeldwart when he taught us about factorials!!!!!!! I suppose I remember?!!! And what is a function?? Under that list was another list taking up most of the page. It was the numbers 1 to 20 and they were spread out so as to leave room to write something next to each one. Even before I could try to make sense of this list, the evil one revealed to me his evil plan. "I will give you exactly four minutes to forego your fearful fate in this fiery inferno forever!" He continued, "You must take the four numbers 4,4,4, and 4 (I thought this was one number) and combine them using any of the seven deadly functions." "Oh, that's easy," I said in a smug tone. "4+4+4=12!" "You must use exactly all four 4s… no more, no less, he countered in an unexpectedly humble manner. I knew his demeanor was not forthright as everyone knows there is no good in that old deceiver. "Fine." I said. "4+4+4+4=16". "Wonderful, wonderful, you have generated one number. Now get the others… the numbers 1 to 20." "Mmmm, can I use fractions or brackets?" I asked. "Alright, I suppose I will allow that. It will make no difference as you will not be able to get all 20 numbers," he said with an evil laugh. "Do you accept my challenge? Or should I find some other victim to torment?" "No, no, I accept…provided you keep your word and let me go if I complete this challenge," I said. "Yes, I give you my word, I will let you go if you can generate all 20 numbers, and remember, you must use all four 4s, [in an evil slow voice] no more, no less. However, now you only have three minutes left!"

9.2 Analysis

9.2.1 Unique Solutions

The following period, students presented to the class their solutions to the Four 4s problem. 18 of the 20 students participating in this study completed solutions for all twenty numbers—although not all of the solutions were correct. As I wrote down each of the twenty numbers on the whiteboard, students offered their solution in hopes of it winning the title of "most unique solution." In most cases, this title went to solutions that incorporated combinations of factorials, square roots, and/or exponents. This was a momentous occasion in the classroom, as everyone seemed to enjoy finding errors in their peer's solutions as well as determining the most unique solutions. Interestingly, not one student made any reference to the story that introduced the assignment.

9.2.2 Story in the Conceptualization Process

When asked to describe *The Four 4s* problem, only 2 out of 20 students made a direct reference to the story in which I introduced the problem. One student wrote:

> The 'four fours' was that you were in front of the devil and given four minutes to do these 20 questions. You must use just fours and only 4 fours. Only using the operations $+, -, \times, \div, !, \sqrt{\ }$, exponent, combine [the last two referring to raising a number or expression to a power, and the use of brackets respectively]. You *must* answer the numbers 0 to 20 [italics mine].

The other student answered:

> You die and go to hell and the devil will let you go if you answer this question using 4 4's and any operations. You have to get all the numbers from 1 to 20. You must use all 4s each time.

In describing the problem, both of these students make no distinction as to what are the imagined events of the story and what is the mathematical task at hand. Clearly, the story is intricately tied to these particular students' conception of the mathematical problem.

Three out of the remaining 18 students, while not making any direct reference to the story, gave descriptive answers using natural language:

> "Use four fours to equal a number between 0 to 20 (if you want higher). *You have only 4 minutes to get all 20* [italics mine]. You may use any operation listed: $\sqrt{\ }, !, \times, +, -, \div, 4^4$"

> "Get all the numbers from 1-20 using 4 4s, *no more*, *no less*, and only using the regular methods $+, -, \div, +, \sqrt{\ }, !,$ ect. [italics mine]."

> "Using four fours, *no more, no less*, you must make them equal 0 through 100 using an equation [italics mine]. $\sqrt{\ }, !, \times, \div, -, +, \frac{n}{d},$ and more."

Note the italics. In the first example, the student describes the problem saying "you have only 4 minutes to get all 20". However, those instructions were given to the character in the story by the devil, whereas the students had two days to complete this assignment. So in this case, the student is, to some degree, identifying herself with the character in the story, taking on the character's problems—and consequently engaging their imaginations with the mathematics embedded in the story. In the last two examples above, the students both echoed phrases that I used when telling the story. Specifically, the four words, "no more, no less" were delivered with great emphasis, in a slow, evil-sounding voice so as to be threatening to the main character, while at the same time giving the students some necessary constraints to the mathematical problem.

The other 15 students however, made neither direct nor indirect references to the story. In fact, instead of describing the problem in familiar natural language, these students simply restated several of their answers to the problem using mathematical notation (i.e. $\frac{4}{4}-\frac{4}{4}=0$, $\frac{4}{4}\times\frac{4}{4}=1$, $\frac{4}{4}+\frac{4}{4}=2$, …). These results suggest that most of the students do not see the story as intertwined with the inherent mathematical content. One possible explanation for this might be related to the way high school students perceive school mathematics—as a set of symbolic, objectifiable facts, algorithms, and processes (Schiro, 2004). Mathematical ideas are rarely associated with oral or written communication, and so students may have answered the question based on what they usually encounter at school and what is normally expected of them (NCTM Standards, 2000). Furthermore, placing a question at the end of an integer review test may not have been conducive to eliciting a natural language response. In this light, students may have been confused as to what was expected of them when answering the question. Another more likely reason students did not associate the mathematical content with the story may have more to do with the type of story used in the study than with the perceptions of students. Unlike *The Legend of Chikara Ni Bai*, where the story explains a mathematical topic, *The Four 4s* simply introduces a subject and is not intertwined with the mathematical content.

9.2.3 Future Experiences

While the results from the question do not seem to link the story to the students' understanding of the mathematical concepts, there is additional evidence to suggest however, that *The Four 4s* story did engage students' imaginations. Consider this exponent story written by Andrew almost one month later:

> ZZZAAAPPP! I have just been hit by a 600,000,000,000,000(trillion) volt lighting bolt. As you can guess I am dead. Slowly my dead body rises to heaven but wait No! No I'm going down, down to the core of the earth. No I am in hell! Why hell? When did I deserve this? Well now I am standing at the firey gates of hell. Man its hot. And now some fat porker walks out to greet me. Holy hell its Mrs. Whitehead! Nope its just Satan. "Welcome to the firey paradise of hell!" says Satan "You will be staying here forever, hahaha! However you may go to heaven if you can complete this task. Make this dough into a cake that has 4,194,304 layers! And it must fit in this small rectangular box!" Says Satan. " Fine you French fried fat porker!" I say. It's easy, I'll just use Mr. Balakrishnan's folding thingymagigi technique. I will fold it once then again and again and so on and so forth, which is a bout 22 times. And it will be really easy cause I'm ripped! While I was doing this some other dude had to do the same thing bus instead he was cutting it into 4.194,304 pieces and staking them. What a dumb bum. It's a good thing I listened to Mr. Mr. Balakrishnon in class. 4 hours later I showed it to Mrs.Whitehead, I mean Satan "How in the hellish hell dud you do that!" Says Satan. I learned it from. Mr. Balakrishnan. And then slowly I ascend to heaven. And I live happily ever after, with a few burn marks.

Clearly, *The Four 4s* story influenced the way Andrew constructed this story. That is, *The Four 4s* lived on in Andrew's imagination. In fact, Andrew's mention of Mrs. Whitehead is another reference to a story I told in class about writing equations that is not a part of this study. This actually illustrates a reoccurring theme found all throughout this research—good stories "live fruitfully and creatively in subsequent experiences" (Dewey, as cited in Gadanidis and Hoogland, 2002, p.3). Not only did the stories I tell resurface in various fragmented forms in many of the written and oral assignments in this study, but they also found their way into many of the classroom activities, informal conversations, and class discussions throughout the year. However, the reader must not forget that my main purpose, as argued extensively throughout this thesis, is to not simply engage students' imaginations, but to engage their imaginations *with the mathematical content*.

10 STUDY IV: SKULL ISLAND

10.1 Treatment

10.1.1 Target and Problematic Issue

Skull Island is a story to introduce an activity on graphing linear equations. According to the planning framework outlined in section 6.3.2.1, the target is graphing linear equations using one of the three forms: slope-*y*-intercept form, general form, or point-slope form. Graphing linear equations at first involves a “leap of the imagination,” as described in section 4.2.1, considering students must connect an abstract equation to a concrete two-dimensional graph. The problematic issue arises when the markers to help make this leap are not readily apparent. For instance, students have difficulty graphing linear equations when they are not in slope-*y*-intercept form, since the slope and *y*-intercept are not always obvious. Consider the following linear equation in slope-*y*-intercept form:

$$y = 2x - 5$$

Students can easily identify the slope as 2 and the *y*-intercept as -5, which makes graphing a relatively simple task. Now consider the same equation written in a slightly different form:

$$2y = 4x - 10$$

In this case, the slope and *y*-intercept are not obvious. Students may mistakenly think the slope is 4 and the *y*-intercept is -10. The problem requires an additional algebra step to put the equation in slope-*y*-intercept form. The same equation in general form is even more problematic:

$$0 = 2x - y - 5$$

The need for explicit markers is based on the tendency of students to look for quick, one-step solutions to graphing problems. Students are often reluctant to perform the necessary intermediate algebra steps in order to find the slope and *y*-intercept. The story of *Skull Island* was designed to encourage students to see each linear equation as a "clue" that needs to be deciphered, and the algebra as part of the steps in the deciphering process. The solution to a mystery (such as one that you might find in a Sherlock Holmes or Nancy Drew mystery) is not always explicit and often requires a careful examination of the evidence and the suspects involved. It was my goal to have students see that in order to graph a linear equation successfully, they usually must perform a series of operations to arrive at a solution, and that intrinsically, the reward is the same—whether solving a murder mystery or finding the intersection of two lines.

This project is a modification of the *The Murder Mystery* activity I developed previously. In *The Murder Mystery* activity, students were put into groups and handed a "secret envelope." Inside the envelope were floor plans of the school with Cartesian graphs superimposed on them and a letter that said the following:

> You have been given the task of tracking down a criminal—me. My crime will be revealed in time, but for now, you will have to use your mathematical prowess in a way you never dreamed. To find me, you will have to carefully solve the following clues using the maps of the school. You may want to bring a ruler and pencils along too! Make sure you keep your work as proof of your skills—otherwise you will surely fail.

> Graph on the lower floor map the linear equation that passes through the point (8,-10) and has a slope of 2. Then on the same map graph the linear equation that passes through the point (10,-9) and has an undefined slope. The intersection of these two lines is your destination. If you hurry you will catch me.

Ten of these envelopes were planted around the school with different teachers. When a group found the solution, they would go to that location and pick up the next set of clues. The activity was set in the context of a murder mystery and students were the "detectives" in charge of tracking down a serial killer. The inspiration for this activity came from a particular scene in the novel, *Angels and Demons* (Brown, 2000), in which the main character, Robert Langdon, determines a point on a map based on the outline of a cross formed by the location of four churches. From a mathematics instructor's point of view, this was a graphing exercise guised as fiction—accordingly, I became consumed with developing a mathematics assignment fashioned in a similar way.

If the level of student participation is any measure of success, *The Murder Mystery* activity was a smashing hit. Students seemed consumed with finding the locations on the map—even those students who normally are not interested in classroom activities. Moreover, many students commented, one way or another, on the story and plot. Many students read the clues out loud and made humorous remarks—yet they all continued to work on the problem with singular intensity. The idea of tracking down a killer was appealing to students and undoubtedly engaged their imaginations. Unfortunately, an activity of this nature requires a great deal of coordination with other teachers and when several students returned to my classroom complaining that the substitute teachers had no idea what was going on, I decided there would have to be some modifications to the activity if I decided to do it again the following year.

10.1.2 Cognitive Tools

Langdon's map problem in *Angels and Demons* as well as *The Murder Mystery* activity, were both imbued with intrigue and suspense, and their speedy solutions motivated by the threat of death—in a manner of speaking. Sounds familiar? Of course, it is that proverbial set of binary opposites once again infiltrating our storylines—good versus evil, ingenuity versus despondency, and so forth. In the *Skull Island* project, we also took this approach and again combined the primary cognitive tool of story with the supporting cognitive tool of binary opposites to engage students with the concept of linear equations. The story is about a cursed map, lost treasure, and pirates—a winning combination with any age of listener. But this tale was not about any pirate—it was about the most evil and feared of all pirates—Bloodhead. This wicked perversion of good fiction represents one side of our binary set, while the reader/listener is the innocent bystander thrust into a quest of supernatural proportions. One might argue that I also used the cognitive tool of knowledge and human meaning as I tried to pass the story off as "authentic" and that the map I presented them with supposedly contained history's first glimpse of Cartesian Coordinates (I specifically made sure the map's date came not long after Descarte's invention). I further used the cognitive tool of jokes and humour to elicit an emotional response from students and connectedly to engage their imaginations (Egan, 2005). Not only is the name "Bloodhead" enough to expose the comedic nature of the story, but the choice to present the story as an actual historical—and personal—event also added to its humorous disposition.

10.1.3 Presentation of Problem and Conclusion

According to our planning framework (Zazkis & Liljedahl, 2005), the presentation of the problem was organized according to the cognitive tools already identified. Once again the primary cognitive tool is story, but this time the story functioned mainly as a means to establish a context for the problem and the related graphing assignment. That is, the story of *Skull Island* simply introduced students to the project. It also served as a rudimentary model for what students were expected to create in the connected assignment. Moreover, much like the familiar board game *Battleship* is based on the premise (or story) that each player is in a sea battle and must graph the locations of their opponent's ships, the story of *Skull Island* served as the premise (or story) for the task of graphing linear equations. After delivering the story of *Skull Island* orally (acting as if the events of the story were real and that I was the recipient of the map), I organized students into groups of two to three and then handed out an outline of the assignment (see Appendix 17.3). Each group had to create their own "treasure map" similar to the *Skull Island* map (I had created a treasure map that I brought to class). The map had to have a story to it that indicates the theme and at least thirteen different locations that were each to be located by graphing two intersecting linear equations. Each linear equation represents a "clue" and was to be described in 1 of 26 different ways—all 26 ways used exactly once (see Appendix 17.3).[11] Students had five class periods to establish their theme/story, design their maps, learn how to do the mathematics, and setup their clues. On the sixth day, the groups exchanged maps and used the "clues" to find the locations on the map.

[11] Students chose their *own* points on their maps and then fashioned their clues using the same wording as those given in the handout.

The *Skull Island* story and project were developed in conjunction with SFU PDP student, Teresa Reed, as a six-period unit. The model for this project is based on an open-ended approach in which there are no scheduled formal lessons, but instead, students meet the learning outcomes as they are working through the project and consulting with the teacher and other students. Class discussions and lectures are impromptu as needed (Boaler, 2002). There are two fundamental differences between *The Murder Mystery* assignment and the *Skull Island* project. First, in the *Skull Island* project, students do not simply graph lines from clues I give them, but they create their own clues which translate into lines. This is a much more difficult task, not only because it requires more preparation time, but also because it requires a more comprehensive and working understanding of linear equations. For instance, consider the following clues students might be presented with:

Graph the equation of the line perpendicular to $2x - 5y + 4 = 0$ (1)
and passing through the same *x*-intercept as $x + 2y - 6 = 0$ (2)

Using this information, students would get the following equation of a line:

$$0 = 5x + 2y - 30 \qquad (3)$$

However, designing these clues, (1) and (2), would be a more difficult task. It would require students to first draw a line on their graph (3) with the particular *x*-intercept and slope they want; Next, they would use the negative reciprocal of the slope, to create a new line which would help establish the first clue (1). For the second clue (2), they would first draw another line sharing the same *x*-intercept as (3) but with a different *y*-intercept; They would then algebraically determine the equation of this new line (2). Clearly, the interplay between the symbolic algebra and the concrete representation of the graph demands students operate at a certain level of mathematical sophistication. The second way the two projects are different is that, unlike the *Murder Mystery* assignment where kinesthetic learning takes place (i.e. students travelled around the school to actual classrooms), in the *Skull Island* project, students only found locations on their maps. I made the decision to sacrifice this component of the project in order to facilitate the more creative, discovery approach as given in the final version outlined here.

10.1.4 Participants

60 students in PMATH10 participated in this assignment, but only 43 students chose to allow their work to serve as data. These students had just finished an assignment where they had to determine the slopes and *y*-intercepts of linear equations in slope-*y*-intercept form and then graph them.

10.1.5 The Story

> It was 15 years to the day when I first saw that cursed map. The woman at the counter had watched me from the moment I walked into the curio shop. It wasn't until I picked up the map and started to examine it that she stumbled over to me and touched my hand. "You are the map-bearer," she said in her thick Jamaican accent. "She's obviously crazy," I thought. But something compelled me to believe her. This was no ordinary map…superimposed on it was a Cartesian coordinate grid—not the regular latitude and longitude lines! There were a few lines that appeared to have been drawn… in blood?!! Written over each line seemed to be markings that resembled linear equations [students laugh]. This was quite possible as the date on the map was 1698 AD—only years after Descartes developed his graphing system. I tried to give the map back to her, but she wouldn't let me. She pushed my hand away and then…right in front of me…she dropped dead! Did I cause her death…or was it that cursed map? I knew something was different about it. Sprawled across the top of the map were the words "Skull Island." A cold shiver went up my spine—almost felt like a hook or a dagger against my back! Okay, my imagination was getting away from me. What was very real however, was that on the map was a big red "X". I knew what that meant—treasure! After all the chaos following the woman's death had passed, I quickly

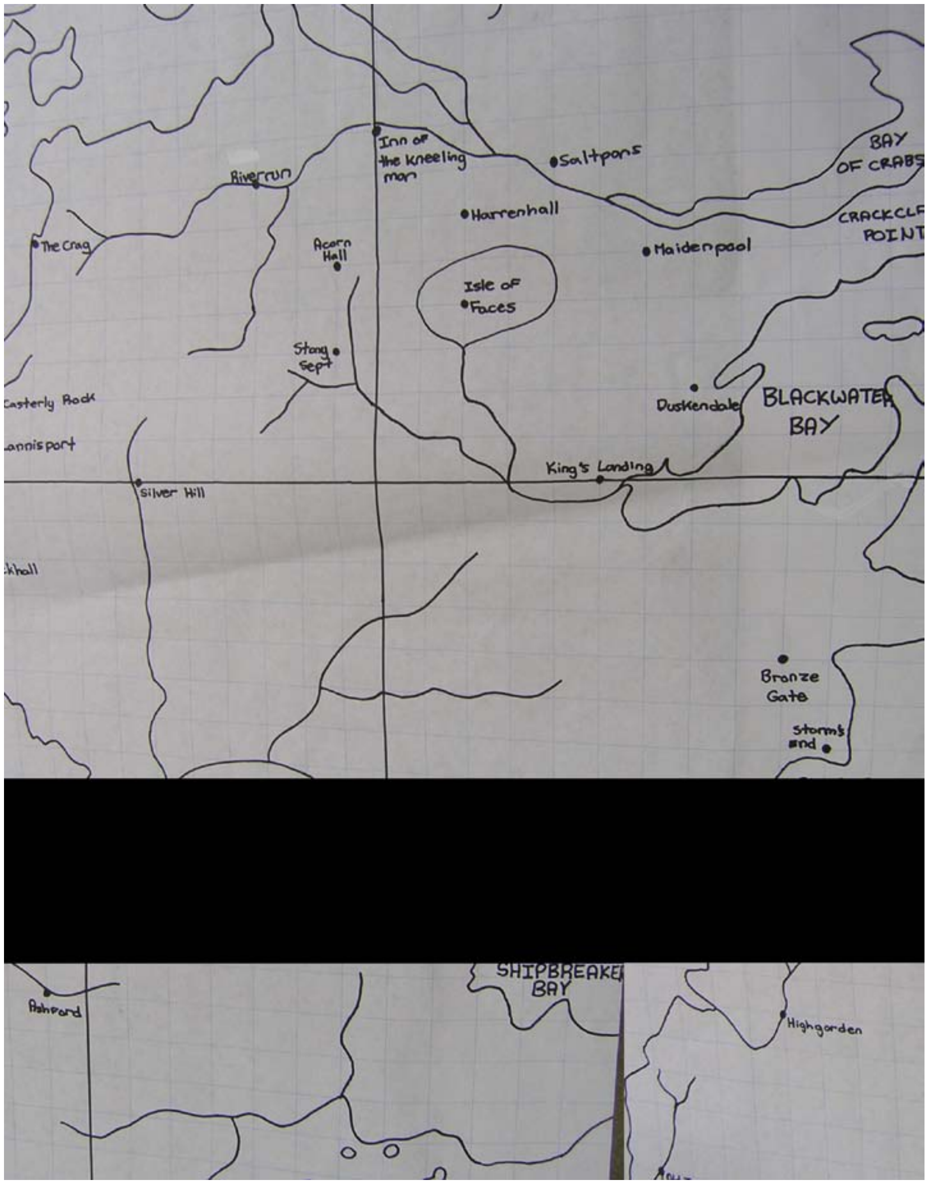
> went to meet my friends in the hotel and show them the map. We tried for several days to find out where the island on the map was located, and how we might go there, but none of the locals seemed willing to talk to us after seeing the map. They were scared—as if they had seen a ghost! "What evil had forged this document?" I wondered. The only option was to go to the library and see if we could find out more about this "Skull Island." Our persistence paid off—well, as much as evil and death and revenge is a good thing…you see, the map was made by the most evil of pirates—the legendary "Bloodhead"! Very little was written about this depraved buccaneer. Apparently, he was the most feared of pirates. Legend has it that Bloodhead was so evil, when some locals kidnapped his wife and children in order to get some of his booty, Bloodhead burst into the home of the kidnappers and shot and killed… his entire family—leaving the kidnappers alive! He was noted as saying, "I would rather see them dead a thousands times over than to have them come between me and my gold." To this day, no one utters the name of Bloodhead... unless they wish a curse upon them… Bloodhead… Bloodhead… Bloodhead…

10.2 Analysis

10.2.1 Themes and Binary Opposites

The *Skull Island* project turned out to be an effective way to blend abstract equations and concrete two-dimensional graphs with the interests and imaginations of students. In the same way children inadvertently learn Cartesian coordinates when playing *Battleship*, students learned—although not so inadvertently—the basics of graphing linear equations and algebraic manipulation through the *Skull Island* project. In *Battleship*, the "story" is war and each player is trying to plot points on a Cartesian graph in order to sink their opponent's ships. In the *Skull Island* project, students came up with their own interesting themes and stories to make the task of graphing 26 linear equations both exciting and memorable. The themes and stories students chose ranged from the modestly familiar to the lavishly fantastic. In *The Search for Isra*, Arianna and Cayley designed a treasure map inspired by the fantasy fiction, *A Song of Ice and Fire,* by R.R. Martin. See Figure10.1.

Figure 11.1: The Search For Isra.



Where the map itself is lacking in aesthetic value, the story makes up for this through extensive character development and carefully crafted plotlines. *The Search for Isra* is a bold example of the inherent binary opposites that dominate most adult and teen fiction, as well as the many interesting stories presented in this study. Like the story of *Skull Island*, students set their themes against the backdrop of such notions as life and death, good and evil, innocence and corruption; ingenuity and despondence, etc. Because the mathematics is so much a part of the story, it is difficult to tell in which direction the "aesthetic flow" is going (Gadanidis & Hoogland, 2002, p.4). In any case, it is clear that the use of binary opposites is in servitude to the greater task of mathematical exposition.

10.2.2 Equations as Clues

Like most of the stories, in *The Search for Isra*, each linear equation was fashioned in a way resembling that of a clue. This is comparable to stories, like *The Twelve Diamonds,* in section 3.6.1, where mathematical patterns are juxtaposed alongside the events in the story. The following excerpts reveal the extent to which this occurred:

Introduction

> Isra, the daughter of a well respected, wealthy Lord (now deceased) and Lady, has gone missing, presumably at the hands of the Samids, an opposing house. The area of Westeros is vast, its lands cast into chaos by its lack of a ruling king; it is, in short, no place for a nine year old girl to be on her own. Of course, Isra is not your typical nine-year-old. Armed with her sword, Sting, and the training to go with it, she is able to dodge guards and tip-toe through unfriendly situations. However, her mother dearly wishes her to be found, and has recruited you and your skills to do so.
>
> To locate the Stark girl you will be given two clues, each of which will give you a line to chart on the provided map. The point at which the lines intersect will reveal a previous or possible location of Isra. Once you have reached the end of your quest, return with Isra to the Ardi Keep in the east. (Not to be charted)

Location One

> At the start of your search your received reports that Isra was last seen in the streets of ____________________(1), which made sense at the time. After all, she was there with her sister, father (before he lost his head) and many men from their homeland far in the north. However, it seemed unlikely that she would remain near the Samids for long. With that in mind you kept your feet moving.

Line one: with a slope of -3/5 and a *y*-intercept of -3.

Line two: with a slope of 4/5 and a *y*-intercept of 4.

Location Two

After fumbling around in the less profitable regions of the city, you came across a few locals that seemed to have associated with the missing girl. They claimed to have seen her flee in the general direction of ____________________(2). Once you arrived, you began a furious search of streets, taverns, and what passed for refugee centers.

Line three: with a slope of -2/7 passing through the origin.

Line four: passing through the point (7, -4) and perpendicular to $y = x – 5$.

Location Three

The search in ____________________(2) proved to be futile. Either Isra caught wind of your search and slipped away in fear, or she simply continued her travels, but in any case her form no longer occupied the city. Grumbling under you breath you spoke to the guards stationed at each gate. Luckily, one of them remembered seeing a girl that matched her description. Or, so he thought. You decided to follow his advice anyway. You came upon ____________________(3) shortly thereafter, where fortune shone upon you and led you to a tavern with a few loose-tongued serving women. The next place to search suddenly seemed a lot closer.

Line five: passing through the point (-2, 3) and parallel to $y = 1x$.

Line six: with the same y-intercept as $y = 3/2x – 4$ and perpendicular to $y = 2/3x - 4$.

Location Eight

____________________(8), as you knew well, had long been sworn to those who opposed the Isra's family. With this in mind you stepped lightly, and used the guise of a Samid solider searching for Isra's head. ____________________(8) inhabitants thought better than to question you, and so offered what information they had freely…and even gave you bread and water, which secured your safety under the time-honoured guest right. However, the information they gave was unnerving at best; Isra was apparently taken to ____________________(9), the Samid stronghold, not long ago. You could only groan in silence and feel for your sword without moving your hand.

Line fifteen: with the same y-intercept as the line 1x-y-3 = 0 and perpendicular to the line -1x-y-3 = 0.

Line sixteen: with the same x-intercept as the line 0x-y+6 = 0 and parallel to the line 0x-y+8 = 0.

Location Nine

Being one of a relatively sane mind you did not even *think* of entering upon the courts of____________________(9). Instead you kept to the streets, biding your time in taverns and drinking with the Samid men. Of course, you kept your disguise, though it was difficult amongst men who knew their army well. All in all, the first few days were not pleasant. By the sixth you were turning to the drinks yourself. By the seventh you were about to give up, when the Gods shined on you and brought your ears to a conversation from ____________________(10). To the road it was.

Line seventeen: with the same x-intercept as the line -2x-y+7 = 0 and perpendicular to the line -2x-y+7 = 0.

Line eightteen: passing through the points (-10, 15) and (-10, -3).

Point Twelve

Reaching ____________________(12) was both a blessing and a curse. You learned from its Lord that Isra had, in fact, been within his walls. However, he continued, and told you that she departed shortly after, for safety's sake; after the boy King's death, ____________________(12) fell under almost constant watch. He also told you, though, that a carriage would be provided to take you to ____________________(13), Isra's current place of residence. Your feet danced you to the carriage doors.

Line twenty-three: passing through the origin and perpendicular to the line $2x - 3y + 18 = 0$.

Line twenty-four: passing through the points (-3, 5) and (-3, -2).

Point Thirteen

At long last your search has come to a close. Within the confines of ____________________(13) is Isra, ill contented as usual with her position amongst nobility, but you can deal with her mood; it's far better than dealing with her when her body holds no mood at all.

Line twenty-five: If A is the point (-4, -5) and B is (-4, -11) determine the equation of the perpendicular bisector of AB.

Line twenty-six: passing through the origin and parallel to the line $8x - y + 5 = 0$.

10.2.3 Realistic and Fantasy Contexts

Many of the stories students wrote were situated within purely fantasy contexts as the one above: In *Nintendo Land*, students take on the role of Mario in search of Princess Peach across a landscape that is shaped much like a Nintendo controller (see Figure 10.2); In *Candyland Island*, students land on the Banana Split Boat Docks, climb Mount Cotton Candy, and traverse the treacherous Oreo Fields in search of tasty treats (see Figure 10.3); In *Salamandastron*, students find themselves in the midst of a *Lord of the Rings* type battle involving ancient rodent-like creatures struggling for control of the coastline (see Figure 10.4).

Figure 10.2: Nintendo Land.

Figure 10.3: CandyLand Island

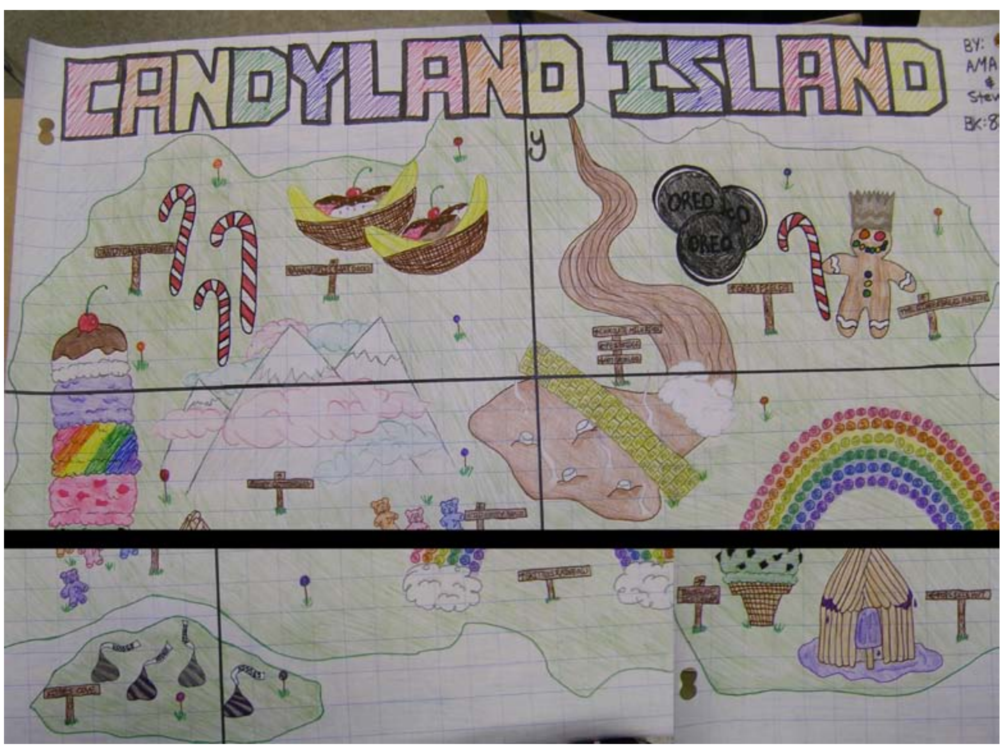
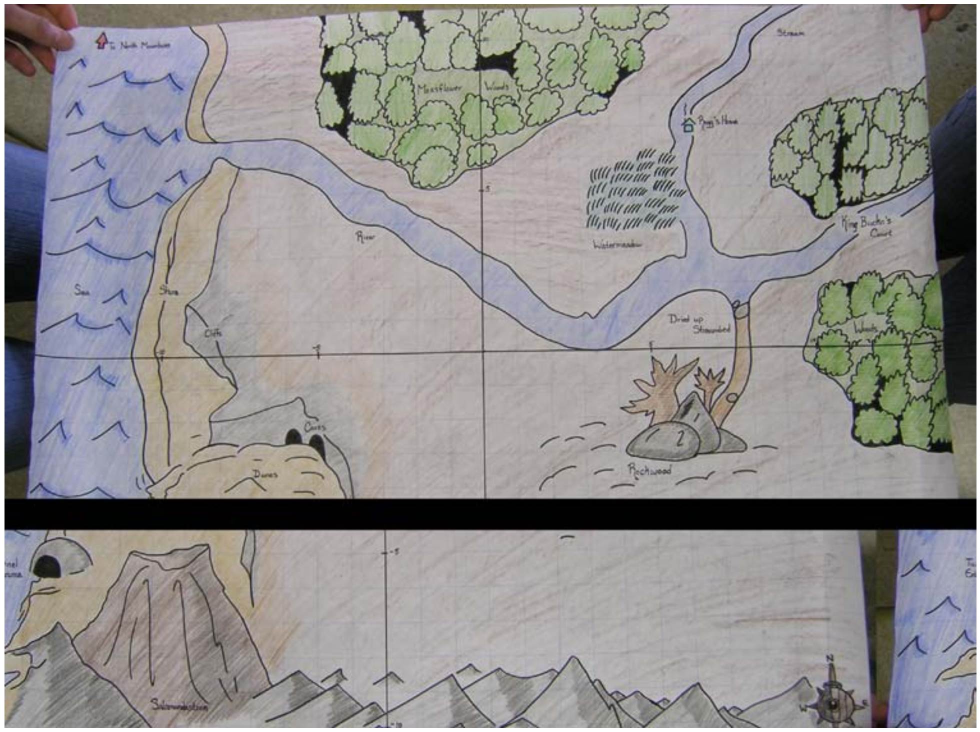
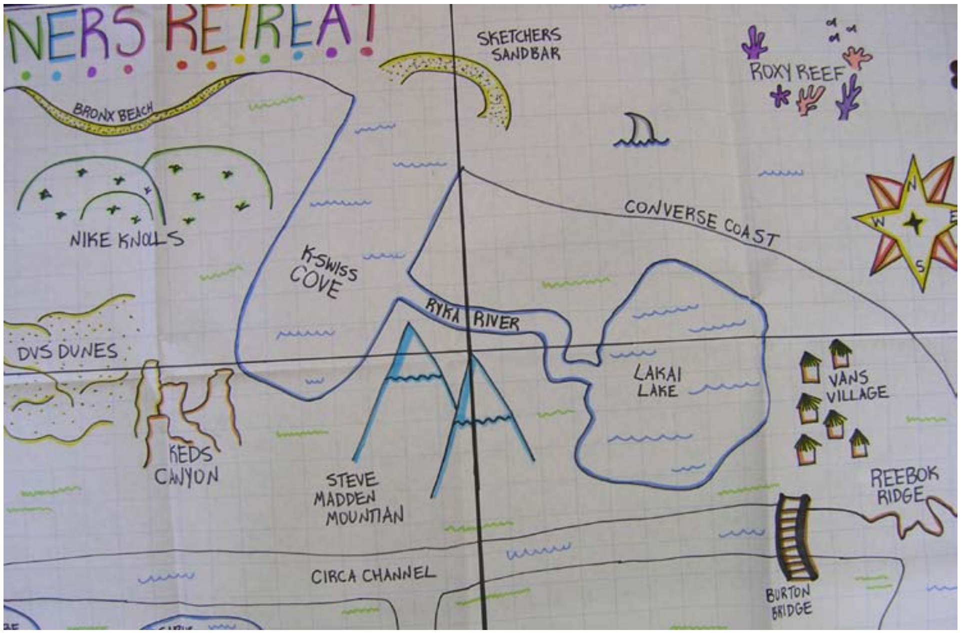


Figure 10.4: Salamandastron.



Not all of the stories students wrote were based on pure fiction. Many were clearly inspired by those things that students are familiar with, interested in, or that might arise in their day-to-day lives. In *Runners Retreat*, Jennifer and Nellie designed a treasure map based on their mutual love for running (see Figure 10.5).

Figure 10.5: Runners Retreat.



The following is the introduction to their accompanying story:

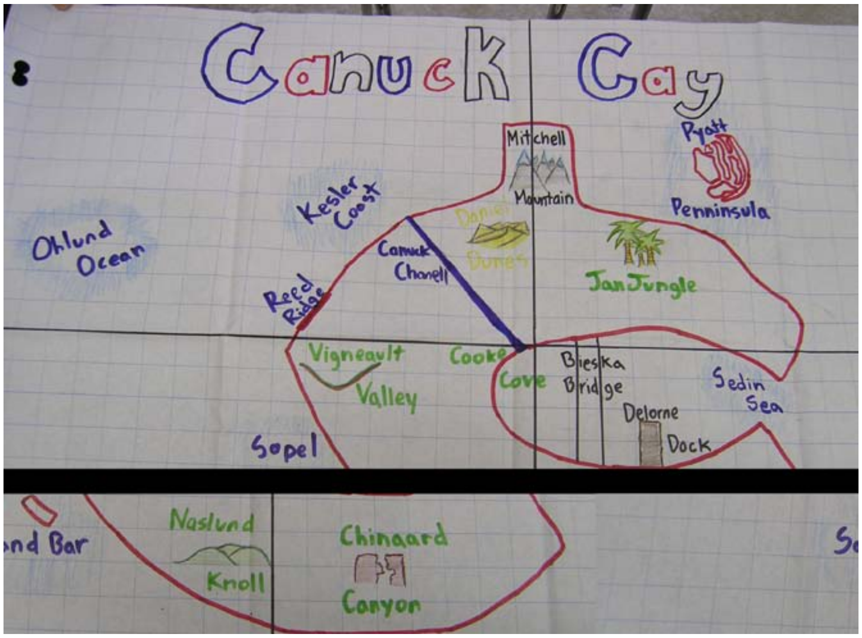
> In the early 1800's, the famous French explorer Charles Chaussure landed on this small island with intentions of making a new colony. Unfortunately, before he his dreams took off, he was brutally murdered by the local natives. Before his death, Chaussure managed to bury a sacred treasure. Even today, people from all over the world come to Runner's Resort in search of this mysterious fortune. Crystal clear ocean, miles of white sandy beaches and lush landscapes are just a hint of things you'll discover while visiting Runner's Resort. Our popular activities include swimming, scuba diving at Roxy Reef, rock climbing at Steve Madden Mountain and Keds Canyon, and sightseeing. Feel free to explore the island and search for Chaussure's treasure. Good luck and have fun!

Shawna and Julie designed *Watermania Island*, a concept that spawned from the activity that dominates both of their free time—water polo. If you are not taking a break from your treasure hunt along the shores of Speedo Lake, you might be scaling Neptune's Peak or making your way through the streets of Flipper City (see Figure 10.6).

Figure 10.6: Watermania Island.

In *Canuck Cay*, students might find themselves sailing across Ohlund's Ocean, stuck in Sopel's Sand Bar, or surrounded by natives in Naslund Knoll (see Figure 10.7).

Figure 10.7: Canuck Cay.



I was especially amused by Kayla and Janine's, *Shrek World*. Here, students assume the roles of Shrek and Donkey in search of lost treasure (see Figure 10.8).

Figure 10.8: Shrek World.

The following are excerpts from their story that illustrate how the mathematical content is intricately connected to each set of clues:

> Welcome to the world of Shrek. This is a once and a life time opportunity. You get a chance to win the Shrek prize. But the question is, are you up for the challenge? To make it more fun, you and your partner can be in character. You and your partner can decide of who will be playing the role of Shrek and who will be Shreks sidekick Donkey! But remember, you don't have to. Now before you begin I will go over some rules.
>
> First rule: NO CHEATING
> Second Rule: Must show all your work
> Third rule: Every time you find a location, come and see us and we will give you the rest of the story bit by bit.
> Fourth Rule: You must read the story that comes with it and when you find where the new location is, come to us and we will give you another piece of the story and two more equations.
> Fifth Rule: HAVE FUN!
>
> Now begin your brilliant journey!
>
> 1. Oh it was a hot, gruesome day. Shrek was so bored. Not only was it hot, but Fiona had left to visit her parents at Far Far Away Kingdom by herself. It wasn't that Shrek wasn't invited, but it was Parent Day and Fiona wanted to spend some quality time with her parents. "Oh Donkey, why must you stay with me? Don't you have any parents you can annoy…I mean visit?" Donkey replied happily, "Yes, but they happened to go far away on vacation for two days," replied Donkey. Shrek rolled his eyes. "Don't you think it's a little strange how they chose to go on vacation right before Parent Day on the last minute?" asked Shrek. Donkey stood there and pondered for a brief second and then just shook his head, "Nope." All of a sudden Donkey had a great idea. He told Shrek of this treasure map he found. Shrek didn't want to go but Donkey wouldn't stop talking so he agreed. They left from Shrek's Swamp and that's where their journey began. The first hints (equations) on the map are….
>
> 1) With slope -1 and *y*-intercept 13 (for one line)
> 2) With slope -0 and *y*-intercept 14 (for another line)
>
> 3. Good job you guys found the next location. Shrek and Donkey have just arrived at the abandoned Farquuad's Castle. Shrek was feeling uneasy about the place so they both tried to find the clue as fast as they could so they could get out of that place. They were lucky because they found a guard who

was wearing a shirt with an equation like thing. They thought that, that couldn't possibly be it but there was nothing else. So they wrote down the equation and ran because the guard started chasing them.

5) Passing through the point (0,10) and parallel to $y = 1x + 10$.
6) With the same y-intercept as $y = - 4x + 3$ and perpendicular to $y = 7x + 4$.

4. It turned out that the equation was right. Shrek and Donkey were happy because they didn't want to go back there again. They just arrived at the Dragon's Lair. This is what changed Shreks and Donkeys life forever. Donkey was happy to see Dragon and his little babies. This was really his home but now and then, he visits Shrek. Before they walked in the castle to say hello they saw an equation engraved on the door.

7) Passing through (0,-9) and parallel to the line which passes through (6, 2) and (17, -4).
8) Passing through (0, -6) and perpendicular to the line which passes through (-5, -3) and (6, -6)

9. Shrek and Donkey have just arrived at the enormous sunflower field. This is where Fiona and he ran around, holding hands for a long time. For Fiona, Shrek grabbed some of the sunflowers to give her later. While Shrek and Donkey were walking around the sunflower field, they noticed that there was one part that had no sunflowers. There in the dirt was two equations written. They are…

17) With the same x-intercept as the line $5x – y + 7 = 0$ and parallel to the line $7x – y + 9 = 0$
18) With the same x-intercept as the line 8x – y + 9 = 0 and perpendicular to the line $– 8x – y + 2 = 0$

10. Their next destination was a scary one for Donkey. They had just reached the Gingerbread House. On the way there Shrek was telling Donkey the Hansel and Gretel story but twisted the ending around. He said that at the end the witch lived and succeeded in eating Hansel and Gretel. Donkey was so scared that he decided to leave this one for Shrek. Shrek laughed and rolled his eyes. This was the exact gingerbread house that Fiona and he went to for their honeymoon. It looked beautiful. He went to knock on the door and an old, ugly lady came and gave him a note. It gave him the equations. They are…

19) Passing through the points (5, -6) and (12, -3)
20) Passing through the points (7, -6) and (4, -8)

11. Shrek and Donkey have arrived at a place they have never been to before, The Palace of Aladdin. When they walked in the palace Shrek had to pull

Donkey the whole way because Donkey wouldn't stop flirting with Jasmine and Aladdin was getting a bit angry. Then they come upon a magic lamp and Donkey
rubbed it. Then a Genie pops up. He says that he'll grant them 3 wishes. Before Shrek could reply, Donkey said food and more food. Shrek quickly blurted out, "We want the equations." Shrek said it fast enough so the Genie gave them food, more food and the equations. They are….

20) Passing through (-5, 8) with slope – 1
21) Passing through the point (0, -9) and perpendicular to the line with the equation $3x + y - 9 = 0$

12. The next destination was another place where Shrek had to pull Donkey along. But not for the beautiful Princess Belle but for the Prince who lived there since Donkey annoyed him. Their destination was Beast's Castle. Shrek understood how the prince felt since the prince seemed to be in a beastly form. The Beast didn't like them walking around his castle at first but the sweet Belle persuaded him to let them. They walked around for an hour or two and finally bumped into Lumière and Clocksworth. They were talking items. They took them to the library and showed them a book with two equations written in them. They are…

22) Passing through the point (0, -3) and parallel to the line with the equation $6x + y + 3 = 0$
23) Passing through the origin and perpendicular to the line $11x + y + 0 = 0$

3 Comprehension Questions

1) Who found the treasure map?

2) What two Disney palaces/castles did they go to?

3) Where was the treasure the whole time and who saw the X that was there but didn't tell the other person?

10.2.4 Context Distraction

While most students entertained the mathematics on par with the artwork, some students focused on the latter and failed to meet the intended learning outcomes for this assignment. As an extreme case, Rory, Leanna, and Amelia developed "The Lighted

Map," a very sophisticated map made with a wooden frame and equipped with wiring and strategically placed coloured lights. After students worked through the clues and found all the locations on the map, they connected some wires and the map lit up showing the correct locations in red. In this way the students working on the map knew whether they found the correct locations or not (see Figure 10.9).

Figure 10.9: Lighted Map.

However, Rory, Leanna, and Amelia only created 5 locations, and failed to negotiate crafting each of the 26 different linear equations. Furthermore, they only attempted graphing linear equations in slope-*y*-intercept form. Much like the girls in Boaler's study (De Bock et al., 2003) as discussed in section 4.4.2 of this thesis, these students became caught up with the context of the problem and failed to address all of the mathematical ideas in question. This is especially disconcerting in light of my research as it suggests that students can become so wrapped up in the emotional aspects of a story, that they might miss some of the deeper mathematical content—in some cases, the story might even help students avoid the mathematics altogether.

The *Skull Island* project invited students to explore the concept of graphing linear equations in a manner that was both imaginatively engaging and intellectually challenging. Undoubtedly, creating clues that translate into linear equations is a much more difficult task than what students might normally be expected to do in a unit on graphing. However, by linking this activity to a story kept students interested and involved in the topic at a surprisingly elevated level—we might liken their enthusiasm for the task to a good game of *Battleship*. In designing their maps, clues, and themes, students expressed themselves in unique ways drawing on their own interests, fantasies, and sense of humour. This study does however bring up one potential pitfall when using stories to teach mathematics—stories can unwontedly draw students away from the deeper mathematical content—even giving students an opportunity to avoid the mathematics altogether. This is a problem that continued to come up throughout the course of my research of which I will address again in later sections of this thesis.

11 STUDY V: THE SKIPPERS

11.1 Treatment

11.1.1 Target, Problematic Issue, and Cognitive Tools

The Skippers is a story meant to ask a question and introduce an activity on word problems. According to the planning framework outlined in section 6.3.2.1, the target, or specific part of the curriculum I wish to address, is word problems—and in particular, uniform motion or rate of work problems. It is a well-known fact that most students find word problems difficult. From section 3.2 we know that the difficulty is mostly in the way students approach word problems. Even before they read a word problem, students feel apprehensive and doubt their abilities to arrive at a correct solution. Therein lies the problematic issue: students come with negative feelings—emotions that are not conducive to imaginative engagement. The word problems students encounter at school are usually lifeless, boring, and have no immediate appeal (Schiro, 2005). Even more, students are expected to transverse the terrain from the concrete to the abstract while at the same time discarding any intuitive strategies they may already possess (Koedinger & Nathan, 2004). This assignment was designed to explore how shaping these word problems into stories might make them more appealing and the experience of solving them more enjoyable. Through careful use of binary opposites, conflicts were introduced in order to make the plotlines more intriguing and the story more imaginatively engaging. In addition, the cognitive tool of jokes and humour was also used to elicit an emotional response from students and consequently engage their imaginations.

11.1.2 Presentation of Problem and Conclusion

Again, I had to consider how to organize the presentation of the problem using the cognitive tools already identified. To introduce the uniform motion problem, I told students the story of *The Skippers*. *The Skippers* is really just a typical uniform motion problem converted into a story—a story with a dynamic plot, interesting characters, and premise students might find both amusing and inviting. Performing this conversion was a relatively easy task. Consider for example the following uniform motion problem:

> A plane leaves Vancouver heading for Toronto travelling at a rate of 200 $^{km}/_{h}$. One hour later, another plane leaves Vancouver heading for Toronto at a rate of 300 $^{km}/_{h}$. After how long will the second plane catch up to the first plane?

This is a fairly simple problem to solve, but like most word problems, it lacks any emotional appeal. Outside of affecting their mark, why would students care whether the second plane catches the first? What if we turned this word problem into a story with a dynamic plot and enticing characters? Perhaps the first plane has a bomb on it planted by some terrorists, and it is set to go off if the plane's speed drops below 200 $^{km}/_{h}$ (a more familiar version of this story involves Keannu Reeves and a runaway bus). As part of a counter-terrorist strike force, you set off one hour later travelling at a maximum speed of 300 $^{km}/_{h}$ in order to intercept the other plane and diffuse the bomb. Knowing the distance between Vancouver and Toronto is approximately 4000 km, will your team make it in time and save all the passengers? Of course, one can add much more to this story, but the stage has been set for some imaginative learning.

The Skippers also provided students with an example of what was expected of them in the forthcoming assignment. After telling the story orally from the front of the classroom, I gave students the task of taking a standard uniform motion or rate of work word problem, like one they might find in their textbook, and turning it into a story—a story with real characters, plotlines, and a setting. The story must pose a question for others to solve. Students had the option of including an illustration with their story. These stories were posted the following class and students picked two stories (one of each type) to try and solve.

11.1.3 Participants

23 students in MATH9 participated in this assignment, but only 15 students chose to allow their work to serve as data.

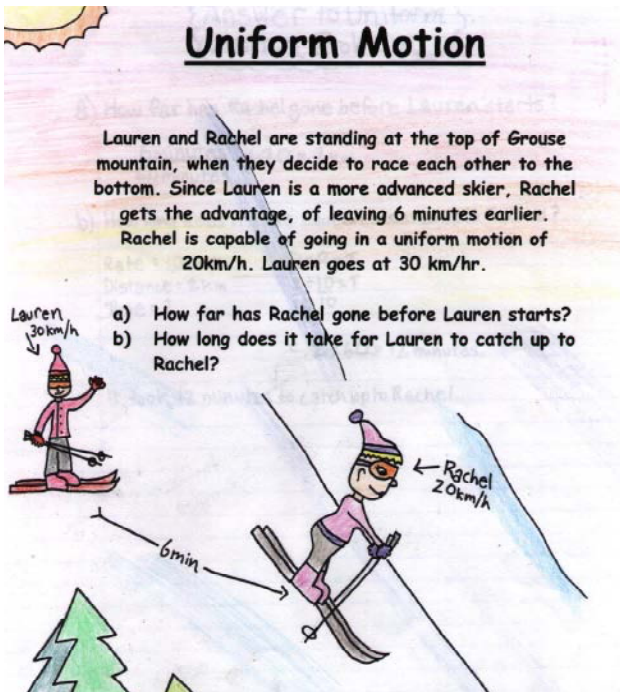
11.1.4 The Story:

> Carey and Brian [I used actual students' names from each of my classes] decided they were going to skip math class today, as they were tired from the party last night. "Hey dude, let's go to the mall where all the cool people hang out" [spoken in a stereotypical California surfer voice]. "Sounds like a plan dude!" So Carey and Brian headed out on foot for the mall at 8:30 am travelling at a rate of 10 $^{km}/_{h}$. Well, Mr. B. realized Carey and Brian were not in class, and asked the other students where they were. Tina [another student in class] ratted them out and told Mr. B. they were heading to the mall. Of course, considering Mr. B. is in top shape and a world-class speed-walker, he also set out on foot but travelling at a rate of 25 $^{km}/_{h}$. Before he could leave however, it took Mr. B. 30 minutes to put on his spandex pants. What time did Mr. B. catch up to Carey and Brian and penalize them for skipping class?

11.2 Analysis

Like many of the studies in this research, *The Skippers* presented some surprising results. The enthusiasm with which students approached this assignment was certainly unexpected. Out of the 15 students that participated in the study, only 4 presented stories that were quite simple—not unlike those found in a typical high school textbook, lacking a dramatic plot and interesting characters. Accompanying each of these four stories however, was a nicely illustrated poster like the one in Figure 11.1.

Figure 11.1: Uniform Motion.



11.2.1 Realistic versus Fantasy Contexts

Most of the stories students wrote however, suggest students were determined to make their uniform motion or rate of work problems appealing by introducing dynamic plots and interesting characters. One of the more surprising results from this study was the number of stories that were situated within realistic contexts—many of which concerned subjects students were already familiar with.[12] 11 out of the 15 stories involved topics such as: working the closing shift at Burger King; planning an after-grad

[12] See section 7.2.3 for a definition of "realistic contexts."

party; and girls and boys running in PE class. In *Watermelon Contest*, Jennifer situates her uniform motion problem in the context of a summer camp activity:

> It was a hot summer's day at Camp Canopy. Chloe and Maddie were kayaking in the near by lake when they heard the lunch bell ring. "Hurry up, or we'll be late!" called Chloe from across the lake. The two girls met each other at the dock three minutes later in a huff. When the girls entered the dining hall, all of the seats were full except for 2 chairs in the car corner. Between the chairs sat Kevin, the groundskeeper of Camp Canopy. He was a very large man, with several chins. The girls looked at each other apprehensively and meandered over to the empty seats. "Excuse me?" Chloe said, poking Kevin on the shoulder, "Um, are these seats taken?" The man stayed focused on the French fries he was shoving in his mouth and grunted something that sounded like "no". The two girls took their seats, and helped themselves to some salad. The girls watched Kevin polish off up to four hamburgers and two large plates of fries. At the end of the meal, the chef came around to each table presenting them with a large watermelon for dessert. Kevin immediately seized his knife and started slashing away at the large watermelon. "Alright," said Kevin breathing heavily, " I challenge you both to a watermelon eating contest". The girls looked at each other surprised. "Okay, you're on!" answered Chloe with a smirk on her face. "Alright", said Kevin, "You know the rules, first one to finish their quarter wins. Ready? One. Two. Three. Go!" At that second, all three of them sunk their teeth into the juice watermelon, slurping and gobbling down every bit as fast as they could. After several, seconds of this, Kevin dropped the ryne on the table and gave a large burp. It took him 20 seconds to finish! Shortly after Chloe finished at 30 seconds, but it took Maddie 2 minutes to finish hers. "Come on! That wasn't fair, Kevin! You're three times our size!" yelled Maddie. "Well I guess your right; it should have been both of you against me. Let's have another contest then" Kevin replied. "No way!" answered Chloe, "I can't eat another! Let's just do the math and see what Maddie and I would have combined!" She grabbed a napkin from the table and a nearby pen, and started scribbling rapidly. "Oh," she said quite glumly, "It would have taken us 24 seconds, so you still would have won Kevin."

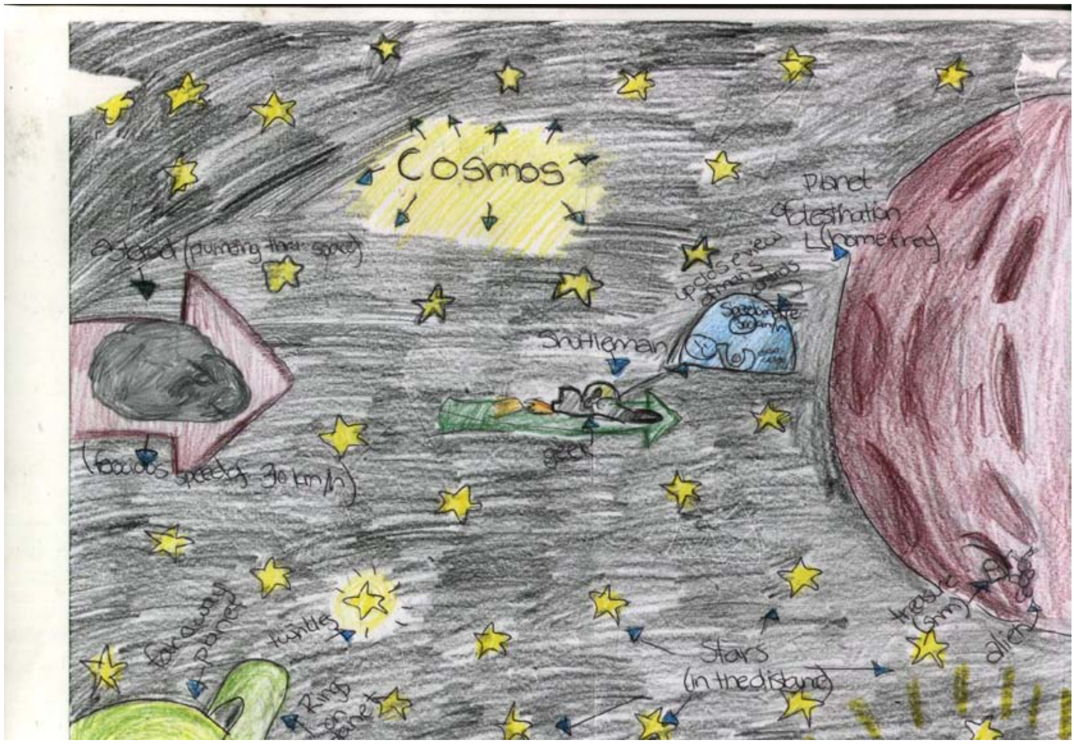
While I had hoped students would "run wild" with their stories—steering clear of the "real-life" word problems like those found in their textbooks—I was mistaken to expect anything more than what I presented students with at the introduction to this study—a real-life story with believable characters and a somewhat plausible setting. This suggests *The Skippers* impacted in a big way how students shaped their stories.

Nevertheless, 4 out of the 15 stories students wrote were situated within mostly fantasy contexts[13] as in Jordan's, *Mars' Treasure*:

> Ancient undiscovered relics are legendarily thought to be hidden in a cave on mars (not really!). To find this treasure would bring ultimate glory to the finder, the relic could be the key to solve a mystery about life on mars. Many daring attempts are made, by individuals, too hungry for riches to avoid foolish mistakes. Oxygen isn't enough to live and dangers fly morbidly throughout the cosmos. In December did one man decide to set off on the risky expedition. For months he built, planned, rationed and researched, to him, his mystery mission was to be fulfilled, then solved, and the relic discovered. With brimming confidence, he launched his craft asteroid. With menacing eyes (so it seemed) he orbited meticulously through space, a fish in a school. As the aircraft came blasting through earth's atmosphere the asteroid became unsettled. He quivered, he quaked. And he began plummeting through space, right after the space shuttle. The shuttle has a one hour head start and has 1,000,000 km till he reaches his destination. His speedometer reads 300km/h. The asteroid is flying turbo (not really) speed at 310 km/h. Will the brave (cocky) traveller have success or is all hope lost with the shattering of his craft? Will he arrive safely before the asteroid catches up or will the voyage await yet another eager soul?

[13] See section 7.2.3 for a definition of "fantasy contexts."

Figure 11.2: Mars' Treasure.



In *The Prince's Ball,* Penny presents another uniform motion problem using a familiar fairytale event:

> Cinderella left the ball at midnight for her stepmother's house and ran at $35\,{}^{km}\!/_{hr}$. Half an hour later, after realizing that Cinderella had mysteriously fled the ball, Prince Charming attempted to catch up to her. Prince Charming ran at $45\,{}^{km}\!/_{hr}$. As you can see he was extremely determined to find her.
>
> > How long did it take Prince Charming to catch up to Cinderella?
> >
> > At what time did he catch up to her?

More like a story from an English Literature course, Heidi's, *This is What I did, My Beloved* includes monsters and mazes to tackle a rate of work problem:

The maze was long, and tiring. He knew that he had to rest soon, or else he wouldn't be able to go on. But where could he rest that was safe? Nowhere. It was like being in the middle of a battlefield, only there was no battle, only creatures that jumped out randomly at him while he was walking. There had been no sign of life for awhile, and almost all his cuts had healed over and the blood was crusted. Knowing that he couldn't give up, he kept walking through the maze, remembering why he had to keep this up.

'Give her back,' he yelled.

The creature only laughed. As it began to walk away with her collapsed in its arms.

'Put her down!' he shouted at it.

'If you want her back, then follow. I promise nothing.'

He tried to run to get her out of its clutches, but was thrown back by the creature.

The memories where fresh, and it kept him going. He was glad, he had to get to her before it was too late. If that creature was anything like the kind he read about, it would torture her until she died.

He shook his head violently, *don't think about that* he told himself sharply. She was alive. She had to be.

He continued walking down the path, when he was overcome by a ground of ugly pixie-looking creatures with acorns.

He took a step back, but that was a mistake. They launched the acorns at his head, and made contact. Laughing in malicious little laughs, they ran into the brush and out of sight. He rubbed his head and quickly went by the part that they ran into, he could hear them laughing in the bush.

He tried to distract his mind from what he was trying to accomplish. He looked up at the sky, several clouds covered it, and every now and then some kind of bird flew in and out of view. He sighed was about to look back at the path when he smashed into something solid, and fell backwards. He rubbed his head as he looked up to see what he ran into, preparing to apologize.

When he looked up he saw the largest creature he had ever seen in his life.

‘Welcome,’ she said in a silky voice that was almost soothing.

‘H-hello,’ he responded.

‘What you seek is beyond me.’ She said matter-of-factly.

He made to go by, but she wouldn’t move.

‘If what I seek if beyond you, why will you not move?’

‘You must pass my tests first.’

He sighed and spoke again.

‘Alright, ask me.’

‘I have one riddle to ask you. If you answer correctly, I will let you pass unscathed. Answer wrongly, you will most-likely die by me. Leave without a word, and I will let you go another way.’

‘Ask me the riddle.’ He said impatiently.

It smiled a crooked smile and spoke.

‘If me and my friend, both do the same job of cleaning up the maze. I do it every second day, and she does it the day after me. Sometimes, we will do the job together. Now, it takes me 8 hours to clean the maze, and it takes her 9. If we were to work together, how long would it take?’

He stood with his mouth hanging open trying to figure out what the woman had just said to him. He tried to work it out in his mind, but he couldn’t figure how old this woman must be. He found a stick on the ground and began etching it into the ground. He looked up from the ground and said to the woman.

‘It would take both of you 4 hours and 23 minutes to clean the maze if you worked together.’ He said.

She blinked and called to her friend.

‘It’s 4 hours and 23 minutes! We should work on it together more often!’ the woman called.

‘Really? Wow. We should!’ the reply came.

He looked at the woman with a confused look on his face.

‘You didn’t know the answer to the question?’ he accused.

She shook her head. ‘Nope.’

‘Then how do you know...’ he shook his head.

She smiled and said ‘ You may now pass through. I hope what you seek is still there.’

He walked away, considering her words, but more irritated that she didn’t know that answer to her own question. And it wasn’t really a riddle either. The irritation quickly turned into anger as he saw the place he had been searching for. The palace was dark and damp. Almost had a castle feel to it, but not as creepy. He entered the palace with his sword clear in its scabbard.

‘Kyra? You here?’ he called.

All he heard was his echo.

As he walked into the main room, he barley paid attention to his surroundings. He had to find her.

‘Well, it seems you followed me here.’ Came an irritated voice from behind him.

He pulled his sword from his scabbard and faced the beast that stole Kyra.

‘Now, now. Let’s not fight,’ a smile came to his vile lips. ‘You have nothing to fight for. Why waste your time?’

‘Where is Kyra?’ he asked ignoring the creatures’ remark.

‘She is here. But to get to her, you must get through me first.’

Suddenly, he became the same horrible beast that was in Kyras’ room when he took her. From his back came two large leather wings, and his head became even more purple and horns grew from the top of his head. The creature bared its teeth, and he saw that they were red. Blood red. Fear crept through his body, as he slowly came towards the creature.

‘Now, Damian, are you sure you want to fight me?’ the creature sneered.

‘How do you know my name?’

‘I know everything about you. Kyra wouldn’t stop talking about you.’ A wicked smile formed over his lips as he said this.

‘Let me through, let me to her!’ he roared as he ran towards the creature with his sword in attacking position under him.

The creature flared its teeth, but wasn’t quick enough. Damian struck the creature in the stomach, making it fall to its crooked knees, and grab its stomach with its clawed hand.

‘You tricky human.’ He said, almost laughing.

Before Damian could say anything, the creature was on top of him, and tearing apart his skin, piece by piece.

Pain seared through him as pieces of his flesh were being ripped from his body. He flung his sword arm, and felt the blade come in contact with something. Eyes, closed, he felt something warm running down his arm. He opened his eyes, to see the damage. He looked up and saw the creature, wide eyed mouth opened, trying to get the sword from its chest, as blood spilled in every direction from both bodies.

Damian let go of the sword, and the creature toppled over him and began rolling on the ground trying to get the piece of metal out of him. Damian rolled over, and saw how much the beast had ripped off. His shirt had been completely destroyed, and there were large marks on his chest that cut deep. Blood seeped from the wounded heavily. He knew that if he didn’t hurry, he would die. And Kyra wouldn’t have a chance.

He stumbled over the creature, and grabbed the sword. The creatures’ eyes became even smaller and wider as Damian twisted the sword to kill the beast. Blood splattered everywhere, and there was a deafening scream of

> pain before it was all quite. Damian started to walk towards the room that the creature was guarding when he heard laughter.
>
> ‘There you go, you won. I hope you enjoy what you find.’ Came the voice of the creature.
>
> Damian turned, but he figured that it was just a lingering part of its spirit. He walked to the door and opened it. As he opened it, he was overcome with the smell of blood. He looked around the room, and saw blood everywhere. On the walls, the floor, the doors. He looked over to the bed and saw a figure laying motionless on it. He staggered over to the figure and was horrified by what he saw.
>
> Kyra was laying there, eyes almost fully opened, mouth opened, and large marks over her chest. Matching the ones on his. Overcome with grief, he laid his head on neck of Kyras’ body. He didn’t know how long he stayed there until darkness overcame him, and never released him.

As in the “Lighted Map” in the previous study, this example also shows how using stories to explore mathematical concepts may not always be productive—the mathematics can be reduced to a supportive role, or even a place of relative insignificance. In this case, Heidi’s strong English-language skills weighed heavily on influencing the content of her story, and as a result the mathematics was lost.

11.2.2 Jokes and Humour

It is also interesting to note the effect the cognitive tool of jokes and humour had on the stories students wrote. At least 5 out of the 15 stories were situated in a purely humorous situation. In a few cases, students accomplished this by including references to me in their stories, as in Mario’s, *Hot Tub Party*:

> One day the class went over to Mr. B’s house for a hot tub party. But he said we had to fill it up. He also said we had to figure out how long it would take to fill it up with 2 hoses. He has a large hose and a small hose. So we went to work. Mr.B told us it takes the large hose 5 hours to fill up the hot tub. And the small hose 9 hours to fill up the hot tub. So we figure out this problem. But we found it takes too long because we all had to be

> home at 7:00. So we got Kyla to ask the neighbours if we could use their hose. The hose was an extra large and the neighbours said it takes them 4 hours to fill up their hot tub. So how long did it takes to full up the tub?

In *Mr. B Has Great Hair*, John describes how quickly I can devour a cake at a party:

> Jason is too busy stuffing cake in his face while Mr. B. is too busy combing is hair… Mr. B. becomes hungry, so he asks Jason politely to share his cakes with him because he has two cakes… All of a sudden, John turned and yells out to Mr. B, "You finished a whole cake in 5 hours!" And John told them, "Can you imagine how fast a cake would disappear if you two pigs worked together?" Orlando comes from nowhere screaming at everyone, "I knew that it will take 1 hour and 52.5 minutes!" Finally, John tells Will to get rid of the pigs while Orlando asks if the DJ named Toby is able to play Celine Dion music. Eventually, Toby says loudly, "His pants are up to his chest like Steve Urkel."
>
> Question: It takes 5 hours for Mr. B. to eat a cake. It takes Jason 3 hours to eat the same cake at the party. How long would it take them if they ate it together?

Of course the incongruity in both of these stories lies in the fact that I would not be at a party with my students. But within a humorous context, students feel comfortable including precarious elements in their stories. According to Willmann (1940), when shocking events or ideas are expressed in such a mild manner, we have an incongruity that may result in a humorous situation. Another important point to consider is how students referenced people from the class in their stories—which is the same tactic I used when telling *The Skippers*. This suggests teacher-modelling plays a significant role in predicting the type of stories students will write.

With the exception of Heidi's story, each of the stories students wrote maintained a focus on the mathematics. That is, the characters, plot, and settings did not overwhelm the mathematical content. In each case, the "aesthetic flow" began with the mathematics

(Gadanidis and Hoogland, 2004, p.4). Maintaining this flow is the challenge in an assignment like this. This is especially pertinent when the student is an accomplished writer but weak in mathematics, as is the case with Heidi. Nevertheless, asking students to take simple word problems and convert them into stories may be an effective way to help them make the gradual transition from solving grounded representations to symbolic ones as described by Koedinger and Nathan in section 3.2.2 of this thesis.

12 STUDY VI: KELLOGGS NEEDS YOUR HELP

12.1 Treatment

12.1.1 Target and Problematic Issues

Kelloggs Needs Your Help is a story to intertwine with the concepts of volume and surface area in relationship to common geometric solids and to introduce an activity to reinforce the ideas presented. According to the planning framework, this is also the target, or specific part of the curriculum I wish to address. Although students at this level are relatively familiar with the ideas of volume and surface area, they are often unaware of the relationship that exists between the two measures and the connections to their two-dimensional counterparts. This can be seen in the way students often mistakenly calculate volume when the surface area is required and visa versa. As a result, a dependence on formulas ensues and students never fully see the true beauty of the interconnectedness of mathematical ideas. Therein lies the problematic issue. Very few students at the secondary level have explored the quantitative relationships that exist *between* various geometric figures. We hope our students will ask questions like: How is the formula for the volume of a cone related to the formula for the volume of a cylinder? Which has the greater volume—a cylinder with diameter x and height x, or a cube with side length x? How are spheres and cylinders alike in terms of surface area? How can one tell from just looking at a particular formula whether it is dealing with two-dimensional or three-dimensional space (i.e. $\frac{4}{3}\pi r^3$ or $4\pi r^2$)? Unfortunately, questions like these rarely surface within the context of the typical classroom activity. Moreover, very few students at the secondary level have considered the relationships that exist *within* a geometric figure. Research by De Bock, Van Dooren, Janssens, and Verschaffel (2002), shows that students tend to use improper proportional reasoning in the context of enlargement and reduction of plane figures and solids. For instance, given a square with side length *x*, the area is x^2. But what happens to the area of the square if the side length is doubled? Students often answer incorrectly saying that the area is also doubled. The problematic nature of this line of reasoning can be traced back to Antiquity to the misconception that quantities are always linked in linear proportions (De Bock et al., 2002). While the focus of this assignment is not specifically centered around this issue—often referred to as the "linear illusion"—it is hoped that students will encounter it during their explorations and negotiate it properly (De Bock et al., 2002).

12.1.2 Cognitive Tools

As usual, I used the cognitive tool of story to present the problem to students in a way that engages their imaginations (Zazkis & Liljedahl, 2005). While we mainly focused on the story of *Kelloggs Needs Your Help* to reinforce the concepts of volume and surface area with regards to geometric solids, and to explore the relationship between these two measures, *The Story of Princess Dido* and the accompanying activities, as described in sections 3.4.1 and 4.1.2 of this thesis, provided students with the necessary two-dimensional foundation in order for them to generalize particular relationships into three dimensions. The cognitive tool of binary opposites once again played a key role in the plot development of both stories and consequently helped keep students engrossed in the mathematical explorations. In *Kelloggs Needs Your Help*, there are at least two ways we see this powerful cognitive tool at work. First, the story is set within the context of the marketing department at Kelloggs. Here, the listener takes on the role of an insignificant employee given the chance to get noticed. Doing a good job on this assignment could be the big career break you [the listener] have been looking for and could win you the respect you deserve. But underneath all the boss' hype, there seems to be a veiled threat—if you [the listener] do not come up with something great, your job may be on the line, or perhaps you'll just remain in relative obscurity the rest of your career. Nevertheless, this is a life-changing event! If trying to make it in the corporate world is not enough motivation, perhaps the environmental issue is. This indicates the other binary conflict inherent in the story—corporations versus the environmentalists, profit versus preservation, greed versus global awareness, and so forth. Students might be motivated to come up with a solution to the problem posed in the story solely based on their sense of environmental concern. For a discussion of the way binary opposites are incorporated into *The Story of Princess Dido,* see section 4.1.2 of this thesis.

12.1.3 Presentation of Problem and Conclusion

I organized the problem using the cognitive tools identified in order to facilitate imaginative engagement with the mathematical content. Students first listened to *The Story of Princess Dido* as I told it orally from the front of the classroom and then they engaged in a slightly modified version of the activities described in section 3.4.1 of this thesis. The objective was for students to understand the relationship between the area and perimeter/circumference of geometric figures. In particular, they should have learned that when given a fixed perimeter, the best configuration to maximize an enclosed area is a square when limited to rectangles, and a circle when there are no constraints. This activity took two class periods after which students were shown the two proofs concerning the area of a rectangle and the area of a circle, given in section 3.5.1 of this thesis.

The story of *Kelloggs Needs Your Help* is largely an extension of *The Story of Princess Dido* into three dimensions, and it was hoped that students would make this connection when designing their cereal boxes. The following class, students listened to a (fictitious) story of how the Kellogg's company needs a new design for their cereal boxes. Like *The Four 4s*, the story drew students into the role of the main character, the insignificant employee of Kelloggs, and I naturally assumed the role of the boss. Through the story, students were instructed to come up with a new marketing strategy (logo, slogan, jingle, mascot), and a new design for the cereal box with the goal to maximize its volume (up to $6000\ \text{cm}^3$), and minimize its surface area. After stepping out of character, I then continued explaining the assignment. They could use any one or combination of the familiar geometric solids discussed previously. They had to come up with at least three different designs (three different polyhedra) showing all measurements and calculations before beginning the construction of their actual models. Their models could be made from any material and must be presented to the class in "pitch" format. Marketing teacher, Carson Power, sat in on the presentations to judge the marketability of the designs. Approximately one week after the presentations, students were asked the following question on a test:

> How did The Story of Princess Dido help you with your cereal box project?

This question was designed to see if students made the connection between the two-dimensional problem given in *The Story of Princess Dido* and the three-dimensional problem given in *Kelloggs Needs Your Help*.

12.1.4 Participants

30 students in MATH9H participated in this assignment, but only 23 students chose to allow their work to serve as data.

12.1.5 The Story

> Allan Harris, the Executive Vice-President and Chief Marketing Officer of *Kelloggs Foods* walked into your remote little office in the tucked away corner of the third floor. "I've been watching you," he said. [When speaking this character's lines, I try to sound like a 1960's American newscaster or politician, something like JFK]. A sense of fear went through your mind. You couldn't help but think that this might be your last day working here. "Kid, I like your designs, your charisma, and your style!" he said. A sense of relief brought you back to your senses. "I want you to head up the new campaign to re-design our familiar rectangular prism-shaped cereal box," he continued. "We have to reduce the surface area of our box to cut production costs, while at the same time staying as close as possible to its current volume of $6000\,\text{cm}^3$. This will be great for publicity, as it will appease the environmentalists who have been demanding we reduce waste in our production process. Now listen kid, I want a whole new look! I want a new type of cereal, with new packaging, a new mascot, new slogan, and even a new jingle! Come up with some design drafts and pitch your ideas to me next week!"

12.2 Analysis

12.2.1 The Cube

Generalizing both the properties of a square to the properties of a cube and the properties of a circle to the properties of a sphere requires a "leap of the imagination" as described in section 10.1.1 of this thesis. This was particularly relevant considering students were not explicitly directed to make this connection. Nevertheless, most of the groups came to the conclusion that the cube would make the best configuration for a cereal box with the greatest volume and the least amount of surface area. Almost every set of design plans the groups submitted included at least one cube. However, only one group actually chose to use the cube for their model. “X-Factor” promises to make one strong since there’s “an ounce of muscle in every bite” (see Figure 12.1).

Figure 12.1: X Factor.

Apparently the cube did not appeal to students in terms of a cereal box design—marketing teacher Carson Power actually gave this group a low score for its cereal box’s lack of aesthetic qualities. The group that created “Hyper Cubes,” improperly reasoned that if one cube will maximize the volume with the least amount of surface area, then two cubes would do even better. Their design puts two cubes together in the shape of a rectangular prism (see Figure 12.2).

Figure 12.2: Hyper Cubes.

While their logic was flawed in terms of the volume/surface area problem, this group's name, artwork, and design plans reveal they actually gave considerable amount of thought to the idea of multiple dimensions.

12.2.2 Geometric Solids

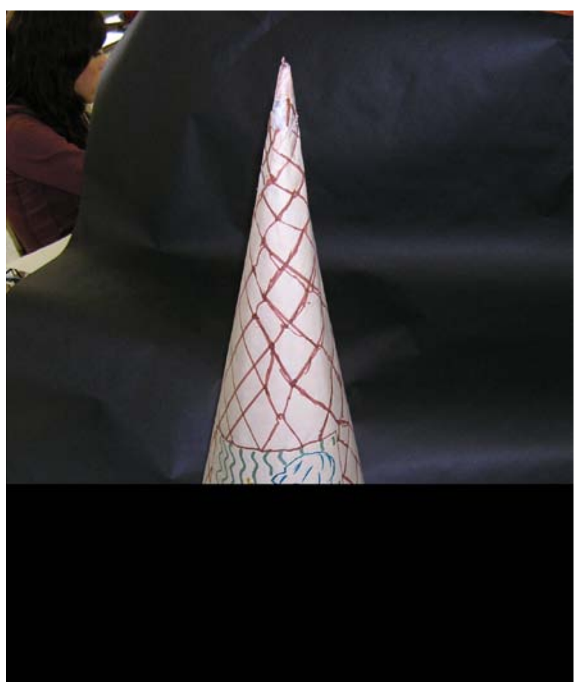
The cereal boxes fell into the full range of geometric solids present in the grade nine curriculum. Several groups even tried their hands at curved surfaces: "B-Puffs" designed a cylinder-shaped cereal box that pokes fun at their mathematics teacher (see

Figure 12.3). It is noteworthy that this group designed their cereal box with almost equal diameter and height suggesting they may have recognized the "cube" effect on maximizing the volume and reducing the surface area; "Cactus Cornucopia," which claims to help one stay regular, also used a cylinder configuration for their cereal box (see Figure 12.4); and "I Scream Cones" wittingly marketed their ice cream flavoured cereal in a box with the configuration of a cone (see Figure 12.5).

Figure 12.3: B-Puffs.

Figure 12.4: Cactus Cornucopia

Figure 12.5: I Scream Cones.



A few groups used a pyramid for their cereal box design: “Berry Fun Equations” made their box into a pyramid shaped strawberry with mathematical problems on each face (see Figure 12.6); and “Pharaoh Puffs” dawns hieroglyphics on the side of its Egyptian-esque cereal box (see Figure 12.7).

Figure 12.6: Berry Fun Equations.

Figure 12.7: Pharaoh Puffs.

12.2.3 Extension of the Initial Problem

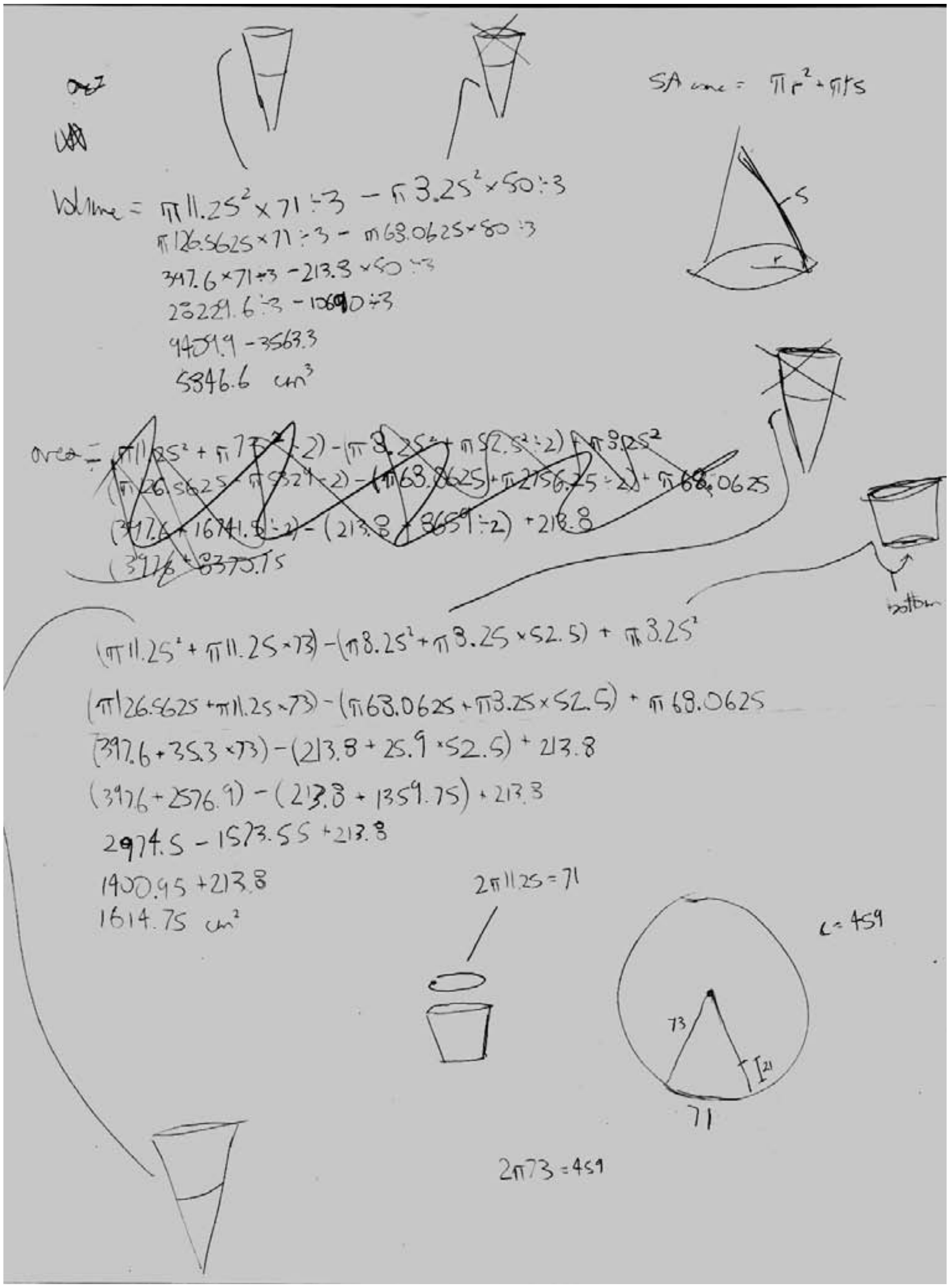
Two groups extended the initial problem to a more difficult one. “Brainios” put their cereal in a hemisphere-shaped box and sported the slogan “E=MC Delicious” (see Figure 12.8). Even more, they decided to present a scale model of the actual cereal box they were proposing. This proved to be a very difficult task as this group had to successfully manoeuvre around the “linear illusion” as described in section 12.1.1 of this thesis. It took three attempts before they finally arrived at the correct dimensions for their scale model.

Figure 12.8: Brainios.

Perhaps the most impressive cereal box design was “KFCereal.” Their desire to make a cereal box that reflected the wholesome greasiness of Kentucky Fried Chicken led them in the direction of a cone—more precisely, a frustrum of a cone (see Figure 12.9). Initially, they posed a “cylinder” design for their KFC bucket-like box hoping I would overlook the slanted sides. When this was rejected, they arrived at the cone design. This group was actually awarded the “contract” for its exhaustive calculations (see Figure 12.10) and overall marketing strategy.

Figure 12.9: KFCereal.

Figure 12.10: KFCereal's Exhaustive Calculations.



12.2.4 Poor Test Results

Where the cereal boxes showed that students were imaginatively engaged with the concepts of volume and surface area, the test results showed that students still had conceptual difficulties and/or difficulties communicating the differences between two and three-dimensional space (see Table 12.1).

Table 12.1 Response Rates for Dido Test Question

Nature of Response	Number (n=23)	% of n
Correctly communicates relationship between 2d Dido Story and 3d Kelloggs Problem	3	13%
Relates 2d Dido Story to 3d Kelloggs Problem but explanation is unclear or incomplete	8	35%
Makes no connection to 3d Kelloggs Problem (uses only 2d terms in explanation)	3	13%
Describes Dido Story using only 3d terms (no reference to Kelloggs Problem)	4	17%
Incorrectly relates 2d and 3d space	2	9%
Not present when Dido Story was told	3	13%

From the table we see that only three students connected the two-dimensional problem in *The Story of Princess Dido* to the three-dimensional problem presented in the story of *Kelloggs Needs Your Help* with relative clarity.[14] Kathy gives this answer:

> For the story of Princess Dido she had to maximize the surface area (or just area) with only a certain amount of bulls hide. This is when we discover that a circle has the biggest area with a fixed perimeter. This helps us because now we know which shape would have the biggest volume with a certain amount of surface area.

James also relates the two problems this way:

[14] A correct answer was taken as one that connects the 2-dimensional properties of a square or circle in terms of maximizing its area to the properties of their 3-dimensional analogues in terms of maximizing their volume. In this regard, the fixed perimeter/circumference of the 2-dimensional object is related to the surface area of the 3-dimensional object, and the maximized area of the 2-dimensional object is related to the volume of the 3-dimensional object.

> The Princess Dido story helped us with the cereal box project by telling us how to make the most area with a certain amount of lining (circle) so that we will use less area (square) in the cereal box project. This helps us make more volume.

> Here is Jane's response:

> It shows us how to maximize the area (in this case volume) with a fixed /minimal perimeter (in this case surface area).

Surprisingly, not one of these students explicitly related the square to the cube or the circle to the sphere, but only alluded to these relationships. We also see that the largest group of students represented in the table (35%) were those that saw a connection between 2d and 3d space, but their explanations were unclear or incomplete. Josiah gives this response:

> It shows us that the shape matters for volume and that circles have the largest area.

> Jeff explains his reasoning this way:

> Because it teaches us which shape can maximize area with a fixed perimeter and we want to minimize area and maximize volume. It helps sometimes.

> Kevin gives this explanation:

> It showed us how to use shapes to our advantage by showing us the perimeter can be less than the area so we applied that to three dimensional shapes.

In some cases, students made no reference at all to the three-dimensional problem as in Kari's response:

> It helped us learn how to maximize area, using perimeter. Which shaped made the most area.

Alternatively, a few students incorrectly thought of the Dido problem strictly in 3-dimensional terms. Matt gives this answer:

> The princess story helps with the cereal box because she was able to maximize volume while minimizing surface area and that's the goal of our cereal box project.

The results of this study suggest that simply telling a story and engaging in a task does not promise that learning will take place. In particular, students had difficulty accurately relating the two-dimensional problem in *The Story of Princess Dido* to the three-dimensional problem given in *Kelloggs Needs Your Help*—even after listening to both stories and engaging in their related activities. However, as discussed in section 8.2.3, while these students had difficulty meeting one particular learning outcome of the activity, their answers to the test question suggest they were beginning to see the "big ideas," concerning two and three-dimensional objects. Big ideas are general principles that transcend the particulars of a concept and can be connected to other areas of study in mathematics:

- Students understood that there is a connection between objects in both dimensions.
- Many saw this as a generalization problem starting with 2-dimensional space and moving to 3-dimensional space.
- They understood that the configuration of these objects affects the space they take up.

- Students understood this configuration problem is connected to an optimization problem—that is, certain configurations would produce the most or least favourable results.

Once again, an understanding of these big ideas might be a great starting place to explore in more detail, the intricacies of the concepts of volume and surface area.

While the story did not seem to help students in answering the test question, it did seem to impact the way students engaged with the concept of volume and surface area. The enthusiasm with which students argued over the designs of their cereal boxes, worked out the appropriate dimensions, and then constructed their final products stand in stark contrast to the excitement generated from the typical task of completing a series of area and volume worksheets. I cannot help but think this was directly related to the motivation imbued in the story. As a result, I believe this study shows that a story can under certain circumstances, not only compliment, but also replace the standard way mathematical concepts are taught at the secondary level. Whether this type of approach will work in the absence of such elaborate activities is not certain.

13 STUDY VII: THE WICKED SONS

13.1 Treatment

13.1.1 Target and Problematic Issue

The Wicked Sons is a story to introduce the concept of division by zero in relationship to simple rational expressions and to explain why division by zero is undefined (Zazkis & Liljedahl, 2005). It is an elaborate re-telling of *The Twelve Diamonds*, written by Zazkis and Liljedahl (2005). In grade nine, students are first introduced to the concept of rational expressions. They learn how to simplify and perform various operations on rational expressions. They learn how to substitute in different values for the variables in a rational expression and evaluate it. According to the planning framework, this is the specific part of the curriculum I wish to address. A good understanding of the nature of rational expressions and how to work with them is a necessary prerequisite for success in grade ten. The problematic issue arises when we look at the non-permissable values for the variables in a rational expression. Consider for example the following rational expression:

$$\frac{x^2+5x+6}{(x+2)(x-5)}$$

We explain to students that the non-permissable values are $x=-2$ and $x=5$. Clearly, when either of these values for *x* is substituted into the expression, we get a zero in the denominator. If we substitute $x=5$ into the expression, the expression becomes

$\frac{56}{0}$, which we tell students is “undefined.” The presence of “non-permissable values” however, often fosters the mistaken idea among students that division by zero is “not permitted.” Who says it is not permitted? What does “undefined” mean? Are there other instances of something being undefined (Zazkis & Liljedahl, 2005)? Unfortunately, many of these questions never surface and consequently students do not get the full meaning behind the expression “non-permissable values.” Nevertheless, underlying the notion of non-permissable values is the fundamental concept of division by zero.

13.1.2 Cognitive Tools

According to the planning framework (Zazkis & Liljedahl, 2005), I used the cognitive tool of story to engage the imaginations of students with the concept of division by zero. Furthermore, the original story, written by Zazkis and Liljedahl (2005), was reshaped in a way that would take advantage of the cognitive tools of rhyme, rhythm, and pattern, binary opposites, and jokes and humour. From section 4.5, we know that patterns play a significant role in the understanding of most mathematical concepts. Recognizing growing or diminishing patterns encourages students to use their imaginations—whether to find specific elements in a pattern, or make generalizations about it (Moyer, 2000). In *The Wicked Sons,* patterns abound, and when students recognize that each death corresponds to a change in the denominator, the patterns will jump out at them. Moreover, placing the story within a binary structure—the good king versus his wicked evil sons—makes this perverse relationship even more memorable. Instead of simply conveying the idea that a change in the denominator corresponds to a death, we hope students will see that each subsequent death is an emotionally charged event—a uniquely crafted murder—coinciding with certain elements in the mathematical pattern.

13.1.3 Presentation of Problem and Conclusion

According to the planning framework, I organized the problem using the cognitive tools already identified in order to facilitate imaginative engagement with the mathematical content. Students first listened to *The Wicked Sons* during the unit on rational numbers. I told this story orally from the front of the classroom which had a profound impact on the final account given here. From my discussion on orally-told stories in section 4.6, I used both non-verbal and kinesic codes to accentuate the story's binary structure (Georges, 1969). Approximately one week later, students were asked the following question on their rational numbers unit test:

What does $\frac{8}{0}$ equal? Explain why you get this answer.

The test question was included to indicate if students, when asked about the concept of division by zero, would use the story to explain their reasoning. Was the story a part of their conceptual framework? Did it play a significant role in establishing the proper meaning of "division by zero?"

13.1.4 Participants

30 students in MATH9H participated in this assignment, but only 23 students chose to allow their work to serve as data.

13.1.5 The Story

King William II was the most righteous and just king that had ever ruled Lichtenstein. He loved his subjects and did everything in his power to win their respect and loyalty. His generosity was well known throughout the land, as he often would be seen among the common folk giving money and food to the poor. Nothing, however, brought the king more respect and honour, than his most prized possessions—12 precious diamonds. These were no ordinary diamonds. They radiated so brilliant a green light that if looked upon would mean almost certain blindness [Students object to the diamonds having this property. Some say they are made of kryptonite]. These diamonds had lasers in them, which emanated outwards from the center, carving a path of destruction wherever they would end up [Students object to anyone having a knowledge of lasers during this time period]. Every other lord or nobleman envied William for his possession of these 12 "laserous" gems. As much as the 12 diamonds brought great joy to the king, however, his 6 sons brought him much sorrow. They were evil men, whose ruthless and merciless ways made them feared by all. There was William III, Charles, Bartholomew, Henry, James, and Edward. When the time of the great king's passing came, his wishes were that that all 12 diamonds were to be distributed equally among his sons—and only his sons! Each son was to get $\frac{12}{6}$ or 2 diamonds. However, each son vowed in his heart to gain possession of all twelve diamonds. That very same night, one of the servants found both Edward and James bludgeoned to death in their beds. Their heads had been cut off. Now there were only 4 sons. Each son was now to receive $\frac{12}{4}$ or 3 diamonds. The next morning as Henry was going for his morning stroll in the garden, someone crept up behind him and stabbed him through the heart. Now each son was to receive $\frac{12}{3}$ or 4 diamonds. Later that day, Bartholomew was found burned alive in the fireplace in the dining room. Now each son was to receive $\frac{12}{2}$ or 6 diamonds. The next day, William III was found drowned in a pool of his own vomit. Now the one remaining son was to receive $\frac{12}{1}$ or all 12 diamonds. Charles was the only son left and he was to, not only receive all 12 diamonds, but he was also to be the next ruler since his brother William III was dead. However, Charles contracted leprosy a week before his coronation and died before receiving any inheritance. Now there were no sons left. So how could the executer of the will carry out William's orders to distribute all 12 diamonds equally among his sons? The orders do not make sense in light of the fact that there are no sons. This situation is undefined and therefore $\frac{12}{0}$ is undefined.

13.2 Analysis

13.2.1 Story in the Conceptualization Process

Unlike *The Four 4s* story, test results indicate that the story of *The Wicked Sons* was intricately connected to the students' understanding of the concept of division by zero. In fact, almost every student when asked to explain why $\frac{8}{0}$ is undefined, incorporated some aspect of the story in their answer. For example, consider Corey's answer:

> Undefined, "$\frac{12}{6}$, 12 diamonds and 6 sons. The diamonds must be given away. Currently each son gets 2 diamonds.
>
> $\frac{12}{3}$, Now, if William dies, and so does another son, everyone else gets 3 diamonds.
>
> $\frac{12}{1}$, Now if the other 3 sons are killed by poles, lasers, or impressive displays of equestrian skills, one son gets all 12 diamonds.
>
> But now if he dies, and the diamonds must be given away, lest they blind no more, we reach a mathematical impossibility. 12 diamonds must be given to the sons, but there are no sons."

Alec actually recounts the gist of the original story in his answer:

> Once upon a time, there was a dead king who had 6 sons and 12 diamonds to give to those sons; each one would get 2 diamonds (1). If 2 of the sons died, each one would get 3 (2). If 4 of the sons died, the remaining would each get 6 (3). But if they all died, who would get the diamonds? The king can't, he's dead.

1) $12 \div 6 = 2$
2) $12 \div 4 = 3$
3) $12 \div 2 = 6$
4) $12 \div 0 = \text{undefined}$

From the two examples above, it is clear that these students associated the story with their concept of division by zero. Moreover, they chose to illustrate their answers using patterns like those in the story. This was, in fact, characteristic of almost all the student responses. This suggests that the story had a profound effect on the way students understood the mathematical patterns—and in turn, how they established meaning for the concept of division by zero. It may however, be that students are good at reproducing the stories I tell them and that they had no other tools to explain such a difficult concept. Nevertheless, students understood how each horrible death in the story was connected in a meaningful way to the denominators of the fractions in the mathematical pattern. It may sound morbid, but during the telling of the story, students sat on the edge of their seats waiting to find out how each son would meet his demise. In fact, at one point in the story, students started to call out their own ideas for how the next son should die—some of them quite humorous. From having their heads ripped off to being disembowelled by the very gems they coveted, the princes were lucky not to have my students holding the pens to their fate. Nevertheless, the dialogue that ensued was quite unique—unlike any other in this study. Students also openly questioned the validity of certain aspects of the story as it was told (i.e. lazerous diamonds, leprosy, etc.). Consequently, the final version presented here is a bizarre composite shaped by both storyteller and audience alike.

It would be pertinent to note here the supporting role the cognitive tool of binary opposites played in connecting the mathematical patterns to the story. Without it, there would be no conniving sons, benevolent monarchs, or gruesome deaths. If this were the case, our story may have provoked only a nominal emotional response, being reduced to a minor player in the sense-making process. However, from the presence of these cognitive tools surfacing in the responses given by students, it is fair to say that they were using their imaginations to engage the concept of division by zero.

13.2.2 Generalizing the Concept of Division by Zero

Upon further analysis of student responses as to why $\frac{8}{0}$ is undefined, several students used an 8 in the numerator as in the test question, instead of a 12 as in the story. For example, Katie writes:

> $\frac{8}{0}$ does not have a definite answer. Relating to the story about the diamonds. $\frac{8}{4}$ each prince would get 2. $\frac{8}{2}$ each prince would get 4. $\frac{8}{1}$ the prince would get them all. But when it comes down to $\frac{8}{0}$ there are no more princes to go to. Therefore the diamonds don't really go anywhere. They cannot be given to no one, leaving us with an undetermined result.

Katherine writes:

> If you have 8 apples and you try to divide them equally between zero people there is no true answer, because there is no one to give them to. So the answer is undefined.

Here is Kevin's explanation:

> $\frac{8}{0}$ = undefined because if you had 8 of something and were going to put it in a cupboard but you had 0 cupboards it would go nowhere so its undefined

It appears these students recognized that the concept of division by zero can be generalized to any numerator and used elements from the story to help in their explanations. Troy explicitly states this in his explanation—even making a connection to rational expressions (see last equation):

> You can't divide by zero because there has to be a denominator, and zero stands for nothing. So if the king dies and leaves in his will that 12 diamonds are to be divided among his 6 sons, and the sons kill each other, save one who dies from leprosy, who can the diamonds [go to]. This is a paradox, because there aren't any sons for diamonds to be given to (sons are denominator, diamonds are numerator).
> $\frac{12}{6}=2, \frac{12}{4}=3, \frac{12}{3}=4, \frac{12}{1}=12, \frac{12}{0}=\text{undefined, so...} \frac{x}{0}=\text{undefined}$

Jason takes a similar approach:

> It is undefined, because if you have 8 *or any number* [italics mine] and you divide by zero, there is no place to put the 8. Like in that story you told about the lasers, the evil sons, the 12 diamonds, and the king.

These students were thinking more flexibly and creatively, drawing connections to other domains—in this case, to rational expressions and the divisibility of natural numbers. (This could, in turn, better facilitate the learning of other new concepts such as factoring encountered later in the year). Some students even attempted to incorporate the more difficult concept of infinity into their explanations—albeit in a very simple way. For example, Calvin touches on it as he describes the patterns in the story:

> With all the king's sons dead, even Bartholomew and Charles, there is no one to divide the diamonds among, so $\frac{8}{0}$ is undefined because as you go down $\frac{8}{4}, \frac{8}{3}, \frac{8}{2}, \ldots$ the numbers get *closer to infinity* and become indivisible. QED [italics mine].

Mark approaches the concept of infinity in a similar fashion:

> When you have a number, let's say 12, and you divide it by 6, that's 2. When you divide 12 by 2, that's 6. Every time the dividend gets smaller, the result gets higher. *When you divide it by 0, it would have to be infinity because it would have to be multiplied infinite times before it reached 12,*

> *which it never would. We can't use ∞ as our answer, though, because that's not a number. Therefore, if you get an answer from that equation, the equation is undefined*. 12 diamonds for 3 people, each person gets 4. 12 diamonds for 2 people, each gets 6. 12 diamonds for 1 person, he gets 12. 12 diamonds for 0 people, the diamonds can't go anywhere [italics mine].

Evidently, their thinking became more flexible, energized, and creative as they struggled to incorporate an unfamiliar idea and draw connections to the concept of division by zero. What is remarkable is that these students chose to include such discourse within the context of a test question.

From the examples above, it is clear that most students associated the story of *The Wicked Sons* with the concept of division by zero. This is quite different from *The Four 4s*, as described in chapter nine of this thesis, where most students failed to make any significant connection to the story in their explanations. One reason for this may be how *The Four 4's* is simply a story to introduce an activity, whereas *The Wicked Sons* is a story to explain a concept (Zazkis & Liljedahl, 2005). While the former type of story invites students to use their imaginations to do the activity, the latter type of story actually asks students to use their imaginations to understand the concept.

Another reason most students associated the story of The Wicked Sons to the concept of division by zero may be due to the way *The Wicked Sons* is set within a patterning environment. In *The Four 4s*, there are no patterns students can identify, follow, and use to predict future terms. Each new equation was distinct and was unrelated to the previous one. In fact, students were partly motivated by the thought of finding some new obscure equation unlike anyone else's. Although *The Four 4s* promotes a sense of wonder and mystery, it neglects students' sensitivities to patterns and pattern's function as a powerful cognitive tool to engage the imagination. Another possible reason for why more students associated the story of *The Wicked Sons* with their concept of division by zero is that in *The Wicked Sons,* the mathematics is intricately tied to the story, whereas in *The Four 4s,* the mathematics is independent of the story. While *The Four 4s* is entertaining and draws students into the problem of the story, solving it does not depend on the events of the story. In *The Wicked Sons,* each horrible death is in a one-to-one correspondence with each new denominator of the fractions in the mathematical pattern. The story and the mathematics are so intricately connected, they are almost indistinguishable from one another! In other words, when the time came for Charles to die, all I would have had to do was write the fraction $\frac{12}{2}$ on the board and students would have known exactly what was about to happen!

The results of this study suggest that stories can be used to explain mathematical concepts at the secondary level. As with *The Legend of Chikara Ni Bai*, this type of story is especially useful when used in conjunction with the cognitive tools of rhyme, rhythm, and pattern and jokes and humour. Students seem to be drawn to mathematical patterns and have little difficulty remembering them when placed in the context of a story. These patterns become especially noticeable when they are associated with humorous elements in the story. Without these, I am not sure how far I would have got before students realised they were being tricked into listening to a fairy tale. While secondary students may not enjoy listening to stories like *Hansel and Gretel* or *Goldilocks and the Three Bears,* a story about kings and princes will capture their attentions when infused with ridiculous characters and silly dialogue.

It is clear from the analysis of student work that the story is intricately connected to students' understanding of the concept of division by zero. It is unclear however, whether their understanding extends beyond the story, as is the case in *The Pythagorean Problem* in chapter eight. The story may have been their only tool to describe the concept of division by zero. In this regard, the story may have "boxed them in" in terms of their understanding and their ability to communicate mathematical ideas. Is this necessarily a problem? Does the story not help students see the mathematics in an alternate way—a perspective that can "add" to their overall understanding? Clearly they found a new way to communicate their understanding of a mathematical idea. Moreover, looking at their responses, I can conjecture that they did not think of division by zero in terms of "not being allowed." The story seemed to dissuade this faulty notion, facilitating a more sound view. In this regard, a carefully shaped story can be used to prevent misconceptions and guide students in the right direction when approaching a mathematical concept. This is not a simple task, as I will show the reader in chapter nine when the cognitive tool of metaphor is solicited.

14 STUDY VIII: MATHEMATICALLY-STRUCTURED STORY

14.1 Treatment

14.1.1 Target and Problematic Issue

The stories students wrote were meant to intertwine with the topic of prime factorization. According to the planning framework, prime factorization is the specific part of the curriculum I wish to address. Prime numbers are essentially the building blocks of our number system. As Dunham (1994) suggests, "Primes play a role analogous to that of the chemical elements, for just as any natural compound can be broken into a combination of the 92 natural elements on the periodic chart… so too can any number be decomposed into its prime factors" (p.3). This property of prime numbers is best summarized in the result known as "The Fundamental Theorem of Arithmetic," which states (Dunham, 1994):

Every positive integer (other than 1) has a unique prime factorization

This theorem contains two very important points worth noting. First, it asserts that every positive integer can be broken down into a product of prime numbers. Continuing the analogy above, a water molecule H_20, can be separated into two hydrogen atoms and one oxygen atom. In the same way, the composite number 60, can be separated into a product of two factors of the prime 2, one factor of the prime 3, and one factor of the prime 5, which can be represented as $2^2 \times 3 \times 5$ (note the similarity in notation) (Dunham, 1994). The beauty of this is that it works for every positive integer (other than 1) –just like it works for every chemical molecule! The second important point worth noting about the theorem is that it guarantees that each decomposition is unique. In other words, no matter how many times we factor 60, it will always break down in the same way—and only in that way—and, no other number will have that same factorization (this is, of course, ignoring the order of the prime factors). The equivalent situation is true of a water molecule and its component elements (Dunham, 1994). What a wonderful discovery it was in science class to find out how different molecules break down and their atoms combine with other molecules to form new compounds. The labs, the experiments—all secretly feeding our sense of wonder and amazement of these chemical elements in action! Even the ominous periodic table, with its array of numbers and colourful symbols, was really just a veiled reminder of the basic structure underlying all matter! This analogical diversion raises some pertinent questions in my mind: Why in high school was I not equally fascinated by prime numbers and their special properties? Why are most high school students not amazed at the way prime numbers combine under multiplication to generate all the of the natural numbers?[15] Why have so few students seen prime numbers depicted in ways other than in numeric form? Why is the discovery of a new large prime number not celebrated in high schools in the same way as in the discovery of some new element or scientific breakthrough? With such a foundational role in mathematics, it is surprising that very little time at the secondary level is devoted to developing an interest in prime numbers as well as the necessary skills to explore them. Consequently, this is the problematic issue I wish to address in this assignment.

[15] Except for the number one.

My love of prime numbers did not come until my third year in university when I took a course in elementary number theory. Sadly, this is where I first learned of the Fundamental Theorem of Arithmetic and the important role the primes play in mathematics. Prime numbers possess majestic abilities working their way into many of the great proofs, theorems, and problems of both the ancient and modern worlds. They are the foot soldiers, solicited to do the tedious yet necessary task of generating the natural numbers. Inconsistent metaphors aside, the primes are a source of great mystery. Many mathematicians have devoted their entire lives to the study of this peculiar set.

> For generations of mathematicians, prime numbers have always had an almost mystical appeal… because, in spite of their apparent simplicity, their properties are extremely elusive. All sorts of basic questions about them remains unanswered, even though they have been scrutinized for generations by the sharpest mathematical minds (Hoffman, 1998, p.35-36).

I hoped that through this assignment, students would begin to see the important role the primes play in the study of mathematics, both for its utilitarian purposes as well as its aesthetic value. If our students could glean even a bit of the wonder and mystery that surrounds the primes, it might encourage further investigation into their properties. At the least, if our students were familiar with the prime factorizations of even the first 100 natural numbers, many mathematical procedures, like multiplying and dividing integers, factoring polynomials, or reducing fractions, would be a relatively painless task.

14.1.2 Cognitive Tools

The supporting cognitive tool of rhyme, rhythm, and pattern again was used to draw students into an exploration of the concept of prime factorization. Continuing from my discussion above, one of the great mysteries of prime numbers is their lack of a simple polynomial pattern. Incidentally, there is currently no way to predict *exactly* where they will appear among the very large natural numbers. That is not to say we cannot predict approximately where they will appear or what properties they potentially hold. Indeed, some of the greatest advances in number theory are a result of these pursuits. Nevertheless, from section 4.5 we know that our students are predisposed to a sense of "rhythm and fit, balance, motion, and symmetry" (Gadanidis and Hoogland , 2002, p.1), and in the absence of these elements, a sense of mystery and wonder flourishes.

14.1.3 Presentation of Problem and Conclusion

I organized the problem using the cognitive tools identified in order to facilitate imaginative engagement with the mathematical content. Students were first taught the Fundamental Theorem of Arithmetic in the traditional sense. One week later, students were asked in a test to state the theorem. The following period, students were shown the first few lines of Gowney's poem that is based on prime factorization (as cited in Lipsey and Pasternuk, 2002. p.2):

cold
winds howl
geese go south
nights long   tea steeps

They were then assigned to create their own poem/story based on the prime factorizations of the numbers from 1 to 20. These poems/stories could be about anything, real or imagined. While the primes themselves do not appear in any simple polynomial pattern, they can act as a pattern for each line of the poem students create. This is not unlike a haiku or sonnet where each line has a prescribed number of syllables and/or other numerical constraints. These were due the following class, at which time, they had to present their stories. Two weeks later, students were asked in a quiz, to state the theorem again. The test question and the quiz were included as part of the study in order to determine whether the story/poem had any impact on students' understanding of prime factorization and The Fundamental Theorem of Arithmetic.

14.1.4 Participants

28 students in MATH8H participated in this assignment, but only 20 students chose to allow their work to serve as data.

14.2 Analysis

This was the second assignment that focused on the cognitive tool of rhyme, rhythm, and pattern to engage the imaginations of students. Initial test results showed that only 10 out of the 20 participants could successfully express the Fundamental Theorem of Arithmetic even after being taught it the previous week. This does not necessarily mean the theorem was not understood. Nevertheless, common mistakes mainly involved the mention of "prime factorization," while failing to point out the "unique" aspect of each factorization, or visa versa. In particular, four of the students that gave an incomplete answer, stated something like, "Every number has a unique '*number*'" or "Every number has a prime factorization."

14.2.1 Eliciting an Emotional Response

In analyzing the stories students wrote, I was interested in whether the topics that students chose had any emotional dimension to them—and thus an indication of imaginative engagement. The results were overwhelming. When given the choice to write a story about something real or imagined, students tended to write about themselves—and in particular, some emotionally-charged event in their lives. For instance, Darren writes about the excitement of Christmas morning:

“Darren
wake up!
It’s Christmas today!”
“Go away * I’m tired”
But Darren, Santa brough presents”
“Okay Brandon * I’ll get up”
My brother could barely contain his excitement
The seven * year old * was dazzled
By all of * the Christmas magic
Down stairs * the Christmas tree lights sparkled
A mountain of decorated parcels lay beneath and the stockings overflowed
“Wow look * at all * the presents Darren!”
“I see them Brandon, you can open yours first if you want to”
Everyone opened * their presents then we all ate breakfast
After finishing breakfast * we cleaned up the dishes
Soon afterwards * we heard * the door * bell ring
It was my grandparents with even more presents for everybody, so once again
we exchanged Christmas presents
Around five * my mom took the turkey out of the oven
She put it on the table and everyone sat down in the living room to eat the delicious meal
It was * the best * Christmas I have ever had.

Lauren writes about his first day in high school:

1 September
2 Grade eight
3 first class math
4 cool pencils, homework already
5 science class with mister Parkensen
6 very boring, want to sleep
7 next is sewing with Miss Jamison
8 pencil crayons for coloring, no sewing
9 next is gym, lots of fun
10 Zed route, not actually that much fun
11 Go home to do homework and rest for the next day
12 Next Day, come to school, FACTS first
13 For Academically Challenged and Troubled Students is what it stands for

14 learn about how not to take drugs, very fun
15 English class, very hyper teacher
16 Next socials, Mister Right, no fun, fall asleep
17 No fun hearing a teacher talk about nothing for 70 minutes worst class yet don't like socials
18 Next is French, with mister Toupin, cool tattoo
19 conjugating all types of verbs in all the tenses takes for ever with an annoying song about IR verbs
20 all classes, finally done, not that horrible

Angela expresses warm feelings for her younger brother:

1	Smiles
2	Every day
3	A newborn child
2^2	So small, so fragile
5	His tiny fingers grasp mine
2×3	I smile, he starts laughing
7	His blonde hair and sparkling blue eyes
2^3	Capture me, I sing, he listens
3^2	Now he crawls, now he walks
2×5	Now babbling, we try teaching him signs
11	His babbling, mixed with a few signs would make communication easier
$2^2 \times 3$	Much bigger, much stronger, he moves swiftly
13	Walking now comes easy for him as he begins to dance and run
2×7	So sneaky, he soon found the pots and pans
3×5	His small hands, discovered new beats and rhythms
2^4	Now smarter, much larger, more brunette, curly locks
17	Running so clumsily he would scare me but even when he fell he got right back up
2×3^2	So brave, opening the door, scaring the monsters
19	For some odd reason he loved vegetables more than fruits his favourite one was peas or was it corn
$2^2 \times 5$	Country music, his favourite, he dances and twirls about

14.2.2 Reversing the Aesthetic Flow

It is clear from the examples above that the stories students wrote are connected to their emotional lives in meaningful ways. This could however, be a result of their familiarity with such literary forms as haikus, sonnets, or limericks that have a dedicated emotional dimension to it. Nevertheless, these examples illustrate how stories can intertwine with the mathematical content in a moving and expressive way. Almost all of the 20 students used proper factorizations in their stories, and were accurate in the way they separated each phrase to match those factorizations—whether by commas, spaces, or even asterisks. From these examples, we can clearly see that the "aesthetic flow" originates in the mathematics (Gadanidis & Hoogland, 2004, p.4).[16] This is particularly apparent in Darren's story, in which little attention was given to making each "phrase" a separate thought, yet he made sure the aestericks were carefully placed to indicate the correct factorizations. Hanna wrote, *Your Birth*, a whimsical, Dr. Seuss-like story about her family:

Mom
Your Mom
Your mom's face
Your mom, your face
Your mom's mom's mom's face
Your mom, your mom's face
Your face's face's face's face's face's mom
Your mom, your face, your dad
Your mom's face, your face's mom
Your mom, your face's mom's face's mom
Your sister's brother's mom's dad's grandma's grandpa's son's daughter's aunt's uncle
Your mom, your face, your mom's face
Your sister's brother's mom's dad's grandma's grandpa's son's daughter's aunt's niece's nephew
Your mom, your face's mom's dad's mom's face's your
Your face's mom, your cat's former owner's face
Your mom, your face, your dad, your cat
Your sister's brother's mom's dad's grandma's grandpa's son's daughter's aunt's niece's nephew's best friend's second cousin
Your mom, your face's dad, your dad's face
Your sister's brother's mom's dad's grandma's grandpa's son's daughter's aunt's niece's nephew's best friend's second cousin twice removed
Your mom, your face, your cat's dad's face's mom

[16] Each story was retyped in the same format as it appeared in the student's work.

Again, an attention to mathematical patterns seems to outweigh any attempts at creating meaningful prose. This becomes even more apparent when reading Hanna's second version of the story in which she tries to illustrate the process she went through in developing *Your Birth*. Clearly, an attention to pattern is central to Hanna's creative processes:

One
Two two
Three three three
Two two, two two
Five five five five five
Two two, three three three
Seven seven seven seven Seven Seven seven
Two two, two two, two two
Three three three, three three three
Two two, five five five five five
Eleven eleven eleven eleven eleven eleven eleven eleven eleven eleven eleven
Two two, two two, three three three
Thirteen Thirteen Thirteen Thirteen Thirteen Thirteen Thirteen Thirteen Thirteen
Thirteen Thirteen Thirteen Thirteen
Two two, seven seven seven seven seven seven seven
Three three three five five five five five
Two two, two two, two two, two two
Seventeen seventeen seventeen seventeen seventeen seventeen seventeen seventeen
seventeen seventeen seventeen seventeen seventeen seventeen seventeen seventeen
seventeen
Two two, three three three, three three, three
Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen
Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen Nineteen
Nineteen
Two two, two two, five five five five five

Writing elegant prose does not necessarily mean the mathematics has been compromised. Consider again Angela's story about her younger brother. Clearly, Angela has an extraordinary ability to express her thoughts and feelings about her sibling in a stylish and artistic manner. At the same time however, it is of particular interest in the way she explicitly includes the factorization with each line of the story. While several other students also indicated their factorizations, Angela was the only student to represent her factorizations using exponents, and in this regard is using a new fresh way to communicate her ideas (Gadanidis & Hoogland, 2002).

Quiz results improved slightly. Of the 20 students that wrote the quiz, 16 correctly stated the theorem. Incorrect answers were once again along the lines of those mentioned previously. This suggests the stories/poems students wrote did not impact their understanding of The Fundamental Theorem of Arithmetic in any meaningful or substantial way. This was certainly unexpected considering the emotional attributes of their stories. Perhaps accurately communicating the theorem is more difficult than actually using the theorem—a similar result found in *Kelloggs Needs Your Help* in chapter twelve. But we also found in that study that while students were engaged in the task of designing and constructing cereal boxes, they failed to make the desired connections between 2 and 3-dimensional space. In *Skull Island*, “The Lighted Map,” while capturing everyone’s attention, incorporated very little mathematics. In each of these studies, students may have forfeited mathematical correctness when the focus was placed on emotion. The implications of this aspect of using stories to teach mathematical concepts at the secondary is quite significant and deserves further study.

15 STUDY IX: THE STORY OF PRINCESS $\sqrt{3}$

15.1 Treatment

15.1.1 Target and Problematic Issue

The Story of Princess $\sqrt{3}$ is a story that intertwines with the subject of radicals and it is a story to introduce an activity. According to the planning framework, radicals is the target, or specific part of the curriculum, I wish to address. Students in grade ten at first have a limited notion of the concept of radicals—it is almost exclusively connected to the idea of square roots—square roots that require evaluation. For example, $\sqrt{16}$ is simply 4, and $\sqrt{24}$ is 4.89. In this regard, students see a square root as a unary operation. Moving beyond this requires a bit of imagination—it requires seeing the radical not as a unary operation, but as a mathematical object with specific properties that can interact with other mathematical objects in different ways. Consequently, this is the problematic issue that usually surfaces when students begin to simplify and perform operations on radicals. Specifically, students have difficulty performing operations on radicals and converting between simple mixed radical form and entire radical form. Additionally, the introduction of higher roots also conflicts with students' initial notion of radicals. (While it is a matter of convention to leave out the index when writing square roots, it is not conducive to extending the concept to cube, fourth, or higher roots in grade ten. Students naturally associate the $\sqrt{\ }$ symbol for square roots even when the index has changed).

15.1.2 Cognitive Tools

Once again, the primary cognitive tool of story was used to engage the imaginations of students with the concept of radicals. This assignment also employs the cognitive tool of metaphor to approach the mathematical content in a new way. Recall from section 4.2.1 that the topic domain is the less well-known context and the vehicle domain is the well-known context. The topic domain for this metaphor is radicals, and in particular, simplifying, multiplying, adding, and subtracting radicals. In order to approach the subject in an imaginative way, I used the vehicle domain of a fairytale, much like the familiar story from the movie, *Shrek*. The primary inspiration for this story and assignment came from *Flatland: A Romance in Many Dimensions*, by Edwin Abbott (1998). *Flatland* showed me that mathematical objects taking on human attributes can bring a new dimension—no pun intended—to a study of almost any subject in mathematics. At the junior level, I would frequently during lessons impose certain human characteristics or activities on mathematical objects or procedures I wanted students to take particular note of. Students seem to respond favourably to a mathematical object, such as a variable, geometric figure, or even a number, referenced using familiar terms like "kid," "she," or "these guys." But it was not until after I taught two lessons—one on "decomposition" and another on "converting mixed radicals to entire radicals"—did I see the extensive power of this metaphorical device in teaching mathematical concepts at the senior level. Both of these metaphors involving decomposition and radicals have to do with dating, so students were particularly interested. In fact, I always conclude each of these lessons by stating, "You see, you can get all the sex ed you need right here in math class." A round of laughter usually breaks out at this point. Of course, my colleagues might argue I am giving my students poor advice as they are trying to negotiate the harrowing path of teen sexuality. I might add however, that one can find a metaphor for almost any aspect of human sexuality within the confines of the mathematics curriculum—to which can be employed with any noble purposes in mind. Nevertheless, taking a boring topic like “decomposition” and turning it into a story in the likes of a daytime soap or an episode from *Melrose Place,* is an exciting endeavour with unfathomable reward.

Normally, to factor a polynomial of the form $ax^2 + bx + c$, where $a \neq 1$, we use decomposition. For instance, to factor $2x^2 + 3x - 9$, we must break up the middle term, so that we have four terms in the polynomial. Why do we do this? Because three is a crowd of course! After some work, students will arrive at $2x^2 - 6x + 3x - 9$. Now here is the dating part. I ask, “What do you do when you see four of your friends all sitting in the cafeteria?” Students usually say something like “nothing” or “talk to them.” I then say, “But look how lonely all four of them are… four lonely people.” At this point, students usually think I am crazy, but they are anxious to find out where I am going with this. After some gestures insinuating that students should know what the next step is, I tell them, “You put them into couples…you group them!” The idea here is that when you have four lonely people, you pair them up so they are not lonely anymore. I usually get a response of “No you don’t!” Ignoring their objections, I continue talking about the virtues of coupling your friends up. Now back to the mathematics. How does this relate to our four-term polynomial? Students quickly see that each term is like one of their friends and to “couple” or “group” them is the same as pairing the four terms up using brackets.

$$\left(2x^2 - 6x\right) + \left(3x - 9\right)$$

Here, the vehicle domain is dating, and the topic domain is factoring using decomposition. This explanation is even more memorable when I show them how they can be paired in different ways—it is most memorable when I use the names of students from my class as the four friends and pay no attention to gender when pairing them! In this example, the four friends represent the four terms of the polynomial. The key, however, is to not simply represent each term with a human, but instead, to give each term human characteristics—to see the mathematical object as a character in a story. Human activity can act as a reference point for how the terms might "behave."

Consider how this might work when converting mixed radicals to entire radicals. After a traditional explanation of the process, I present the following problem to students: Convert $2\sqrt{5}$ to an entire radical. I elaborate on the procedure with a story:

> 2 would love to date $\sqrt{5}$, but $\sqrt{5}$ is very different from 2. She's a "radical"—you can tell by her big black *radical sign*! If 2 is going to have any chance at all, he's going to have to at least dress like $\sqrt{5}$. How can 2 be dressed like $\sqrt{5}$? He needs a radical sign. $2=\sqrt{?}$. [Students easily determine $2=\sqrt{4}$]. Now that 2 is dressed up as $\sqrt{4}$, he has a chance with $\sqrt{5}$. $\sqrt{4}$ meets $\sqrt{5}$ and one thing leads to another and… $\sqrt{4}\cdot\sqrt{5}$ … and bam! … they become $\sqrt{20}$. [The students recognize the sexual connotations of this product and seemed to especially enjoy this part].

I later extend the metaphor to include other elements of the topic domain such as cube roots and fourth roots and connect them to attributes like "preps" and "jocks" of the vehicle domain.

In *The Story of Princess* $\sqrt{3}$, a domain-level mapping occurs between the vehicle of the fairytale and the topic of radicals. That is, several elements in the vehicle domain can easily be mapped to elements in the topic domain without much manoeuvring. The central metaphor is the mapping of "compatible creatures" in the vehicle domain to "like radicals" in the topic domain. What makes two characters in our story compatible? First, as in all traditional fairytales, males and females fall in love. So how does one tell the difference between a male and a female? In the fairytale, this goes without saying, but in the world of radicals, the presence of a numerical coefficient is the determining factor—I call this "the penis metaphor." Of course, I do not refer to it in this way with my students, but my choice of potential suitors for the princess—such as the very large $20\sqrt{2}$—is sure to get a rise out of a few of the more learned students!

Second, the radicand in the topic domain has something to do with whether two characters in the vehicle domain are compatible. A very big radicand translates into a giant or ogre. The closest anyone comes to matching the ogre's size is the curvaceous, but brutish maiden, $\sqrt{33}$. Luckily for us, and the fairytale we have stumbled upon, he is not even remotely interested in her. Unfortunately for $2\sqrt{75}$, a large radicand not only means exile from the village, but it also thwarts any chance of his winning the princess' hand in marriage. Presumably, one is not compatible with someone whose radicand is quite different in magnitude. In fact, one is not compatible with someone whose radicand is different at all! This is a very important point in terms of the topic domain—in order to add two radicals together, their radicands must be the same (that is, if we wish to combine them into one radical). So either the two radicals are like radicals, or they have different radicands and we must turn them into like radicals. In *The Story of Princess* $\sqrt{3}$, the process of an ogre being transformed into a handsome prince in the vehicle domain is mapped to the process of a mixed radical being converted to a mixed radical in simplest form, in the topic domain. This latter process can be represented by:

$$2\sqrt{75} = 2\sqrt{25 \times 3} = 2\sqrt{25}\sqrt{3} = 2 \cdot 5\sqrt{3} = 10\sqrt{3}$$

This transformation process seems to be the one event the story seems to hinge on. It is much like the transformation that takes place in *Shrek 2* (Katzenberg, 2004) when Shrek drinks the magic potion and turns into a handsome prince.

15.1.3 Presentation of Problem and Conclusion

Using the cognitive tools already identified, I organized the presentation of the problem according our planning framework. However, this time I took a slightly different approach. I first told students *The Story of Princess* $\sqrt{3}$, and then instructed them to finish the story, modify the story, or create a new story, using the same characters. So in this case, I wanted students to build on the story I had already developed. Their stories had to be at least one paragraph long, and had to be mathematically sound. In other words, their stories could neither alter the mathematical properties of mathematical objects nor alter the rules and operations that are used in conjunction with them.

15.1.4 Participants

60 students in PMATH10 participated in this assignment, but only 43 students chose to allow their work to serve as data. The participants had recently completed assignments on simplifying radicals, adding and subtracting radicals, and multiplying and dividing radicals.

15.1.5 The Story

> Once upon a time in the land of Radicals, lived the Princess $\sqrt{3}$. She was the fairest maiden in all the land and all who saw her were amazed at her beauty. Even the lovely $\sqrt{6}$ or $\sqrt{33}$, who were all curves, paled in comparison to her, or $\sqrt{17}$ or $\sqrt{11}$, whose long legs drove knights and

princes wild, were jealous of the attention the princess received. Each day, suitors from all the land would come to try and win Princess $\sqrt{3}$'s hand in marriage. But she was very particular and showed little interest in their boyish antics. $2\sqrt{8}$ tried to dazzle her with his good manners and offered her flowers. But she immediately began to sneeze and bloat up when he presented them to her. $2\sqrt{5}$ rode in on a magnificent white stallion, but the princess was not impressed when he fell off the beast and landed in a pile of manure. $8\sqrt{2}$ recited poetry and sang love songs, but that only put her to sleep. $20\sqrt{2}$ tried to impress her with his large coefficient, but being so young and innocent, she paid no attention to those sort of thing. A glimmer of hope appeared with the arrival of $2\sqrt{9}$, but Princess $\sqrt{3}$ soon discovered he was not a radical at all and so she bid him to leave her land. It seemed as though no one was compatible with the Princess [no one could be *added* to the Princess]. This was a terrible predicament for the kingdom, since the time of the Princess' coronation grew near and all knew she would have to forfeit her crown if she had not found a husband. But there was someone, a monstrous and hideous beast, who also desired the Princess… $2\sqrt{75}$! His enormous radicand had made all the people fear him and so he banished himself to the swamp just beyond the palace walls. He had watched from afar, the Princess $\sqrt{3}$ play in the garden, sing to the birds, and pick flowers since she was a child, and he had grown to love her. Although they had never met, the ogre knew in his heart that she was the perfect radical. One day the Princess wandered beyond the safety of her garden and was met by some bastard thieves. The wretches knew the Princess would fetch a handsome ransom and attempted to accost her. It was then that fate would enter our story, for as always, $2\sqrt{75}$ had been watching the Princess and at the very moment of the evil deed he intervened and rescued her. The Princess looked up at her deliverer and…[Would they fall in love? Was he compatible with her? Why were the others not? What magic or event could change his appearance so that he could marry the Princess? Finish the story. You may change the story or characters in any way you like, but your math must be accurate!].

15.2 Analysis

15.2.1 Metaphor and the Nature of Radicals

Mapping the attributes of fairytale characters and their activities to the attributes of radicals and various mathematical operations is an imaginative process that allows students to explore the nature of radicals in a way that simple number exercises cannot. It forces students to take the perspective of the mathematical object and learn its boundaries and limits in the way that it interacts with other characters and its environment. Using metaphor like this is quite different from the typical way this cognitive tool is used in the learning of mathematical concepts. In most cases, metaphor seems restricted to learning about processes and algorithms. For instance, the idea that an equation is like a teeter-totter that needs to stay balanced, helps students understand how to solve an equation, but does little to shed light on the nature of the elements in a mathematical equation. In *The Story of Princess* $\sqrt{3}$, students, not only learn how to simplify, add, and subtract radicals, but they also explore the very nature of radicals. For example, in *Princess* $\sqrt{3}$, Haley displays her understanding of the relationship between the size of a radicand and a character's place in society:

> The ogre, $2\sqrt{75}$ was so in love with Princess $\sqrt{3}$ that he decided he would do anything to get her to notice him. He thought of everything possible. He wanted to go see $\sqrt[3]{225}$, the great and mighty warlock, who could help him be worthy of $\sqrt{3}$'s love. But $2\sqrt{75}$ couldn't go because that would mean travelling into the village where he was not welcome.
>
> $2\sqrt{75}$ was walking on the bridge near his cave early one morning when he heard whimpering from the river below. He crept down to see who it was making the sound. Once he reached the river's edge, he was surprised to see $\sqrt{\frac{1}{3}}$, the castle's smallest and clumsiest leprechaun. He had slipped and fell into a hole. By this time, $\sqrt{\frac{1}{3}}$ had spotted $2\sqrt{75}$ and began to get very excited. $2\sqrt{75}$ was excited as well. Maybe the leprechaun could help him? Indeed, $\sqrt{\frac{1}{3}}$ could. He offered to help the ogre if he would help him out of the hole. $2\sqrt{75}$ lifted $\sqrt{\frac{1}{3}}$ out of the hole and told him all about his dilemna with Princess $\sqrt{3}$. $\sqrt{\frac{1}{3}}$ immediately knew what to do. He wrinkled his nose, jumped in a circle three times and then yelled "Rumplestiltskin!" $2\sqrt{75}$ was then transformed into $2\sqrt{25\times 3}$

> and then again into $10\sqrt{3}$! $10\sqrt{3}$ then ran into the village to find $\sqrt{3}$. Upon seeing him, she knew he was the one for her. He was exactly the suitor she was looking for. She fell in love, they got married, and lived happily ever after.

Not only does Haley equate a large radicand with an ogre, but she also extends this metaphor by associating a small radicand with the leprechaun $\sqrt{\frac{1}{3}}$. In this way, Haley uses her imagination to introduce new elements in the vehicle domain that map to contrived elements in the topic domain—without compromising the mathematical content. By contrived, I mean that Holly took an "imaginative leap"—and succeeded—when she introduced some element of the topic domain that was neither previously discussed in the telling of the story or in the introduction to the assignment. Even more remarkable, however, is the presence of the great and mighty warlock, $\sqrt[3]{225}$. Not only does the introduction of this character demonstrate Haley's ability to distinguish between square roots and cube roots (it is surprising how many students ignore or forget to include the indices when performing operations with multiple types of roots), but it also suggests she sees cube roots—and perhaps even higher roots—as more difficult to simplify, hence the distinction of "great and mighty" given to this eccentric inhabitant of Radicaland.

A similar attention to the nature of radicals and their radicands is seen in Damian's, $\sqrt{3}$ *and the Ogre* $2\sqrt{75}$:

> It was the middle ages. Queen $\sqrt{3}$ needed a husband. Many knights tried to impress her, $\sqrt{6}$, $\sqrt{8}$, $\sqrt{5}$, and $\sqrt{2}$ all failed. Four noble knights tried to impress her. $2\sqrt{5}$ the horse rider failed, $4\sqrt{7}$ the sword-handler failed, $8\sqrt{2}$ the poet failed, and $3\sqrt{6}$ didn't do much better. Little did $\sqrt{3}$ know, when she was a child the fierce, scary ogre, $2\sqrt{75}$ had been watching her from the forest. The forest was an area few townspeople went for fear of being eaten by a monster or attacked by a gang of thieves. The old witch, $\sqrt{100}$ had been making a magic potion that brought out the beauty or image of what the person

> truly is on the inside. She drank some herself and became the young lady 10. One day the queen was picking flowers and remembered a flower from a story that was found only in the forest. The four noble knights $2\sqrt{5}$, $4\sqrt{7}$, $8\sqrt{2}$, and $3\sqrt{6}$ agreed to come. They traveled deep into the forest where very little daylight rested during the day. Suddenly, nine thieves surrounded them from the surroundings. The knights fought bravely, but failed to defeat them all. $4\sqrt{7}$ lasted the longest, but alas fell as well. $\sqrt{3}$ screamed and ran, then tripped. When all seemed grim, a deafening roar was heard. The giant ogre $2\sqrt{75}$ flung the remaining thieves aside with the brute strength of ten men he possessed. Helping her up, he treated her respectfully and showed no signs of thinking of eatingher eventually gaining her trust. They returned to the town and she explained to the frightened townspeople what happened. The once an old witch 10 came from the crowd and offered the ogre $2\sqrt{75}$ some of the potion she made. He accepted it and turned into $2\sqrt{3}\cdot\sqrt{25}$, then $2\sqrt{3}\cdot 5$, and then $10\sqrt{3}$, a handsome man with the strength high above an average man's. $10\sqrt{3}$ and $\sqrt{3}$ had much in common. Eventually they married and lived happy.

I was particularly interested in the witch, $\sqrt{100}$. Once again, the size of her radicand makes her a peripheral member of society. More importantly, however, her true nature is that of a whole number, whereby Damian skilfully works this attribute into his storyline by way of a magic potion. Drawing connections and making distinctions between different types of number systems in this manner suggests Damian is engaged with the mathematical content, and connecting such abstract concepts to plotlines in a fairytale is, undoubtedly, a work of the imagination (Rose, 1996). This concern with the "true nature" of radicals is particularly significant. Consider the following paragraph from Damian's story:

> The old witch, $\sqrt{100}$ had been making a magic potion that brought out the beauty or image of what the person truly is on the inside. She drank some herself and became the young lady 10.

Metaphorically, what Damian is alluding to is the mathematical principle that simplifying a radical does not change its "value," but only its "representation." This is an important idea at almost any level of mathematical inquiry (i.e. multiplying both the numerator and denominator of a fraction by the same number or mathematical object only changes the fraction's form and not its value). Similar reasoning can be seen in Jesse's, *The Princess and the Ogre*:

> One day the ogre decided to go down to the castle to meet the princess. He knew the princess was very picky, but he thought he had the qualities she was looking for. He knew the princess would never speak to him if she saw he was an ogre, so he needed a disguise. "The princess will know I am an ogre if she sees my huge radicand!" said the ogre. The ogre learned that with a few changes to his attire, he looked like a handsome $10\sqrt{3}$! Suddenly he looks like someone Princess $\sqrt{3}$ would like! … She [the princess] saw the handsome $10\sqrt{3}$ and knew she was the right one for her.

Like Damian, Jesse maintains that the Ogre is still the same person, but only appears different. Instead of a magic potion however, Jesse simply has the character change his clothes. In terms of the mathematics, $2\sqrt{75}$ and $10\sqrt{3}$ have the same value, but only appear very different. However, it is only in this latter form that one can seamlessly combine $2\sqrt{75}$ with $\sqrt{3}$. In the same way, it is only after the physical transformation in the story takes place, that the two characters become compatible. Nevertheless, Jesse is taking serious leaps in her exploration of the nature of radicals.

Two other aspects of Damian's story are worth noting. First, it is clear that the "penis metaphor" mentioned previously, had no significant place in Damian's conceptualization of the story. We see this in the way he describes knights both with and without coefficients. In fact, this result was consistent with all of the stories students wrote. Not one student made any reference to the "penis metaphor" when telling their stories. While this is both surprising and disappointing—considering that my wayward brilliance went unnoticed—it renewed my faith in the relative purity and innocence of adolescence.[17]

A second aspect in Damian's story worth noting, is the way he used the literary device of foreshadowing when he wrote, "The giant ogre $2\sqrt{75}$ flung the remaining thieves aside with the brute strength of *ten* men he possessed" [italics mine]. Was this coincidence, or was it a hint to the reader as to the true nature of the ogre, ($\mathit{10}\sqrt{3}$)? [Italics mine]. In any case, the mathematical content is so intertwined with the story, that it becomes difficult for the reader to distinguish one domain from the other. This situation is particularly noticeable in Paul's, *The Matheival Tale*, where his blending of both natural and mathematical language makes the lines between the two domains somewhat unclear (note the inserted italics):

> The Princess $\sqrt{3}$ had to kiss the ogre $2\sqrt{75}$ to *turn him into* a $2\sqrt{25\times 3}$. The $2\sqrt{25\times 3}$ ogre could then *simplify himself* to $5\cdot 2\sqrt{3}$, which *makes* $10\sqrt{3}$. The princess and the ogre could then be *added together* ($\sqrt{3}+10\sqrt{3}=11\sqrt{3}$). Then they both lived happily ever after…and then they had babies ($1\sqrt{3}$ and $2\sqrt{3}$). The End.

While Paul uses terms from each domain interchangeably, he does not compromise the integrity of the mathematics involved. I am certainly not encouraging students to become careless in the way they communicate mathematical ideas. On the contrary, by encouraging students to write stories that connect the two domains, I hope they will first explore the properties of both domains; develop a good understanding of the target domain; and finally, communicate the mathematical ideas in the target domain accurately. Put another way, I am not suggesting that students simply abandon the use of

[17] This may simply be due to the fact that students did not have a printed copy of the story and so may not have picked up on this slight nuance deviously weaved into the story.

precise definitions and accurate mathematical terms, but they should develop an understanding of the mathematical content—demonstrated through the use of informal language—before trying to communicate their ideas in a formal way (NCTM Standards, 2000).

15.2.2 The Transformation Process

Perhaps the most common element in every fairytale is a touch of magic—and it usually arrives at that crucial moment in the story when the villain has the upper hand and everything seems lost. It is then that some type of transformation takes place that enables the hero to rise above his/her own weaknesses or insecurities and confront the villain. In *Cinderella*, it is that moment in the garden when the Fairy God-Mother shows up and gives Cinderella a full makeover; In *Sleeping Beauty*, it is the kiss of Prince Phillip that breaks the curse and awakens Princess Aurora from her 100-year sleep. In *The Story of Princess* $\sqrt{3}$, something enchanting also needs to happen. Fortunately for $2\sqrt{75}$, the story is deviously postured for a magical event to take place—and fortunately for us, this magical event can easily be mapped to the prescribed mathematical process. In *The Story of Princess* $\sqrt{3}$, the transformation of the ogre $2\sqrt{75}$ into the handsome Prince $10\sqrt{3}$ corresponds to the conversion of a mixed radical to a mixed radical in simplest form. This transformation is what the outcome of the whole story depends on—and it is this conversion process with radicals that we want students to fully understand and perform. In order to finish their stories, students had to connect these two processes through a carefully constructed metaphor—and this involved a leap of the imagination. Consider how this transformation process seems to dominate every aspect of Janine's, *Princess* $\sqrt{3}$. The settings, the developing plotlines, and the introduction of new characters, all seem to serve a single purpose:

> The ogre tried to think of a way to win over Princess $\sqrt{3}$'s heart, but no matter what he did, he couldn't come up with anything. The next morning, he set out to find anyone he could that might help him. He came across a family of bears that gave him very little help, but they did give him a bowl of porridge to hold him over a bit longer. Little did the ogre know the porridge had an ingredient in it that was okay for a bear to eat, but for an ogre it was considered "slightly" poisonous. After a few minutes, he started to feel very dizzy. He stopped and sat on a rock next to a small lake. He felt weird. He looked in the water to see his reflection, and realized that he was no longer $2\sqrt{75}$, but $2\sqrt{25 \cdot 3}$. He was changing. He figured that being $2\sqrt{25 \cdot 3}$ was not going to win over the princess' heart, and he had to either disguise himself, or change himself. He decided he would try working out. So he started by lifting branches that were about 50 pounds each. After about a few weeks, he was lifting entire trees. He was very anxious to see if this was working so he looked in the lake again. He had changed again. His body was breaking down. He was now $10\sqrt{3}$. He couldn't believe his eyes. He wondered whether this would do the trick or not. He decided he would give it a try. The next morning, he made his way over to the castle. After everything he had gone through, he was too scared to face Princess $\sqrt{3}$. He sat in the bushes outside her room and thought about what to do, and without realizing, he fell asleep. A few hours later he woke up to a beautiful sight. It was the Princess $\sqrt{3}$. She had heard his snoring, and then found him fast asleep behind the bushes. "Who are you?" she asked. "I'm the ogre $2\sqrt{75}$." he said. "No you're not" she said, "You're the ogre $10\sqrt{3}$!" The ogre didn't know what to do. He stood up, brushed himself off, and started to walk away. "Where are you going?" asked the princess. "I should go. I didn't mean to disturb you." he said. "Disturb me?" she asked, shocked. "You haven't disturbed me. Please don't leave."

Where Janine spends a great deal of time developing the fairytale side of the metaphor, in *A Fairy Tale Story*, Jessika is clearly more concerned with expounding on the mathematical content. Her explicit outline of the steps in converting $2\sqrt{75}$ to $10\sqrt{3}$ leaves little room for guesswork. Even her use of such literary devices as flashbacks/flash-forwards only adds to the developing plotline that appears to be centered on the mathematical transformation that is taking place:

> Once upon a time there lived a beautiful princess named $\sqrt{3}$. She was elegant, simple and very well known among the kingdom. $\sqrt{3}$ lived with her parents $\sqrt{6}$ and $\sqrt{2}$ in a massive castle looking over the vast lands and kingdom. King $\sqrt{8}$ always insisted that $\sqrt{3}$ should start dating handsome nobles, such as $2\sqrt{5}$, $4\sqrt{7}$ and $3\sqrt{2}$. Princess $\sqrt{3}$ wanted nothing to do with these men. One day, while painting in the garden, $\sqrt{3}$ stumbled across a disgusting wart-covered toad. $\sqrt{3}$ was frightened at first, until she recognized those light emerald green eyes she had seen before. You see, many years ago, Queen $\sqrt{2}$ sent her daughter to a very well known, expensive summer camp. While visiting this camp, children learnt proper manners, etiquette, posture, and stanza. There $\sqrt{3}$ met a handsome, polite boy named $10\sqrt{3}$. This boy happened to have breath-taking emerald eyes. During their stay at camp, $10\sqrt{3}$ and $\sqrt{3}$ became very close and gained a friendship they would never forget. One day $10\sqrt{3}$ suddenly went missing. $\sqrt{3}$ was devastated! There were horrible rumors circulating that $10\sqrt{3}$ had gone home sick or disappeared in he forest. $\sqrt{3}$ never forgot that day, and still awaited her prince to one day come back to her.
>
> The princess was amazed to see such beautiful eyes. It brought back such fine memories. The princess learned that the toad's name was $2\sqrt{75}$. $2\sqrt{75}$ was once a handsome young boy, until he stumbled across an old shack while playing hide and seek in the forest. The evil witch that owned the house immediately changed the boy into a disgusting toad. "Oh $10\sqrt{3}$! Is that really you?" exclaimed the princess. She leapt up and kissed the frog. A tall attractive looking man stood dazed in front of $\sqrt{3}$. ($2\sqrt{75} \rightarrow 2\sqrt{25}\sqrt{3} \rightarrow 2 \cdot 5\sqrt{3} \rightarrow 10\sqrt{3}$). Both of the adults were extremely happy to see each other. Soon after they both had a child, named $11\sqrt{3}$ ($10\sqrt{3} + \sqrt{3} = 11\sqrt{3}$) and lived happily ever after. The End!

Everyone knows that if an ogre can magically transform into a handsome prince, then the reverse is also true—a handsome prince can magically transform back into an ogre—remember Shrek 2? Knowing this made it easy for Nicola, in *Princess Story*, to generalize the more familiar concept of magical transformations in the vehicle domain to the less familiar concept of radical simplification in the topic domain (Dialogue, 2005):

> The princess went home hoping that one day she would find her true love. She was now getting worried she would never find anyone—until she received a letter from someone she has never known before. The letter read:
>
>> Dear Princess $\sqrt{3}$, you don't know who I am but I know we should be together. We must meet," signed $10\sqrt{3}$.
>
> Time was running short, the princes had to find a prince before her 19th birthday which was in 1 month. She figured she must see this $10\sqrt{3}$ as soon as possible. She got her servant to search the town looking for $10\sqrt{3}$ and to tell him to meet her by the lake, Saturday night at 10 o'clock. Saturday finally came, and the princess got ready and went and sat down on a bench by the lake. It was 9:57 and she was getting nervous. She heard someone coming behind her and sure enough it was a handsome young man. "Hi, I'm $10\sqrt{3}$" he said. "It's a pleasure to meet you" said the princess. They went on talking the whole night until midnight when $10\sqrt{3}$ started changing in look. He started to look all green and lumpy then all of a sudden he turned into $2\sqrt{75}$, and ugly ogre! The princess got scared and turned and ran away. She never found a prince in time so she never became queen.

Nicola appears to be thinking with some degree of mathematical abstraction. According to Harel and Tall (1991), "An abstraction process occurs when the subject focuses attention on specific properties of a given object and then considers these properties in isolation from the original" (p.39).

15.2.3 Misconceptions and Algebraic Errors Revealed

From section 3.7, we know that writing activities can reveal what students know or do not know about a subject and whether they can communicate that knowledge effectively (Halpern & Halpern, 2006). Many of the stories students wrote revealed misconceptions they had about the nature of radicals and the difficulties they encountered when performing operations with radicals. For instance, in the first few lines of *The*

Fairy, Sarah, through her use of language, shows that she does not fully understand what it means to simplify a radical:

> The princess takes one look at $2\sqrt{75}$ and shews him away. Then he goes home all sad and just wishes that there was a way he could *become* $\sqrt{3}$. He thinks of an idea. What if he could go and find someone and make a wish that he was *compatible* with $\sqrt{3}$? [italics mine].

Clearly, Sarah confuses being "compatible" with $\sqrt{3}$ with "becoming" $\sqrt{3}$. Sarah also shows she is unsure of the steps involved when simplifying a mixed radical. Notice below, in the last part of her story, when she factors 75, she places the 3 outside the radical—yet she still arrives at the correct answer (when I followed this up with her she admitted to soliciting the final answer from another student and then attempted to work backwards):

> He thinks long and hard and goes to find the nearest fairy that will grant him his wish. When he finally gets there, the fairy grants him his wish, but with a price. If he does not get the princess to fall in love with him in the next week, he will go back to $2\sqrt{75}$! He says yes and $2\sqrt{75}\ldots 2\sqrt{25}\times 3\ldots$ $2\cdot 5\sqrt{3}\ldots$ is now $10\sqrt{3}$. He goes and finds the princess and she looks at him and they fall in love, and live happily ever after.

In this excerpt from *Tender Transformations*, Janet reveals that she is still unsure about how to add $2\sqrt{75}$ and $\sqrt{3}$ (see italics):

> $\sqrt{3}$ heard a loud bang come from the palace courtyards. She walked towards her window and looked down to see an outrageously giant ogre fighting off her guards. $\sqrt{3}$ was no wimp, she was in PETA, and we all know how they roll. She ran down the stairs until she finally came to the courtyard and screamed, "No! Stop the madness… can't we all just get along?" All eyes were now on the princess, and an awkward silence fell upon the entire crowd. The ogre took a couple of steps closer to $\sqrt{3}$. "$\sqrt{3}\ldots$ we were made for each other, *we can be combined together to equal* $10\sqrt{3}$. And the other suitors who came for you were not meant for you the way I was. Don't be afraid, I love you." The princess ran into the

> ogre's giant arms and immediately fell in love. See the ogre would honestly turn himself into a human once they kissed but fear of being sued by the makers of Shrek caused the author of this story to change the plot line, so the ogre stayed an ogre. *Many years later a new number was born to rule the land, and his name was* $10\sqrt{3}$, and peace once again reigned.

In *The Fairy Tale of* $\sqrt{3}$ *and* $2\sqrt{75}$, Bruce fails to successfully simplify $2\sqrt{75}$:

> … A massive green ogre appeared in front of the princess. "What…what…what do you want?" quivered the princess. " Come to me my lady, just touch my hand and all will be well," calmly expressed the ogre. The princess nervously reached out and touched his palm. A bright light emerged and the ogre was changed from $2\sqrt{75}$ to the magnificent $11\sqrt{3}$. He was now a prince, the most compatible prince that the princess has ever seen. "How?" Asked the princess "Well you see no one has touched my hand (except a sword) I have been trapped and you have freed me. We can now live to the best of our lives. The prince and the princess lived happily ever after. (Until Mt. Shreddies erupted and killed everyone on the island.)

Karrie, in *Princess* $\sqrt{3}$, recognizes that metaphorical interpretation is necessary, but has problems selecting what aspects of the two domains connect to each other (Cameron, 2002). Her eagerness to make the mappings fit her interpretation of the metaphor may have contributed to her computational error in reducing $\sqrt{25}$ to $\sqrt{5}$:

> So one day, $2\sqrt{5}$ made fun of $2\sqrt{75}$ for his bad looks, so he went to the market and got a new wardrobe ($2\sqrt{3\cdot 25}$). He got 2 shirts and ties ($2\sqrt{3}\cdot\sqrt{25}$) and ripped one of the shirts in 2 to make a headband ($2\sqrt{3}\cdot\sqrt{5}$). …The ogre decreased in style ($10\sqrt{3}$). The princess agreed. ($10\sqrt{3}+1\sqrt{3}$) …They both lived happily ever after ($11\sqrt{3}$).

Following up on this computational error revealed that Karrie believed $\sqrt{25}$ breaks down to $\sqrt{5}$ (instead of 5), and like Sarah, solicited the final answer from another student. Conceptual errors like the examples above might go unnoticed if students were not first asked to express their ideas in familiar natural language before attempting abstract symbolic notation (Koedinger & Nathan, 2004). Aside from the last example where the metaphor may have contributed to misconceptions, in most cases, the metaphor helped students better understand the nature of radicals and how to successfully simplify or combine them with other radicals.

Metaphor is a powerful cognitive tool that has the potential to engage the imaginations of students with the content of the secondary mathematics curriculum—particularly when it used to bring inanimate objects or ideas to life. Metaphors are challenging, both within a mathematics context and without a mathematics context. The key is to keep the metaphors simple and choose domains that naturally and clearly map to one another. Complicated metaphors are much like “Algebra Tiles” in the same way they require a great deal of figuring out before they can be useful in helping students understand a particular concept from the curriculum. Complicated metaphors are best left for the English or writing class.

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16 CONCLUSION

I have tried to demonstrate that stories have the ability to engage the imaginations of students with the secondary mathematics curriculum. Cast in the light of a Greek mythology, situated within a humorous tale about bloodthirsty pirates, or thrust into a medieval romance involving ogres and knights, mathematical concepts take on real emotional value and consequently find a meaningful place in the student's world—a world mostly outside the mathematics classroom. Topics and concepts once thought boring, difficult, or even too abstract at the secondary level become quite approachable when viewed through a narrative lens. Each of the nine studies presented in chapters seven through fifteen depicts a slightly different way stories can be integrated into regular instructional practice. Of course, these studies by no means address the full range of ways narratives can assist in the learning of mathematical concepts. Instead, we might view these studies as starting points for further, more detailed investigations. While each of the studies has its own independent analysis and in some cases a brief conclusion, I have for the most part, waited up to now to tie all of the data from the studies together into one unified conclusion. This was both necessary and strategic as it was only after studying the constituent parts, did I begin to fully grasp some of the underlying themes relevant to my two research questions described in sections 5.4 and 5.5 of this thesis. Accordingly, the substance of this conclusion consists primarily of a discussion of these themes and how they apply to mathematical instruction at the secondary level.

16.1 Themes

Several themes emerged in the process of this research. The first theme has to do with the way stories promote healthy teacher-student interactions. Using stories in the classroom opens up new ways for teachers and students to interact with one another and step out of their traditional classroom roles. For some instructors of mathematics this may be an uncomfortable position to be in, while for others it may give them a unique opportunity to reach students with different learning styles. I believe the dialogue that occurs in the midst of these storytelling events is conducive to a healthy learning environment. The next two themes have to do with the way stories compel students to go beyond their current understanding of mathematical ideas. This may involve generalizing new concepts or taking on a new or different perspective. In this way, students take "imaginative leaps" as they try to make sense of their creative endeavours. Connectedly, the next theme addresses how stories better facilitate the learning of "big ideas." Understanding these overriding, more general principles first, can make it easier for students to work with the particular, more detailed concepts and procedures. The next theme relates to how stories can better facilitate the communication of mathematical ideas. Stories provide a natural way for students to express their ideas in their own words before moving on to more formal mathematical language. Additionally, what students write, reveals what they know or do not know about a subject and how they approach mathematical concepts. Furthermore, what students leave out of their stories also implies what conceptual difficulties they might have. The last theme has to do with how stories need to be shaped in order to be effective at the high school level. For this, we return to our cognitive tools described in chapter five and discuss what role each one plays in this process.

16.1.1 Stories Facilitate the Dynamic Interactions Between Students and Teacher.

The first theme has to do with what goes on between the storyteller and audience. Every storytelling event is a unique and complex occurrence that involves the dynamic exchange of ideas and emotions. Stories allow teachers and students to interact in ways typical lectures or activities do not. In *The Four 4s,* I became the devil, delivering an order of operations assignment to a classroom full of tormented souls; in *Kelloggs Needs Your Help,* I was the boss, challenging my young marketing staff to save the corporation with their innovative cereal box designs; in *Skull Island*, I spoke with fear, trepidation, and with tears in my eyes, as I recounted that horrible day when I came into the possession of Bloodhead's cursed map. The linguistic, paralinguistic, and kinesic codes teachers transmit to their audiences create a distinctive mood and give rhythm to the stories they tell—something the printed word cannot easily do (Georges, 1969). When describing the great warrior's return in *The Legend of Chikara Ni Bai*, I paced back and forth across the room, eventually falling to my knees after painting the picture of Ni Bai's dead, decapitated son. I created suspense and dramatic tension by whispering certain lines, talking very slowly, or glaring at members in the audience as though they themselves had committed the heinous crimes. At certain points in the story, I would pause, and wait for students to take on the role of Ni Bai or the swordsmith and deliver blood-curdling screams. (The screaming actually became a source of comedic relief from the mounting tension—or rather, silliness—building in the classroom). A similar type of exchange occurred in *The Wicked Sons,* where students offered their ideas as to how each son was to meet his doom—corresponding to the changing denominator in the related mathematical pattern. A round of applause erupted as we, together—storyteller and audience—decided upon each son's fate. It was exchanges like these that fuelled my resolve to never again deliver stories in print form only. Telling stories involves a certain degree of openness and intimacy—both of which operate within the confines of certain established rules (Georges, 1969). Here, in the context of a storytelling event, the teacher steps away from the safety of the podium and reveals a part of themselves through the characters they take on and the storylines they choose to follow. In the same way they become privy to vital information about their students—the way students respond to characters and storylines reveals their specific interests and needs (Teaching Storytelling, 2007). What is important is that both parties can step back into their formal classroom roles once outside the storytelling event. The only things we hope have changed are a better understanding of the concepts, and a richer, more developed relationship between teacher and student.

16.1.2 Stories Lead Towards the Generalization of Mathematical Ideas.

Perhaps the most significant finding from the research is the way stories encourage students to apply a particular concept or idea in a wider context (Harel & Tall, 1991). The ability to generalize mathematical concepts is a particularly useful skill for early algebra students at the secondary level as many of the concepts they encounter are extensions of those learned in earlier grades. For example, to understand the concept of polynomial division introduced in grade ten, requires an expansive generalization of the concept of long division with whole numbers, usually learned in grade six (Harel & Tall, 1991). In several of the studies, while students did not explicitly generalize mathematical concepts, they used variation—an initial step in the generalization process. In *The Legend of Chikara Ni Bai*, we saw how a large number of students, when writing about exponents, expanded the range of the base in exponential growth to include higher bases. The story I presented to students was relatively simple, with a base of 2, yet several students chose to vary the base to 3 or even higher when writing their own stories—solely for the purpose of creating an interesting and unique story. In *The Pythagorean Problem*, one group of students extended the notion of irrationality, situating it within the wider context of real-life. Instead of simply taking the triangle and discussing the irrationality of one of its sides, they wrote a story about ancient Greek sailors trying to keep their warships in perfect triangular formation. In *The Story of Princess* $\sqrt{3}$, the ogre's transformation into a handsome prince helped students perform the reverse process of converting a mixed radical in simplest form back to a mixed radical not in simplest form (i.e. $10\sqrt{3} \rightarrow 2\sqrt{75}$). In *The Wicked Sons*, students actually made generalizations when exploring the concept of division by zero with a dividend of 12, to include other dividends—some students even used rational expressions to illustrate their abstract thinking. In each case, students took a particular schema and expanded its applicability range to meet the needs of their story (Harel & Tall, 1991). The exciting part is that, unlike typical classroom assignments where teachers often must prompt their students to take these type of steps, stories facilitate the natural progression of ideas into new, previously unchartered territory. Even more, students want to take these "imaginative leaps" as they express themselves in their own unique ways and go beyond what is presented to them by the teacher.

16.1.3 Stories Present Mathematical Ideas from a Different Perspective.

Stories take the facts, algorithms, and rules students learn in school and place them in new and exciting contexts. The unique set of circumstances found within each story gives students a different perspective and discourages the notion that mathematical ideas are static concepts applicable in only one domain. The desire to make their own stories exciting and dynamic often leads students to think more creatively and flexibly about the mathematics they encounter. In *Kelloggs Needs Your Help*, one group of students painstakingly negotiated volume, surface area, and proportions as they constructed a scaled down version of their hemisphere-shaped cereal box. Another group took on the more difficult task of constructing a cereal box in the shape of a frustrum of a cone. In *Skull Island*, students meticulously devised sets of clues that translated into pairs of linear equations—their intersections indicating locations on a Cartesian grid. In *The Story of Princess* $\sqrt{3}$, students perceived radicals in terms of human qualities and behaviour. The courtship of a fairytale princess with an ugly ogre became the means for students to explore the nature of radicals and their associated operations. This is a slightly different approach from the one taken in *Mathematics as Metaphor*, where mathematical objects are simply mapped to other objects or ideas (i.e. people throwing snowballs or places in the school students are not permitted to go). In *The Story of Princess* $\sqrt{3}$, students actually "became" radicals and explored their operations and boundaries using human actions as a reference point. This was a difficult task, as it involved successfully discriminating which elements in the vehicle domain (humanity) map to elements in the topic domain (radicals)—without compromising the mathematics. In *The Wicked Sons*, most students explained the concept of division by zero with a story similar to the one I told, but contextualized it in creative new ways—one student actually described the mathematical pattern in terms of putting things into a decreasing number of cupboards. Other students touched on the notion of infinity in their discussions of division by zero. Whether shaping their stories to fit particular mathematical patterns or extending patterns to fit developing plotlines, students used their imaginations in new and complex ways. In *The Legend of Chikara Ni Bai*, students linked mathematical patterns to specific events in their stories. Aliens poking a boy with a stick, trees growing at an increasing rate, or guests going to a party at their math teacher's house, were just a few of the subjects students chose that highlighted the beautiful patterns found in exponential growth. In *Mathematically-Structured Story*, students described emotionally-charged events in their lives using a structure based on prime numbers. I suspect their fascination with the "Christmas tree" look of their stories is indicative of the universal desire to find pattern in the world. This attention to pattern is certainly a topic worth further investigation. I am especially interested in the effects of positioning mathematical ideas against the backdrop of particular narrative patterns. Dramatic structures, such as exposition, rising action, climax, and denouement, are consistent elements found in all good fictional narratives. How can these already proven aesthetically pleasing patterns be intertwined with the abstract mathematical concepts taught at the secondary level where students already have more developed English-language comprehension skills? For an answer to this question, we might take a closer look at such fictional narratives as, Thomas Pynchon's *Gravity's Rainbow* described in section 4.2.2 b of this thesis (Koehler, 1982).

16.1.4 Stories Encourage Students to See the "Big Ideas."

Stories encourage students to see the big ideas behind many of the topics and concepts they encounter at school. From section 8.2.3, I know that when students first understand the general principles behind some topic or concept, it makes it easier for them to work with particular instances of those principles later. Using an analogy, it would be like telling students the theme is "Disneyland" and then giving them a connect-the-dots puzzle where the solution is in the shape of Mickey Mouse (Cech, 2007). In *The Pythagorean Problem*, students could not explicitly describe irrational numbers, but their use of words like "perfect," "whole," and "fit," suggests they were developing a more general notion of the concept—that there are sets and subsets and classes of objects. In *Kelloggs Needs your Help*, students faced a similar problem in accurately describing the relationship between objects in two and three-dimensional space. Yet some students displayed a sense of some of the more general principles that surround the topic, such as generalization, optimization, and configuration.

16.1.5 Stories Encourage the Communication of Mathematical Ideas.

Stories, by their very nature, elicit a response from the listener (Hoogland, 1998). They stir up images and emotions; they inspire fresh ideas and conjure up distant memories. Upon hearing a story, students will look for connections between elements in the story and the events of their lives. It follows that writing stories is a natural and familiar way for students to explore these connections and relate them to the mathematical concepts they encounter at school. When students write stories, they grapple with new ideas and communicate their findings using their own words and in their own unique way. This, in turn, leads to a sense of ownership and ultimately a better understanding of precise mathematical language. In *Mathematically-Structured Story*, students wrote stories that reflected those things that stir their emotions and that dominate their thought lives. More significantly, they related these emotions to the mathematical concepts they had learned, using their own words. As is the case with Angela, where she represented her factorizations using exponents, students will eventually advance to more formal representations.

Writing stories or writing in response to stories reveals any conceptual and/or procedural difficulties students might have with the mathematics they encounter: In *The Pythagorean Problem*, students made procedural errors in the way they substituted values into the variables in the Pythagorean Theorem; In *Kelloggs Needs Your Help*, we saw how several students still had difficulties communicating their understanding of two and three-dimensional space; In *The Story of Princess* $\sqrt{3}$, students revealed their continuing difficulties with adding radicals as well as the process of converting a mixed radical to a mixed radical in simplest form. In the midst of these errors however, students explored the nature of radicals in a way that captured their imaginations. For example, Karrie incorrectly mapped particular clothing items to each of the factors of the radicand in $2\sqrt{75}$. Undoubtedly, her interests in fashion inspired her to take these "imaginative leaps." Because Karrie was emotionally tied to her story, it was not difficult to help her see where she had made the mistake and correct it.

Although speculative, what students chose not to include in their stories may also gives us some insight as to the degree to which they understand the mathematical content: In *The Legend of Chikara Ni Bai*, we saw how a few students avoided making reference to the mathematical patterns in exponential growth. Moreover, the students that did expound on the mathematical patterns avoided using zero and negative exponents, which suggests they might find these concepts difficult; In *The Pythagorean Problem*, most groups contextualized irrational numbers simply in terms of the classroom setting (i.e. triangle with one side that “does not fit”). Only one group actually applied the notion of irrational numbers to a larger setting outside the classroom; In *Skull Island*, one group of students suffered from “context distraction” and avoided most of the mathematical content altogether; In *The Skippers*, Heidi wrote an enchanting story about a magical creature guarding an ancient maze. While the mathematics is sound, it appears more like a “minor inconvenience” wedged deep within a beautiful literary exposition.

Reading a story is simply not enough to guarantee that students’ imaginations will be engaged with the mathematical content. The results from each of the ten studies strongly suggest that students must engage in some type of writing or other creative activity in response to the stories they hear if the mathematics is to find a meaningful place in their lives—it is certainly a necessary prerequisite if there is to be an evaluation of sorts. The exception to this might be when the story is used to explain a concept (see section 3.6). As in the case of *The Wicked Sons* where there was no writing assignment or accompanying activity, students perceived the concept of division by zero almost exclusively in terms of the story.

16.1.6 Stories Need to be Carefully Designed If they are to be Effective at Engaging the Imaginations of Students with the Secondary Mathematics Curriculum

From the outset of my research, I understood that stories need shaping in order to appeal to students at the secondary level. While I address this in chapter four of this thesis, putting theory to practice is not always a straightforward procedure. Moulding stories to maximize their emotional impact, as well as finding ways to implement them in the classroom—without transforming my mathematics assignments into a collection of writing or English exercises—was a significant challenge. Undoubtedly, each of the cognitive tools described in the literature review plays a unique role in ascribing emotional value to certain mathematical concepts.

It is almost a moot point that the use of binary opposites was a necessary prerequisite for imaginative engagement. Without some type of conflict, the stories I told would have had little effect on the way students approached mathematical ideas. In *The Legend of Chikara Ni Bai*, the clash between the forces of good and evil creates a tension in the story that culminates in an explicit display of exponential growth. The same is true of *The Four 4s* where the student is pitted against the devil himself. Without these conflicts, these students may not have become emotionally engaged with the mathematics in any meaningful way. However, while the presence of binary opposites initially drew students into an exploration of the mathematical ideas in *The Four 4s*, it was of a purely ornamental nature—the mathematics was not intricately connected to these conflicts (Mazur, 2005). Results from this study suggest students abandoned the story altogether in their conceptualization of order of operations. The same however, is not true of *The Legend of Chikara Ni Bai*, where the stories students wrote, consistently included binary structures intertwined with the mathematical content.[18] This can be tied to other factors, which we will discuss in the next paragraph. Unlike the two stories mentioned above, in *The Pythagorean Problem*, the conflict is rooted in the mathematics itself. In this case, a divergence of theories sets the stage for the dramatic tension that ensues. Even more

[18] This could however, be related to the fact that students did not simply solve a mathematics problem, as in *The Four 4's*, but they engaged in a writing activity.

potent, is the conflict between the ogre and the townspeople in *The Story of Princess* $\sqrt{3}$. Here, the mathematics itself is postured in such a way that it is inseparable from the story's binary structure.

If I were told that I could use only one cognitive tool to shape my stories, it would be that of rhyme, rhythm, and pattern. From section 5.5 we saw how an attention to pattern is fundamental to cognitive development and plays a significant role in the recognition of beauty (Moyer, 2000; Egan, 2005). It is no surprise then, that each of the stories, when infused with a burst of patterns, captured the imaginations of students in a way that most of the other stories did not: The pattern of exponential growth revealed in the making of the magical sword in *The Legend of Chikara Ni Bai*, may account for the way students wrote stories that were intricately connected to the mathematical content; In *The Wicked Sons*, students categorically perceived the concept of division by zero in terms of the mathematical patterns inherent in the story; and in *Mathematically-Structured Story*, the lack of patterns may account for the way students approached the concept of prime factorization with such wonder and emotion.[19]

Perhaps the cognitive tool I use the most in my stories—and the one I probably give the least attention to in this discourse—is the cognitive tool of jokes and humour. Knowing that a burst of emotion almost always accompanies even the slightest bit of comedy in this irreverent venue, it is almost a prerequisite that humour, in some form or another, is weaved into the stories I write. Most instructors of mathematics however, tend to avoid situations where emotions run wild and laughter takes over—it seems irrational and borders on the chaotic. Yet some of the most powerful learning

[19] Students commented extensively on the "Christmas tree" look of their factorization stories.

opportunities can be found in the midst of a joke or humorous situation. In *The Legend of Chikara Ni Bai*, students rested on every word anticipating another horrifying scream or the introduction of a silly character—unaware I was building them up for the climactic scene championing exponential growth. A ridiculously-named pirate and a treasure map misplaced in the hands of a mathematics instructor clash with the earnest works of Descarte in *Skull Island*. Incongruities of this nature cannot help but elicit an emotional response from students while at the same time inadvertently drawing them into a closer examination of the mathematical topic.

The cognitive tool of knowledge and human meaning also played a key role in this study. Students love to hear about real people in real situations, and when mathematics is presented in the context of the human emotions and motivation involved in its discovery, it takes on real meaning (Egan, 2005). Students at this age are interested in the past and want to know more. Interestingly, high school students do not expect to learn about history in their mathematics class. They expect abstract language and theoretical number work—so at the slightest hint of reality, their interests become peaked. Additionally, when the most extreme elements of reality are highlighted, the effect is multiplied (Egan, 2005). In *The Pythagorean Problem*, students were fascinated by the ancient sect's weird rituals and beliefs. In *Skull Island*, they doubted the validity of the legendary pirate, Bloodhead, but revelled in his outlandish behaviour.[20] In *Kellogg's Needs Your Help,* students asked "Why would a big company like that ask a

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[20] Without some attention to the cognitive tool of jokes and humour, this story would not have made it past the first sentence. In fact, many of the stories I told required the listeners to suspend all critical analyses and take a tongue-in-cheek approach. For instance, in *The Story of Princess* $\sqrt{3}$, my plea to childhood fairytales was a source of great amusement—along with its obvious borrowings from Disney's *Shrek 2*. But without this type of approach however, I am not sure how this fairytale would have faired within the high school context.

bunch of high school students to help them design a new cereal box?" Yet they undertook the project with the fervour of young marketing execs.

Of all the studies presented in this thesis, *The Story of Princess* $\sqrt{3}$ stands alone in its potential for further exploration. The use of metaphor in this way has received little attention in the educational literature. It was my hope that through this research, we might better understand this powerful cognitive tool and how it might be implemented at the high school level to teach mathematical concepts. The idea of taking mathematical objects—especially those of a more abstract nature—and turning them into characters with human qualities has both a certain profundity and childlike appeal. The stories students wrote displayed various degrees of mathematical proficiency as well as a penchant for risk-taking in their mappings of human qualities to mathematical ideas. Unfortunately the second study concerning the explicit use of metaphor, *Mathematics as Metaphor*, was not well planned out. The associated assignment was too difficult and the metaphors were quite complex. This was more like an exercise in advanced writing than a grade ten assignment in mathematics.

16.2 Final Remarks

There were many more stories I told that did not make it into the pages of this thesis (i.e. *Mrs. Whitehead and The Big Lawn*, *Monkeys Throwing Poop*, *The Dating Equation*, *Eratosthenes*) —a few of which might shed even more light on this burgeoning new field in secondary mathematics education. We all have stories to tell, whether they be drawn from the events and circumstance of our daily lives, or from the nonsensical images that consume our imaginations. Stories require very little time to prepare, and good stories require just a bit more. So where shall we take our students today? What far off worlds might we explore whose elements intertwine with the concept of exponential growth? What shady characters might enlighten our students to the virtues of polynomial division? What historical figures may we resurrect in order to put passion back into the mindless facts, algorithms, and processes we present to our students day in and day out? After all, the imagination has no limits.

17 APPENDICES

17.1 Ethical Approval

17.2 Consent Forms

17.3 Skull Island Handouts

Skull Island Project

A location on a map can be determined using the point of intersection of two lines. In this project you will work in groups of 2 or 3 to create a large treasure map with a series of locations each determined by two linear equations. Linear equations may be in any of the three forms:

General form	$Ax + By + C = 0$
Slope-y-intercept form	$y = mx + b$
Point-slope form	$y - y_1 = m(x - x_1)$

Each location on your map is a step towards the final location where the treasure is located. When you have finished your map, you will exchange it with another group that will attempt to graph the equations you give them and find the treasure that you provide.

The Group:

You may work in groups of no more than 3. Remember that you can divide the task between you, but each person is responsible for knowing the mathematics in the project and will be tested on it daily. Make sure that you are skilled in solving all the problems in the workbook exercises and the types of problems on the posters corresponding to each day. Your group must provide a "treasure" for the group working on your treasure map.

The Evaluation:

1. Small map showing graphed equations and locations (26 marks)

You must have a properly labelled coordinate plane superimposed on your map. You must have 13 different locations on your map with 26 different equations graphed. The equations you use must be numbered corresponding to the numbers on the posters around the room. Your equations must not be exactly the same as the ones on the poster, but must be of the same form. Each line must be labelled clearly (use a legend if necessary). All written work must be shown on a separate attached sheet of paper. Your written work must be clearly identified corresponding to the graphs of the equations. (Each person in the group must do this).

2. Large map showing locations (30 marks)

This map must be a scale drawing of your small map. This is the map you will exchange with the other group. It must also have a coordinate plane superimposed on it clearly labelled. All 13 locations must have names as well as any other notable sites. Your map must have a theme that is clearly identifiable. It does not have to be a pirate treasure map. For instance, it could be: a map of a city and you are on the trail of a serial killer; a map of a shopping mall and you are trying to get away from zombies; a map of England and France and you are trying to get away from the *Opus Dei*; or a map of whatever else you can imagine. You must describe the theme of the map, the characters involved, and what they are trying to find. This can be on a separate sheet of paper or as part of your large map that will be given to the group that attempts to solve your "treasure hunt."

3. Quizzes (20 marks)
Sometime during each class, you will be expected to stop what you are doing and write a quiz on the preceding period's objectives (see schedule of project).

Schedule of Project:
You will have exactly 5 periods to finish your map, and learn the material from each section. Late marks will be accrued after that.

Day 1:
Determine group, theme.
Sketch rough draft of small map.
Begin rough draft of large map.
Learn: "Writing equations using slope-y-intercept form" (workbook: p.381-387).

Day 2:
Draw good copy of small map.
Sketch rough draft of large map.
Graph lines on small map using form (equations 1-14 on posters).
Learn: "The general form equation " (workbook: p.389-396).
Quiz on "Writing equations using slope-y-intercept form" (workbook: p.381-387).

Day 3:
Draw good copy of large map.
Graph lines on small map using (equations 15-19 on posters).
Learn: "Writing equations using general form" (workbook: p.397-402).
Quiz on "The general form equation " (workbook: p.389-396).

Day 4:
Draw good copy of large map.
Graph lines on small map using (equations 15-19 on posters).
Learn: "Point-slope form " (workbook: p.403-411).
Quiz on "Writing equations using general form" (workbook: p.397-402).

Day 5:
Finish large map completely.
Graph lines on small map using (equations 20-25 on posters).
Learn: "Further practice with linear equations" (workbook: p.413-420).
Quiz on "Point-slope form " (workbook: p.403-411).

Day 6:
Exchange maps with another group and find the treasure.
Hand in small map, work notes, and theme write-up.

Hand in large map after group is finished

Slope Y-Intercept

$$y = mx + b$$

We have learned that the graph of an equation in the form $y = mx + b$ is a straight line with slope m and y-intercept b.

The following is a list of examples of different ways you can determine the equation of a line:

Write the equation of the line:

1. With slope 4 and y-intercept 6
2. With slope 3 and y-intercept 4
3. With slope $\frac{-3}{5}$ passing through the origin
4. Passing through the point (0,1) and perpendicular to $y = 4x - 2$
5. Passing through the point (0,1) and parallel to $y = \frac{1}{10}x + 24$
6. With the same y-intercept as $y = 2x - 3$ and perpendicular to $y = \frac{7}{3}x - 2$
7. Passing through (0,6) and parallel to the line which passes through (0,1) and (4,-6)
8. Passing through (1,3) and perpendicular to the line which passes through (7,-2) and (12,-3)
9. Passing through (3,-9) and parallel to the x-axis
10. Passing through (3,-9) and parallel to the y-axis
11. Passing through (1,4) and perpendicular to the x-axis
12. Passing through (1,4) and perpendicular to the y-axis
13. the x-axis
14. the y-axis

General Form

$$Ax + By + C = 0$$

The general form of the equation of a line is an equation where all the terms are collected to the left side of the equation. The general form equation is $Ax + By + C = 0$, where A, B and C are expressed as integers if possible.

The following is a list of examples of different ways you can determine the equation of a line:

Write the equation of the line:

15. With the same y-intercept as the line $22x - 3y - 18 = 0$ and parallel to the line

    $2x - 3y + 9 = 0$

16. With the same y-intercept as the line $3x - y + 18 = 0$ and perpendicular to the line

    $3x - 2y + 5 = 0$

17. With the same x-intercept as the line $4x - 3y + 4 = 0$ and parallel to the line

    $x - 4y + 8 = 0$

18. With the same x-intercept as the line $4x - 3y + 4 = 0$ and perpendicular to the line

    $x - 4y + 8 = 0$

19. Passing through the points (7,5) and (3,1)

Point-Slope Form

$$\boxed{y - y_1 = m(x - x_1)}$$

The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$ where m is the slope of the line and (x_1, y_1) represents a point on the line.

The following is a list of examples of different ways you can determine the equation of a line:

Write the equation of the line:

20. Passing through (2,-1) with slope 2
21. Passing through the point (5,0) and perpendicular to the line with equation

    $3x - 5y + 17 = 0$
22. Passing through the point (2,4) and parallel to the line with the equation

    $3x - 5y + 17 = 0$
23. Passing through the origin and perpendicular to the line $4x + y - 34 = 0$
24. Passing through the origin and parallel to the line $4x + y - 34 = 0$
25. If A is the point (-6, 8) and B is (-10,-4) determine the equation of the

    perpendicular bisector of AB.

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