UNDERGRADUATE STUDENTS' CONCEPTIONS OF INEQUALITIES

by

Elena Filofteia Halmaghi
Licence in Mathematics, University of Bucharest, Romania, 1985

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Abstract

Inequalities are vital in the production of mathematics. They are employed as specialized tools in the study of functions, in proving equalities, and in approximation or optimization studies, to enumerate only a few areas of mathematics where inequalities are put to work. The concept of inequality, however, is problematic for high school and university students alike. Moreover, the school curriculum seems disconnected from the role of inequalities in mathematics and mostly presents inequalities as a subsection of equations. The placement of inequalities in the school curriculum and the disconnect between school mathematics inequalities and a mathematician's approach to inequalities take the blame of research in mathematics education reporting on students' misconceptions when dealing with this concept. This study moves from the theory of misconceptions to a framework of undergraduate students' conceptions of inequalities. In an effort to learn more about what students 'see' when dealing with inequalities, three research questions are pursued: *What are undergraduate students' conceptions of inequalities? What influences the construction of the concept of inequalities? How can undergraduate students' conceptions of inequalities expand our insight into students' understanding of, and meaningful engagement with, inequalities?*

Data for this study was produced mostly through learner-generated examples of inequalities that satisfy certain conditions. The participants in the research were undergraduate students enrolled in two mathematics courses – a foundations of mathematics course and a precalculus course. The results of this research are five conceptions of inequalities. It was also found that the undergraduate students' conceptions of inequalities mostly occupy the lower regions of Tall's 'Three Mental Worlds of Mathematics'. The speculation is that what Tall calls the 'met-befores' as well as what I call the 'missed-befores' influence the construction of the concept of inequalities. Curriculum suggestions for preparing the ground for the work on and with inequalities are presented. This study contributes to ongoing research on mathematics concept formation.

Keywords: inequality; conception; understanding; learner-generated examples; met-before; missed-before

To the loving memory of my mom, Ana, my first partner in problem solving.

To the person who declared me a mathematician in grade 5 and taught me how to prove by ‘reductio ad absurdum’ in grade 6, Professor Lazar Samoila, my middle school mathematics specialist teacher.

Domnului Profesor Lazar Samoila, cel care m-a declarat matematician in clasa a 5-a iar in clasa a 6-a m-a invatat sa demonstrez prin reducere la absurd.

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Beginnings

> The *equality indicates a boundary*, but we are really concerned with what lies *inside* and *outside*. The equality is like a fine ceremonial dress, beautiful for show; but you get into your shirt-sleeves for the real work. In fact, that seems to be the keynote of the situation; we like to present our *finished mathematics*, mathematics for show to the public, as much as possible in *equality form*, but in the mathematical workshop, **inequalities** are the standard tools. (Tanner, 1961, p.294)

1.1 Introduction

1.1.1 The Protagonist

The main character in this work is Inequality.

I was often asked during my study "what is your research area in education?" If my answer was simply 'inequalities', my interlocutor would normally think that the focus of my study was social class inequalities, or gender inequalities, or racial inequalities, or maybe disadvantaged groups of students, or many, many other inequalities related to education. I had to explain that I was actually working on mathematical inequalities. Even this qualification – mathematical – would not be sufficient to imagine my study. "You mean – social inequalities that are the product of having a good or a weak background in mathematics?” some people would reply. Then I would have to explain that I was concerned with the plain concept of inequalities studied in mathematics and the major question that I would like to answer is: Why are mathematical inequalities so hard to be meaningfully addressed by students taking undergraduate mathematics courses?

Depending on the background of my conversation partner, the discussion about my work would follow two distinct paths. Some people would continue with the questioning and astonishment: “Mathematical inequalities? Hmm! Is this concept rich enough for such a big study? I mean, I met inequalities once in one of my undergrad courses, as an odd relative of equations. I do not recall any meaningful context or applications for inequalities. I wonder what they can offer you.” Some other people, such as Dr. J, who is an award-winning mathematics instructor would reply instantly: “But they are hard!” Moreover, Dr. D, who is a much-respected researcher in chaos theory said: “Inequalities are tricky objects to work with. Even productive and versatile mathematics researchers are very careful when using an inequality to prove something. For example, whenever I need an inequality in my work, I check the inequality part more often than most other parts of my paper. After the last editing, I don’t send the paper to publisher, but I put it aside for a few days, and then I start from fresh working on the inequality again. I send the paper only after it passes this one more verification step."

In a similar way, commenting on students’ difficulties in proving by mathematical induction, one of the mathematicians in Nardi’s (2007) research acknowledged that when dealing with inequalities there is an extra level of inconvenience in their work: “Inequalities are difficult because there is no right hand side in them which students would know how big or small they ought to make the expression on that side in order to continue producing the chain" (p.84).

Everybody could acknowledge that inequality – not equality – is the most prevalent relationship between the people or the objects surrounding us. Most people could spot an abundance of inequalities in the educational context alone, not to mention the other aspects of someone's life. Moreover, the mathematicians or mathematics instructors I spoke with were able to relate their experience with inequalities to a mathematics education study concerning inequalities.

Inequalities play an enormous role in mathematics. They are not just some peculiar relative of equations – they are one of the most powerful tools for producing equalities. Inequalities are a means of expressing constraints and solving linear programming or optimization problems, and they provide a way of expressing the domain of a function, of solving limits, or of setting up research questions that relate equations to special cases. The whole complexity of mathematical inequalities correlated with the fragments or pieces of inequalities encountered by undergraduate students is of interest to me in this research. The focus, therefore, is the composition and decomposition of the inequality concept juxtaposed with my students' images of inequalities in an effort to identify their conceptions. My study contributes both to research in mathematics education in general and to curriculum in particular.

In what follows, I outline some experiences that have guided my research interest toward inequalities and present some of the most important ideas that are involved in this study. This background introduces the initial question, narrates the search for a niche and leads toward the research questions that guided the study.

1.1.2 Background

My recollections of mathematics go far back in time. As a pre-schooler, during the long winter evenings, I played cards and Nine Men's Morris[1] with my brothers and friends. Therefore, calculations and mental strategies were a daily routine. In addition, when I was only five, I taught myself how to read in a few hours of personal effort by decoding the lines of a poem I knew by heart. The lines became words and the words became sounds. Finally, the sounds were transformed into letters. In grade five, I used to think of myself as a mathematician. This was not only because I was good at calculations, but also because I was very good at problem solving. My primary school experience was full of memories of working on exercises requiring several hours of work, followed by great moments of illumination. Since elementary school, not a day passed by during my school years – and this is no exaggeration – that I did not spend at least a few moments explaining a mathematical concept or helping a peer solve a mathematics task. Not only did I consider myself a mathematician, but I also declared, in grade fie, that I would become a mathematics teacher.

As a high school student, I continued my individual work on mathematics and, at the same time, I studied to become a classroom teacher. Then, as a university student in mathematics, I made geometry the object of my thesis and I received an education as a specialist teacher of middle, high school and college mathematics.

[1] Nine Men's Morris is a strategy board game for two players. Each player has nine pieces, or 'men', which move along board's twenty-four spots. The strategy is to form rows of three 'men' along the board's lines. The player with a row of three pieces has a 'mill' and removes one of the adversary's pieces. The player who has fewer than three pieces on the board loses the game.

When I graduated from university, I felt very well prepared for my first job as a mathematics specialist teacher for middle and secondary school in rural Romania. There, I worked for four years, teaching mathematics to students in grades 5 to 10. The mathematics I covered with those students is similar to an in-depth version of BC's 7 to 12 curriculum. I say 'in-depth' because in Romania, formal geometry starts in grade 6. By the end of grade 10, students cover Euclidian geometry in 2D and 3D, subjects that are studied only at the university level in BC. Feeling well prepared for teaching and possessing a solid content knowledge background, I embarked on my first job with the hope that my students would be exposed to meaningful and engaging mathematical experiences. I sincerely wished that my teachings would help their understanding of mathematics, augment their appreciation for the process of doing mathematics and spur their love for mathematics. With a belief in discovery learning, I strove to attend to the individual needs of each child. I worked hard and had great expectations from my students. However, I soon realized that my efforts were not being rewarded. Only a small number of my students were able to learn and appreciate our work in the classroom. The majority of my students were performing below the expectations I had set for them. Since then, the question about why some students are able to learn better than others has been haunting me.

Following my first job, I got an appointment at a vocational college where I taught mathematics for almost eight years. There, I continued to encourage my students to at least appreciate mathematics, if not love it as much as I did. Again, I had some very good students who were able to keep up with me; however, the majority of them were lost most of the time. Questions such as, what makes some concepts harder to grasp than others, or what makes some students better at learning than others, resurfaced. During that time, however, a new question arose – is there such a thing as a perfect curriculum? My question was rooted in my suspicion that the school curriculum was too jam-packed and many of the concepts, such as proofs of divisibility rules in grade 5, were too abstract for the young minds I was teaching.

From the year 2000 until today, I have been teaching undergraduate mathematics courses at Simon Fraser University and elsewhere. I have had the opportunity to teach Calculus, Precalculus, Discrete Mathematics and Mathematics Foundations courses for which I had to employ various teaching strategies – from lecturing to huge classes to facilitating learning and discovery in seminar settings. History seems to be repeating itself with my students at SFU: while some are enjoying the classes and are growing with me during the course, others remain confused and continue to underperform. All these experiences have made me look to the past to find possible explanations for the fact that some students are better at learning mathematics than others.

Teaching elementary mathematics in Romania, I also observed that doing mathematics in some classes was such an enjoyable task, whereas it was quite painful in other classes. Some groups of students could model real-life situations mathematically, while others were panicked and were distressed whenever given a word problem to solve. In the staff room, I was able to meet, discuss with and get to know the school's teachers. From these interactions, I could make connections between my current students and their previous teachers.[2] This way, I could conclude that the classes for which mathematics

[2] In Romania, in most of the schools, students remain in the same cohort for all their primary school years. Moreover, from grade one to grade four, they will have the same classroom teacher.

was enjoyable had a primary teacher who loved mathematics and was able to convey this sentiment to his/her students throughout their time together. If a class had a primary teacher who either did not like mathematics or mathematics was not one of his/her strongest skills, it was very likely that many of the students coming from that teacher will either have math anxiety or no enthusiasm or pleasure in doing any mathematical activity. As soon as I realized that, I got involved in helping my fellow teachers with number representations, proofs, problem-solving techniques and math puzzles: I became my colleagues' resource whenever they had a problem to solve or a concept to be introduced to young students in a meaningful way.

Thus, the difficulties my students had in understanding mathematics triggered my interest in delving deeper into the root of their problems. Is the poor understanding of a specific concept due to the nature of the concept or to their mathematics background? If the problem is indeed their mathematics foundation, how far back in time should I go to discover the roots of their troubles? Connecting my old and new experiences, I came to suspect again that elementary teachers play a crucial role in a child's mathematical representations. Reaching that conclusion, I felt that I had to go to the place where teachers are prepared and help them see the importance of their job. It is this thought that has inspired me to embark on the journey to pursue a PhD in mathematics education.

During my graduate studies, alongside the development of my understanding of the mathematics education field, I continued to teach undergraduate mathematics. Being intensely exposed to teaching and becoming fluent in the language of interpreting my undergraduate students' work in mathematics, I decided to make the focus of my research a topic that I frequently encounter in my teaching. This was not a difficult decision, since for some years; I had observed one such concept – inequalities! What makes the inequality concept so hard for undergraduate students to understand and manipulate became one of the major questions that helped shape my study.

My interest in pursuing this study was partly an attempt to answer the above specific question related to inequalities. The other aspect was the insight that my findings may transfer to the area of mathematics education in general. Finding the root of the problem in understanding the inequality concept, thus, became the focus of my work.

1.1.3 Finding a Niche

In this sub-section, I begin with a time line story of my practice – the purpose is to introduce a series of initial questions leading toward my research questions. Then, I present the questions that guided the study. In the last part, I foreshadow the structure of this thesis.

Teaching undergraduate mathematics, I encountered many moments when my students did not seem to show the vaguest sign of understanding the concept we were working on. Students memorizing procedures and reproducing parts of our class work without making any connections between concepts were some of the darkest aspects of my teaching career. After many years of teaching, I have gathered a repertoire of misconceptions, common mistakes, and pitfalls that I make my new students aware of when teaching a specific concept. In time, I have noticed my students acknowledging some of my advice and successfully remediating some misconceptions. However, there is one concept in my repertoire of semesterly teaching that, no matter how carefully I present its many aspects, only a very small number of students master it at the end of the course. This concept is inequality. Inequality is a rich concept; it can be traced down in all areas of mathematics, from comparing numbers in first grade to approximating areas in tertiary mathematics. Even though there are not too many major studies with inequality as the sole protagonist, education researchers have a special interest in inequalities. In the study presented here, however, the focus is exclusively on inequalities.

At the beginning of the study, the question that triggered my work on inequalities was why is it so hard for so many students to understand inequalities? The question was too general to indicate a clear direction to pursue or the tools I should use toward reaching an answer. Employing the discipline of noticing (Mason, 2002), I have educated myself in seeing links to inequalities when teaching other concepts or provoking connections to inequalities when they were not evident to my students. Finally, I focused on special aspects of this concept and was able to narrow down my observations to more researchable questions such as: What makes the concept of inequalities so hard for undergrad students to understand and manipulate? What stands in the way of understanding inequalities? Is there something in the prior knowledge that conflicts with the new concept? Is there an old definition, image or schema that hinders the understanding of inequalities? Is there something in the composition of the concept that makes it inaccessible to the population of students that I deal with? These were all questions that were pursued during my practical and theoretical involvement with inequalities.

The search for an answer to some of the questions followed three distinct paths: (1) looking for epistemological obstacles inside the concept itself or in the history of the concept (Cornu, 1991); (2) checking for didactical obstacles due to the placement of the concept in the curriculum (Radford, 1997); (3) searching my students' work for for instances for me to depict and interpret their concept images of inequalities. The three paths were not completely separate or travelled in any chronological order. They intertwined with each other and were travelled simultaneously. Moreover, the experience accumulated on a specific journey improved the experiences that followed when walking on another path. In the dissertation, I present the three paths in different chapters. The questions guiding the study will set the natural progression of the chapters. More precisely, the questions remaining unanswered at the end of a chapter will lead the discussion into the next chapter.

1.2 Narrowing Down the Topic

The work on narrowing down the topic followed an outline summarised in some major questions pertaining to inequalities, questions to which I considered important to have the answer. In what follows, I introduce the reader to those questions and I foreshadow some answers.

<u>What are inequalities?</u>

> Our subject is difficult to define precisely, but belongs partly to 'algebra' and partly to 'analysis'. Algebra or analysis, like geometry, may be treated axiomatically. (Hardy, Littlewood & Polya, 1934, p.4)

Inequality is a term used very often in mathematics contexts and real-life situations. If one opens a college mathematics book, inequalities are often on that very page, explicitly used to express relationships between numbers or to write restrictions for quantities or implicitly embedded in the domain or behaviour of functions. If one skims through a mathematics journal, it may not be a surprise to see the inequality symbol in every single article. Moreover, the daily news, which rarely presents some explicit mathematics, would show an abundance of inequalities; they might be called inequity, disparity, unfairness, unevenness, irregularity or dissimilarity.

The Canadian Oxford Dictionary defines inequality as the lack of equality between persons and things, disparity in size, number, quality, etc. This could be a good definition to start thinking about inequalities; even to start basic work on inequalities in a mathematics context. However, for doing advanced mathematics, as well as for the development of this study, a more workable definition is required. Moreover, a comprehensive definition of inequality must incorporate or connect many discrete elements, such as symbols, conventions or concepts intimately related to inequality. Figure 1.1 is a concept map of the concept of inequality, one which incorporates many aspects of inequalities. The formal definition of inequality, the one adopted from the Encyclopaedia of Mathematics, and the axioms related to inequalities are the object of sub-section 2.1.1.

The concept map represented by Figure 1.1 is not a definition of inequality; it is more of an indication of the bits and pieces of the concept image of inequality that will add to the discussion in this study. The map shows that working with inequalities links not only to algebra or functions; number sense, logical reasoning, experience with proofs, embodied dynamic or static inequality experiences are all parts of the inequality schema. How many of those meaningful experiences are offered to students prior to starting heavy work on inequalities, as well as how and when that training is provided, are questions whose answers will be attempted in this study.

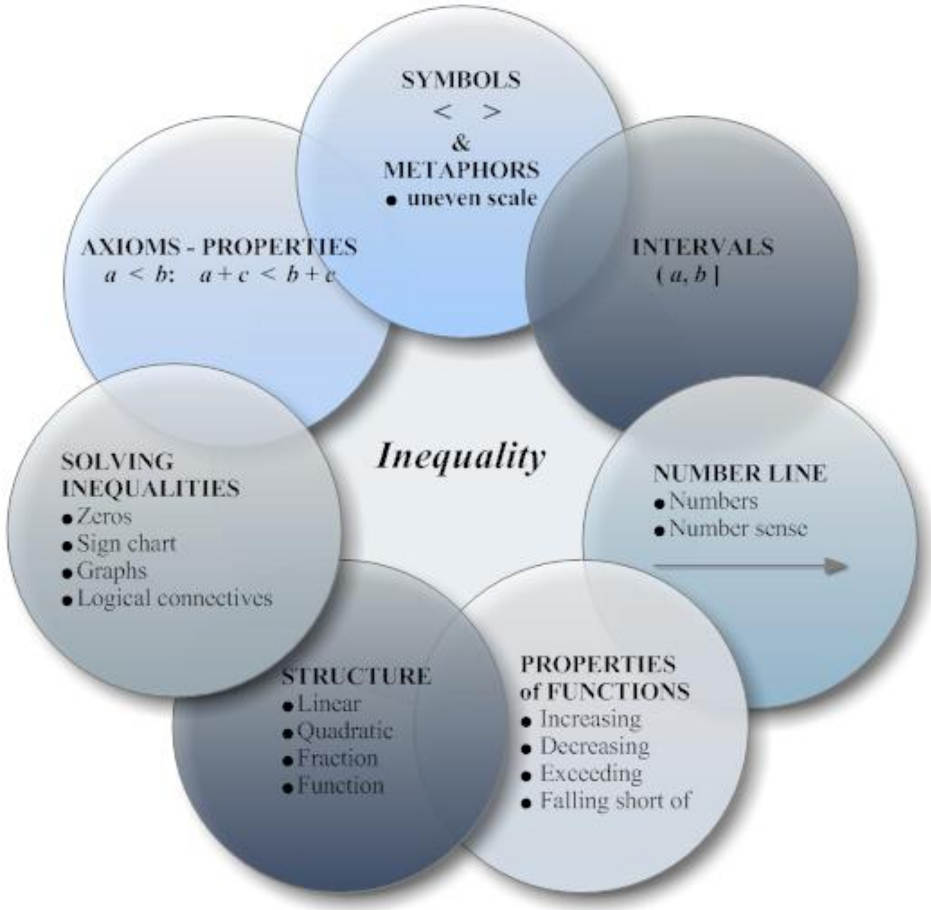


Figure 1.1: A concept map of inequality

<u>How did inequalities come to be?</u>

Euclid recognized the conditions when one length was larger than another and expressed that in words such as ‘falls short’ of or ‘is in excess of’. However, no indication of inequalities for numbers, except in Archimedes’ work in science, mathematics and engineering, is found in the ancient texts (Fink, 2000). Several inequalities, such as the inequality of the arithmetic-geometric means of two quantities or triangle’s inequality, which were known to ancient mathematicians, are found in geometric contexts. Then, up to Newton and Cauchy, there is not much reference to or work on inequalities. Hardy, Littlewood and Polya (1934) introduced the mathematics community to the concept of inequality, as we know it today. Section 2.2 presents a detailed summary of the important events that helped inequalities come to be.

Where are inequalities located in mathematics?

In higher levels of mathematics, inequalities are everywhere. In analysis, for example, inequalities are a means of proof (Burn, 2005). In optimisation, inequalities off all sorts are the standard tools. In function theory, inequalities are a means of writing constraints and deriving domains. In axiomatic geometry, the axioms of order are inequalities. In field theory, inequalities with their four properties – reflexivity, antisymmetry, transitivity, and comparability – serve as a model for the totally ordered set. There is gossip among mathematicians, informing that analysts spend half their time searching for inequalities they need for their work's advancement. The same folklore speculates that the same mathematicians may not be able to use those inequalities, for not being able to prove them (Hardy, 1929). In Hardy's words, "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove" (Hardy, 1929, p.64). These accounts inform that inequalities are in great need in many areas of mathematics where they help the advancement of theories. In section 2.1, inequality is formally introduced and observed at work.

Where are inequalities located in the K-12 Curriculum?

In school mathematics inequalities are studied mainly in algebra courses. Moreover, they are studied as a subsection of equations. Section 2.3 comparatively presents inequalities in school mathematics in British Columbia and Romania, two school systems I have been part of for four decades.

What does the research literature say about the teaching and learning of inequalities?

Many of the published studies present inequalities in parallel with equations (e.g., Kieran, 2004; Dreyfus & Hoch, 2004; Vaiyavutjamai & Clements, 2006a&b). There are a few studies where inequalities were connected to the study of functions (e.g., Boero, Bazzini, & Garuti, 2001; Garuti, Bazinni, & Boero,. 2001; Sackur, 2004). Presenting results from inequalities either in relation with equations or in connection with functions, most of the studies report on weaknesses of the pedagogy employed for teaching inequalities. The learning of inequalities encounters multiple obstacles and the students' work on inequalities exhibits misconceptions. To the best of my knowledge, no study on inequalities focused on students' conceptions of inequalities, rather than reporting on misconceptions. With no further ado, it seems that a new study on inequalities is necessary. And here I start the journey.

1.3 Outline of the Dissertation

The dissertation begins with a consideration of the journey toward choosing and narrowing down the topic. Chapter 2, whose organization is foreshadowed in section 1.2, presents inequalities in three major contexts – mathematics, history of mathematics, and curriculum. The formal definition of inequality, a snapshot of mathematician's work on and using inequalities, a historical account of inequalities, and an exposition of inequalities in the school curriculum are among the topics touched upon. Chapter 3 presents prior educational research on inequalities and identifies areas where more research on inequalities seems to be necessary. The research questions guiding the study conclude this chapter. Chapter 4 presents a brief overview of theoretical frameworks and constructs that informed my research.

Chapter 5 presents the discipline of noticing as a research practice and gives the rationale for choosing learner-generated examples as an instrument for collecting the data and phenomenography as the method for interpreting the data. This chapter also describes the participants in the research and introduces the two empirical studies conducted in an attempt to answer the research questions. Chapters 6 and 7 present two separate studies. Chapter 6 is devoted to the results and analysis of the preliminary data. The major result of the preliminary study is the description of five conceptions of inequalities, which becomes the framework for the main study. Chapter 7 validates the five conceptions of inequalities and gives more substance to the description and names of the emerged conceptions of inequalities.

The summary of the findings and the interpretation of the outcomes of the two studies are presented in Chapter 8. This chapter sheds light on the contributions of this study, highlighting the developed *Conceptions of Inequalities* (COIN) framework. It also offers some pedagogical suggestions in preparation for studying inequalities. The last part of this chapter is devoted to conclusions at the end of the journey.

Chapter 2:

Inequalities in Mathematics, History of Mathematics, and Mathematics Education Research

> The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the ideas, like the colours or the words, must fit together in harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy, 1992, p.85)

2.1 What Are Inequalities?

2.1.1 Definitions

An (algebraic)[3] inequality is a mathematical statement that one quantity is greater than or less than another one. For example, "a is less than b" and "a is greater than b" are inequalities. Using symbols, the inequality "a is less than b" is denoted $a < b$. Using the same pattern, "a is greater than b" is denoted $a > b$, "a is less than or equal to b" is

[3] When I first began this work, I believed that inequalities are algebraic in nature. The mathematics encyclopaedia that I consulted for the definition contained the word 'algebraic' in its explanation of the concept of inequality. Hardy, et al. (1934) confessed that the "subject is difficult to define precisely, but belongs partly to 'algebra' and partly to 'analysis'. Algebra or analysis, like geometry, may be treated axiomatically" (p. 4). Having grown in my understanding of the nature of inequalities, I am now more cautious about this aspect.

denoted $a \le b$, "a is greater than or equal to b" is denoted $a \ge b$, and "a is not equal to b" is denoted $a \ne b$ (Gellert, Kustner, & Hellwich, 1975; Postelnicu & Coatu, 1980). For short, we say that an inequality is defined whenever we have two expressions linked with one of the five symbols: $<$, $\le$, $>$, $\ge$ or $\ne$.

For example $2x+6>0$, $(a-2)^2 \ge 0$, $3 \le 7$, $x-y<2$, or $1 \ne 3$ are all inequalities. Carefully looking at the above examples, it can be distinguished that, even if all of them are called 'inequalities', they are different in nature. In the above examples, some inequalities have variables: $2x+6>0$, $(a-2)^2 \ge 0$, $x-y<2$; while others do not have variables in their composition: $3 \le 7$ or $1 \ne 3$. In mathematical logic, inequalities with variables are called *predicates* and inequalities without variables are called *propositions*. With propositions, one can be sure immediately if the inequality is true or false. Predicates, however, are of two types; the classification is done by the role that the variable plays in the inequality (Postelnicu & Coatu, 1980).

The inequalities $2x+6>0$ and $x-y<2$ are linear inequalities in one and two variables, respectively. In some languages, such as Romanian, French or Italian (Bagni, 2005) for example, the two inequalities will fall into a category, which is called inequations. The name *inequations* was attributed, possibly, by analogy with equations, with which they look similar in structure, except for the symbol connecting the two sides of the algebraic expressions involved in the structure. The work that is usually done on this type of objects is 'solving for their solutions'. To solve for the solution of an inequation, one must write it in equivalent forms, as simple as possible, such that the solution to be easily read from that last form. For example, to solve the inequation

$2x+6>0$, equivalent forms must be derived, such as $2x>-6$ followed by $x>-3$. The solution can be read from the last one. As for the second type of predicates, in those three Romance languages, $(a-2)^2 \geq 0$ is an inequality. The difference between inequation and inequality is that they parallel 'equation' and 'identity,' respectively. In the case of inequality one does not have to solve for a solution, but has to prove that the statement is true. $(a-2)^2 \geq 0$ is a universal truth: This statement holds for every real number. More examples of inequalities and the particularities of manipulating inequations as well as inequalities are discussed in more detail in the following section.

2.1.2 Inequality Manipulation

The properties of inequalities that help transform a given inequality[4] into an equivalent inequality are the following:

> Suppose that *a* and *b* are (real) numbers such that $a<b$, and *c* is another (real) number different than zero. Then the inequality $a<b$ is equivalent to:
>
> 1. $a+c<b+c$
>
> 2. $ac<bc$ for $c>0$
>
> 3. $ac>bc$ for $c<0$

The first of the above properties reveals that whenever you have an inequality, if you add (or subtract) the same amount to the left and to the right side of it, you will get a

[4]I will use the term *inequality* to describe both *inequations* and *inequalities*. However, when the distinction between the two objects is important, I will refer back to *inequations* and *inequalities*.

new inequality that is equivalent to the initial one. If you multiply an inequality by a positive number, the new inequality is equivalent to the initial one. However, when multiplying an inequality by a negative number, the equivalent inequality will have the direction of the inequality symbol reversed (Zill & Dewar, 2007).

Using property (1) and then (2), the inequality $2x+6>0$ becomes $x>-3$, which can be interpreted as every real number greater than -3 is a solution of the inequality. Very often, the solution of an inequality defined on a real domain has an interval form, as well as a graphical representation. The interval solution of the inequality $2x+6>0$ is $(-3,+\infty)$ and the graphical representation is:

.

Every number greater than -3 will satisfy this inequality. For example, if number 4 is chosen, then: $2(4)+6=14>0$. The inequality holds true. The predicate becomes a true proposition. Every number less than or equal to -3 will make the inequality false. If we try -3, then: $2(-3)+6=0<0$. The specific inequality is false. The predicate becomes a false proposition.

The inequality $x-y<2$ has a different trajectory to finding a solution. In addition, the solution has a strange shape: it is a set of pairs of real numbers that make it true. For example, the pair (1, 2) produces $1-2=-1<2$ (true), therefore it is an element of the solution set. Formally, the solution is given using set notation: $S=\{\,(x,y)/\,y>x-2\,\}$. A graph of the solution accompanies the formal solution of a two-variable linear inequality. The above solution is represented by the graph in Figure

2.1, which shows a dotted line $y = x - 2$ that separates the plane into two regions, neither of which contain the line. The shadowed region, including the line, is not part of the solution set[5]. The points situated in the upper part of the plane will define the solution.

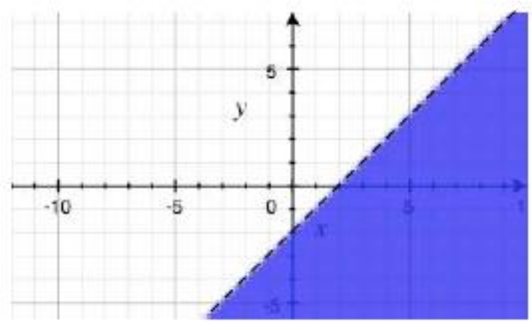


Figure 2.1: Linear inequality in two variables

When the coefficient of the variable is not a known number, the inequality is said to be parametrical. For example $ax < 2 - x$ is a parametric inequality. To solve this inequality, after rewriting it in the simple form $(a+1)x < 2$, one must separately consider three cases: (1) when $a > -1$ the solution is $x > \frac{2}{a+1}$; (2) when $a < -1$ the solution is $x < \frac{2}{a+1}$; and (3) when $a = -1$ the inequality reads $0 < 2$. This statement is true; therefore the inequality is true for all real numbers *x*.

[5] In some books, the convention is to shade the solution region. However, in optimisation problems, it is very important to clearly see the solution region coming from many restrictions, because you have to further use the vertices of the solution region to capture the optimal value. Therefore, I prefer the convention of leaving the solution region blank.

2.1.3 School Methods of Solving Inequalities

When solving linear inequalities, either in secondary or tertiary mathematics, there is not much discussion about what methods to choose. The classical pattern of solving equations, with the amendment that division or multiplication by a negative number results in an inequality with the symbol reversed, is applied. For example, solving $-3x+3\le 6$ results in $-3x\le 3$ which gives the simple-solution-discernable form $x\ge -1$. In interval form, the solution is $[-1,+\infty)$ and the graphical representation of the solution is as follows: 

.

Quadratic, polynomial or rational inequalities can be solved by various methods - three of them are identified and discussed in research on inequalities studies: the graphic, the sign-chart (and the improved sign-chart), and the logical connectives method. Rational inequalities may also be solved by multiplying the inequality by the square of the least common denominator and then using the sign-chart to solve the resulting polynomial inequality.

The graphic method usually consists of creating a function associated with the inequality, graphing the function, comparing the *y* with the *x*-axis (or another *y* in some cases), reading the x values for the appropriate y, and giving the solution (Sackur, 2004). For example, using the graphic method to solve the inequality $\frac{1}{8}x(x+2)(x-3)^3\ge 0$, one must graph the polynomial function $f(x)=\frac{1}{8}x(x+2)(x-3)^3$ (see Figure 2.2) and then read from the graph the values of x for which the graph is above the *x*-axis or touching it. The solution, in interval form, is $(-\infty,-2]\cup\{0\}\cup[3,+\infty)$. The graph of the function used to solve the inequality can be produced either by hand or by using a graphing calculator. Reading the solution from the graph falls entirely on the students' shoulders.

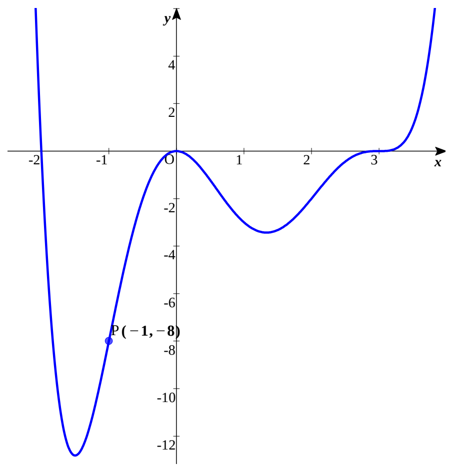


Figure 2.2: The graph of a polynomial function

Abramovich and Ehrlich (2007) recommend a graphic tool that could graph relations, and therefore, inequalities as well. The difference between the previous mode of using functions to solve inequalities and the new approach is that the solution of the proposed inequality is identified by the machine itself and visually represented on the graph. For example, in Figure 2.3, the inequality $x > x^2$ is represented by the graphs of two functions $y = x$ and $y = x^2$. On the x-axis, one can see the interval (0, 1), representing the solution of the inequality, highlighted.

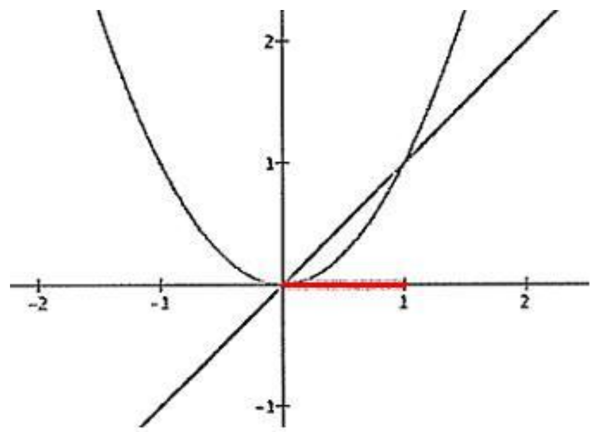


Figure 2.3: Inequality (from Abramovich & Ehrlich, 2007, p.187)

A sign-chart used to solve inequalities consists of finding the intervals where the evaluated expression in the composition of the inequality is either greater than or less than zero. The intervals are bounded by all the zeros of the associated equations (from numerator and denominator in the case of rational inequalities) aligned on a number line. The positive-negative behaviour of the expression is given by the positive-negative number calculated by taking the test values for each interval. For example, solving $\frac{x(x-1)}{(x+2)} \leq 0$ consists in getting the zeros from numerator and denominator – these are -2, 0 and 1 – then writing the associated function $f(x) = \frac{x(x-1)}{(x+2)}$. Test values are taken from each interval determined by locating the zeros on the number line. The values of the function are then calculated for every test value. For the sign chart method, we keep only the sign of the value of the function at each of the test values. In this example, $f(-3) = -12$ (negative), $f(-1) = 2$ (positive), $f(0.5) = -.1$ (negative), and $f(2) = 0.5$

(positive). See Figure 2.4 for the sign chart representing the inequality $\frac{x(x-1)}{(x+2)} \le 0$, from where the solution reads $(-\infty,-2)\cup[0,1]$.

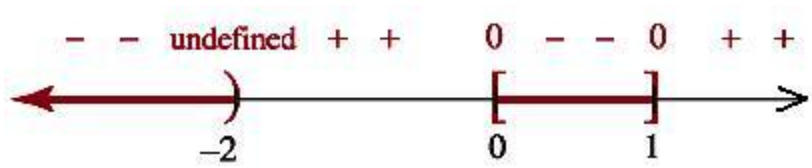


Figure 2.4: Sign chart (McLaurin, 1985)

Calculating the value of the algebraic expression at each point – as in the previous example – could be tedious; therefore, an improved sign-chart method is used more often for the advantage of working with linear components, a method which allows us to calculate the sign of the expression. When using the improved sign-chart (Rivera & Becker, 2004; Zill & Dewar, 2007) to solve the rational inequality $\frac{x(x-1)}{(x+2)} \le 0,$ the work consists of solving for the zero values from the numerator and denominator, and then completing the table. The solution is given by the union of the intervals $(-\infty,-2)\cup[0,1]$.

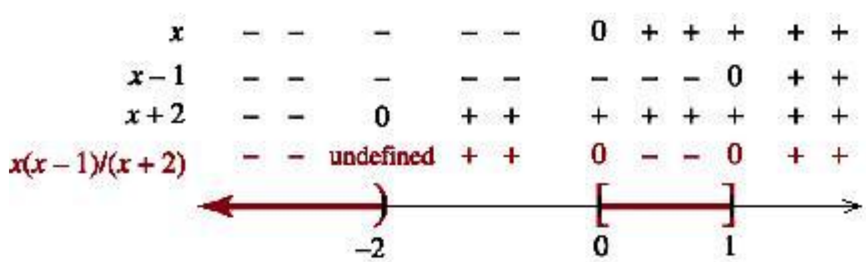


Figure 2.5: The improved sign chart (Zill & Dewar, 2007).

Given a rational inequality of the form $\frac{x}{(x+2)} \leq 0$, multiplying it by the square of the denominator, $(x+2)^2$, the inequality becomes a quadratic one $x(x+2) \leq 0$. This manipulation is allowed for all real numbers except -2. Since the number we multiply with is positive, the inequality keeps the same direction and holds. The advantage of this new form is the multiple range of methods of solving a quadratic inequality: (1) using a sign-chart, (2) a graph, or (3) the leading coefficient. The first two methods have been discussed so far. The leading coefficient uses the fact that the graph of a quadratic function – the parabola – opens up or down, depending on the coefficient of x^2. The inequality $x(x+2) \leq 0$ has a positive leading coefficient, thus, the function is negative on the interval between the zeros. The solution for the quadratic inequality is $[-2, 0]$. However, keeping in mind that -2 is not in the domain of the initial inequality, the solution becomes $(-2, 0]$.

The 'logical connectives' approach to solving an inequality – for example $\frac{x}{(x+2)} \leq 0$ – consists in using the structure of the fraction and logically analyzing all the possibilities of getting numerical values that are less than or equal to zero for the given fraction: $\frac{x}{(x+2)} \leq 0 \Leftrightarrow ((x \leq 0 \; and \; x+2>0) \; or \; (x \geq 0 \; and \; x+2<0))$. This is true when $(x \leq 0 \; and \; x>-2) \; or \; (x \geq 0 \; and \; x<-2)$. The first bracket results in solution $(-2, 0]$ and the second one is ϕ (the empty set). The logical connective or placed in between the two solution parts translates into the union of the two solutions: $(-2, 0] \cup \phi$. Therefore, the solution of the given inequality is $(-2, 0]$.

2.1.4 Famous Inequalities

The inequality $(a-2)^2 \geq 0$ is different from the inequalities presented and solved in the previous section. There is nothing to solve here. This inequality presents a universal truth. In other words, the square $(a-2)^2$ is greater than or equal to zero for every real number a (Postelnicu & Coatu, 1980). In the same category as this inequality, we can find many famous inequalities worth mentioning:

- the absolute value inequality: $\| a | - | b \| \leq | a + b |$;
- the triangle inequality: $| a + b | \leq | a | + | b |$;
- the generalized triangle inequality:

  $| a_1 + a_2 + a_3 + ...a_n | \leq | a_1 | + | a_2 | + | a_3 | + ... | a_n |$;
- the inequality of the arithmetic and geometric means:

  $$ab \leq \left(\frac{a+b}{2}\right)^2, \quad \sqrt{ab} \leq \frac{a+b}{2}, \text{ when } a \geq 0, b \geq 0;$$
- the generalized inequality of the arithmetic and geometric means:

  $$\sqrt[n]{a_1 a_2 ... a_n} \leq \frac{a_1 + a_2 + ...a_n}{2}, \quad \text{when } a_1, a_2, ...a_n \geq 0;$$
- the isoperimetric inequality in the plane $4\pi A \leq L^2$, where A is the area and L is the perimeter of a geometric shape;
- the Cauchy-Schwarz inequality for real numbers:

  $$| a_1 b_1 + a_2 b_2 + a_3 b_3 + ...a_n b_n | \leq \sqrt{a_1^2 + a_2^2 + a_3^2 + ...a_n^2} \sqrt{b_1^2 + b_2^2 + b_3^2 + ...b_n^2}$$

  and when $n = 2$, dropping the subscript and taking a, b, x, y as four real numbers and using the triangle inequality, the inequality becomes:

  $$| ax + by | \leq | a \| x | + | b \| y | \leq \left(\sqrt{a^2 + b^2}\right)\left(\sqrt{x^2 + y^2}\right);$$
- the Cauchy-Schwarz inequality using sigma notation: $\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$

  .

Fink (2000) distinguished between two types of inequalities in the category of universal truths: ad hoc and general inequalities. He used the name ad-hoc inequalities to describe the tools someone needs when proving a result. For example, $-1 \le \cos x \le 1$ is an ad-hoc inequality that one will build upon when proving $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ using the squeeze rule. The inequality that bears the name 'Polya's dream' $1 + x \le e^x$ is another ad-hoc inequality which was used to prove the arithmetic mean-geometric mean inequality (Steele, 2004). General inequalities are famous inequalities that can stand alone as mathematical results or are frequently used in estimates of numbers, functions, or integrals. Examples of general inequalities include the inequalities of the means or the Cauchy-Schwarz inequality (Fink, 2000). The next section provides a sneak peek into the mathematician's laboratory, where the 'loved' Cauchy-Schwarz inequality is followed through different proofs.

2.1.5 A Mathematician's Approach to Inequalities: A 'Loved' Inequality

"Every mathematician loves an inequality," is often heard in mathematics circles. What do they mean by that? Does the mathematician like the form of the inequality? Does the mathematician love the aesthetics of the symbolic shape of the inequality? Does he/she embrace the practical aspect of an inequality? Does he/she like what the inequality stands for? Or does the mathematician love the elegant proof of an inequality?

Many mathematicians love the Cauchy-Schwarz inequality (Steele, 2004). The inequality does look quite interesting: $|ax + by| \le |a||x| + |b||y| \le \left(\sqrt{a^2 + b^2}\right)\left(\sqrt{x^2 + y^2}\right)$. It expresses a relationship between four quantities. The way the variables are taken away from the initial cluster and paired to create the last part of the inequality reflects power. It must express an important relationship. It forces one to think and see more than just four numbers that are operated and compared. Is there a use or an application for this inequality? For a taste of the work and a glimpse of the aesthetics of inequalities, let us follow the Cauchy-Schwarz inequality through several proofs. The first two proofs are adapted from Steele (2004) and the third one comes from Nelsen (2003).

Proof 1, algebraic: Using the properties of absolute value, one can easily prove the first of the two simultaneous inequalities: $|ax+by| \le |ax| + |by| = |a||x| + |b||y|$.

The second inequality, (let's label it (2)) $|a||x| + |b||y| \le \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right)$, especially if one looks at it as being the sum of two different quantities, can be seen as originating from the inequality of the means $ab \le \left(\frac{a+b}{2}\right)^2, \quad when\ a \ge 0, b \ge 0$. So, let's play a bit with the inequality of the means, using all four variables that are in our inequality to be proved:

Thus, $|abxy|$ can be seen as the inequality of the mean for *ay* and *bx*: $|abxy| \le \left(\frac{ay+bx}{2}\right)^2 = \frac{a^2y^2 + 2|abxy| + b^2x^2}{4}$. Leaving out the intermediary step, we can write:

$|abxy| \le \frac{a^2y^2 + 2|abxy| + b^2x^2}{4}$. Multiplying all by 4 we get:

$4|abxy| \le a^2y^2 + 2|abxy| + b^2x^2$. Subtract $2|abxy|$ from both sides we get:

$2|abxy| \le a^2y^2 + b^2x^2$ (*)

But: $\left(a^2+b^2\right)\left(x^2+y^2\right) = a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$ (**)

Squaring the left side of (2) and using (*) we get:

$$\left(|a||x| + |b||y|\right)^2 = a^2x^2 + 2|abxy| + b^2y^2 = a^2x^2 + b^2y^2 + 2|abxy| \le$$

$$\le a^2x^2 + b^2y^2 + a^2y^2 + b^2x^2$$

Keeping only the extreme members of the above inequality, we have:

$\left(|a||x| + |b||y|\right)^2 \le a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$. Using (**), this becomes:

$$\left(|a||x| + |b||y|\right)^2 \le \left(a^2+b^2\right)\left(x^2+y^2\right).$$

Taking square root on both sides, we get:

$$\therefore \quad |a||x| + |b||y| \le \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right).$$

For simplicity of the proof, I could have restricted the four numbers to be positive and dropped the absolute value.

Proof 2, algebraic: Let's start again with the second part of the inequality: $|a||x| + |b||y| \le \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right)$. Squaring both sides of this inequality we get an equivalent form: $\left(|a||x| + |b||y|\right)^2 \le \left(a^2+b^2\right)\left(x^2+y^2\right)$. Expanding both sides, we get: $a^2x^2 + 2abxy + b^2y^2 \le a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$. Cancelling the terms that are on both sides and collecting the like terms on one side, we get:

$0 \le a^2y^2 - 2abxy + b^2x^2$. The right side of this inequality is a perfect square:

$0 \leq (ay - bx)^2$ This is true. Following the chain of equivalences, the last inequality being true, all other inequalities will be true, including the one that we have to prove (Steele, 2004).

The difference between the two algebraic proofs is that in the first proof, the work started with a true inequality (a general inequality, more precisely) to build on all the other equivalent forms up to the one that must be proved. In the second proof, work was done on the inequality that needed to be proved up to the point where a general true inequality could be seen. The last inequality being a recognized truth, all the other equivalent ones were true.

Proof 3, a proof without words: Nelsen ingeniously proved the Cauchy-Schwarz inequality, by using a tiling and the fact that a parallelogram has an area smaller than the area of a rectangle whose sides are equal to the sides of the parallelogram. He claims that a mathematically inclined person will see mathematics everywhere. Moreover, in the tilings made by artisans and preserved over the centuries either in old buildings or in paintings, a trained mind could identify proofs of well-known theorems. There is no surprise, therefore, in seeing the Pythagorean Theorem as well as many other theorems proved by expressing relationships between different areas in tilings. However, there is a big surprise to see tiling proofs of some very famous inequalities, like the inequality of the means or Cauchy-Schwarz inequality. Thus, inspired by the tiling found in The Courtyard of a House in Delft, by Pieter de Hooch, changing the size of the tiles, Nelsen 'sees' without words the proof of the Cauchy-Schwarz inequality (Nelsen, 1997).

The tiling shows how the sum of the areas of the two different rectangles, with sides of $|a|$, $|x|$ and $|b|$, $|y|$ respectively, is the same as the area of the parallelogram with sides $\sqrt{a^2+b^2}$ and $\sqrt{x^2+y^2}$. A new rectangle with sides equal to the sides of the parallelogram, $\sqrt{a^2+b^2}$ and $\sqrt{x^2+y^2}$, is introduced in the picture. The area of this new rectangle is greater than (or equal to) the area of the parallelogram, which proves the inequality $|a||x|+|b||y| \le \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right)$.

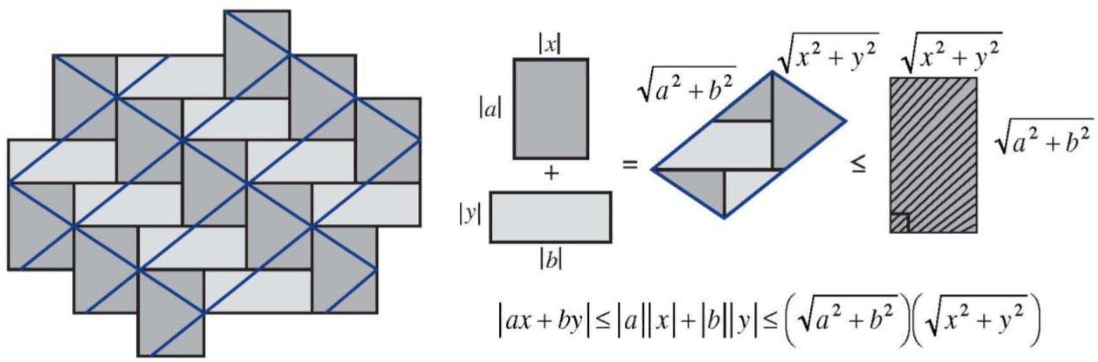


Figure 2.6: Proof without words (from Nelsen, 2003, p.8)

The repertoire of proofs for the above inequality is impressive the Cauchy-Schwarz inequality is the basis for many concepts in many areas of mathematics from which algebra, geometry, trigonometry, calculus, linear algebra, or probability can be easily mentioned. Three proofs were presented here with a double agenda: (1) the aesthetic aspect of each proof and (2) to give a taste of the work involved in manipulating inequalities.

Reflecting on the philosophy of manipulating inequalities, Hardy stated that "an inequality that is elementary should be given an elementary proof, the proof "should" be inside the theory it belongs to, and finally the proof should try to settle the case of equality" (Fink, 2000, p.129). It is very well known in mathematics that the work is not over when you put Q.E.D. at the end of a proof. Looking back for extensions or for a more simple and elegant proof is a step equal in importance to solving the problem within the given conditions (Polya, 1957). Therefore, mathematicians are first looking for inequalities, then they are trying to prove them, and, after proving an inequality, in many cases, they are probing its utility.

In summary, the work on inequalities is varied and rewarding. As could be experienced from this section, the many variations of one inequality, its many proofs and applications could be enough material for a complete study. This section gave only a taste of mathematicians' work on inequalities, using the Cauchy-Schwarz inequality as an example. The following section is dedicated to the major events that are marking the history of the inequality, from Antiquity to the beginning of the 20th Century.

2.2 How Did Inequalities Come to Be?

> If we wish to foresee the future of mathematics our proper course is to study the history and present condition of the science. (Henri Poincaré as quoted in Kline, 1972, p.vii)
>
> Mathematics begins with Inequality. (Tanner, 1962)

Mathematicians as well as mathematics educators can attest to the importance of inequalities in the study of mathematics. Considered building blocks for many mathematical areas, inequalities are studied for three particular reasons: practical, theoretical and aesthetic (Bellman, in Mitrinović, Pečarić, & Fink, 1991). At the practical level, from the early involvement with problem solving, one must learn to bind variables or to write restrictions for the unknowns - and these are expressed as inequalities (Bagni, 2005). In theory, inequalities are used to express the domain of a function, to prove limits, to set up research questions that relate equations to special cases that are inequalities or to prove equality by means of inequalities (Burn, 2005). Moreover, mathematicians testify that there are many aesthetic aspects in inequalities, as well as in some of their proofs (Mitrinovic et al., 1991). Aesthetics in mathematics encompasses an appreciation of the beauty, elegance and significance of mathematical entities (Sinclair, 2004); a generation of harmonious and permanent patterns (Sinclair, 2004; Hardy, 1940/1992); a perseverance in continuing a journey even when one is lost in misleading propositions (Zeitz, 1999; Papert, 1980). All these aspects are present in the work of a mathematician concerned with inequalities.

If only for the three mentioned reasons, one could expect to see an abundance of literature related to the history of inequalities. However, not many History of Mathematics books have inequalities in the index of topics. In Mathematical Thought from Ancient to Modern Times, by Morris Kline (1972), for example, there is no such entry. In Boyer's (1968) A History of Mathematics, there is one entry on inequalities – Inequality, Bernoulli, 416 (p.704) – whereas, for equations, there are more than fifty entries. Why? Are inequalities so insignificant in the mathematics context to be completely ignored when writing a history of mathematics thinking book? Is this because inequalities are solely the product of contemporary mathematical thought? Inequalities are absent from the index of history of mathematics books that cover ancient mathematics, or even modern mathematics, and become present in the history of mathematics books that focus on more recent mathematical thought. Were inequalities foreign to ancient mathematicians? The present chapter will shade some light on these questions.

This following sections attempt to illustrate the importance of inequalities to mathematics by looking at the historical development of inequalities. Ariadne's thread in section 2.2.1 deals with the perpetual question: Why do mathematics educators and mathematics education researchers engage in checking the history of mathematics? Section 2.2.2 takes on the challenge of following inequalities throughout history, starting with Antiquity and then continuing to the Modern mathematics era when Algebra flourished. The implicit questions when presenting the most important developmental stages of inequalities are: What helped inequalities get noticed in the old mathematics texts? Could inequalities have evolved from Algebra?

2.2.1 Why Educators and Researchers in Education Are Concerned with the History of Mathematics

> It is useful to study the history of [a] concept to locate periods of slow development and the difficulties which arose which may indicate the presence of epistemological obstacles. (Cornu, 1991, p.159)

When the teaching, learning, or understanding of a concept encounters problems, there is a tradition in mathematics education research to turn the search for the answer to the problem toward the history of the concept (Cornu, 1991). In the development of the concept, one may find information about periods of slow development. There could be an indication somewhere that the concept had been creating problems to mathematicians first. As it is well known, Hippasus probably died for discovering irrational numbers.

Even if the Greek mathematicians had previously experienced incommensurability, they had problems accepting that the world is not fully explained by whole numbers and the relationship between them (Kline, 1972). Such incidents inform us about epistemological obstacles associated with irrational numbers.

Epistemological obstacles are a "way to interpret some of the recurrent and non-aleatorical mistakes that students make when they learn a specific topic" (Radford, 1997, p.8). Radford follows Brousseau's classification of recurrent mistakes in order to have a clear picture of epistemological obstacles:

> (1) an *ontogenetic source* (related to the students' own cognitive capacities, according to their development);
>
> (2) a *didactic source* (related to the teaching choices);
>
> (3) an *epistemological source* (related to knowledge itself) (Radford, 1997, p.9).

Epistemological obstacles could be identified for the concept under observation by confronting the obstacles found in historical texts with the recurring mistakes our students are making when working on that concept. Teaching a concept linked to epistemological obstacles and being aware of that, the educator could plan when and how it would be more appropriate to introduce that concept to the students to avoid unnecessary hardship. Also, the teacher can work on training the students to practice that concept even when the meaning is obscured and manipulation mistakes are persistent (Sfard, 1995).

Several studies indicate that the history of mathematics is valued not for detecting epistemological obstacles only, but for informing the teaching of mathematics as well as research in mathematics education in various other ways (Bagni, 2005; Burn, 2005). Historical anecdotes used by many teachers as motivators for the study of a concept are viewed as a naïve but powerful approach of history to didactics (Radford, 1997). Recapitulation, or presenting topics through their historical development, is another way of using the history of mathematics in class, since the method essentially sets the stage for the students to recreate the concept (Radford, 1997). Moreover, similarities between the historical development of a concept and its cognitive growth have been observed (Bagni, 2005). For example, Harper (1987) used the history of algebra to predict young students' intuitive algebraic thinking. The task was to solve a Diophantine problem requiring students to find two numbers when their sum and their difference were known. Nearly half of his participants used rhetorical or syncopated algebraic arguments to solve the problem. The Vietan symbolic algebra was the least popular method for solving the problem, since the students were unable to use the parametric setting to obtain the general solution. The argument here is that, historically, from rhetorical to symbolic algebra, there were centuries of new accumulations in mathematics. In parallel, didactically, students experience a huge jump in understanding and manipulating algebraic concepts when going from simple equations to parametric ones (Sfard, 1995).

Burn (2005) argues that the historical development of a mathematical concept "can reveal actual steps of success in learning" (p.271). This exploration could be applied when the research in education reveals that the understanding of a concept is not "consonant with students' intuitions" (p.271), which seems to be the case with inequalities (Bazzini & Tsamir, 2003). Burn uses the history of mathematics to fill the gap between a "pre-Zeno mindset to Weierstrassian viewpoint" (p.271) relative to the limits concept. The history of a mathematical concept has been successfully used as a longitudinal study of learning. What is very interesting for my study is not only the general idea of using the history of mathematics to improve teaching and learning, but also Burn's findings that "[p]roofs of equality by means of inequalities precede the notion of limit" (Burn, 2005, p.1). Exposing his students to the classical Greek mathematicians' use of inequalities, in a pre-real numbers context, to help them see the power of inequalities in a more intuitive and, at the same time, rigorous way, Burn (2005) argues that the understanding of the rigorous proof of limits improved. For the purpose of my study, Burn's findings are twofold: (1) they motivate my search into the history of mathematics for informing teaching and (2) they provide leads for further digging for inequalities in the history of mathematics.

2.2.2 The Evolution of the Inequality Concept

As a discipline of study, inequalities do not have a long history. As a mathematical concept, however, they were not foreign at all to ancient mathematicians (Bagni, 2005). The ancients knew "the triangle inequality as a geometric fact" (Fink, 2000, p.120). They were also aware of the arithmetic-geometric mean inequality, as well as the "isoperimetric inequality in the plane" (Fink, 2000)[6]. Euclid used words 'alike exceed', 'alike fall short' or 'alike are in excess of' to compare magnitudes (Kline, 1972, p.69). The definition, "The greater is a **multiple** of the less when it is measured by the less," (Katz, 2009, p.74) shows that the mathematicians of ancient times were adept at comparing magnitudes and expressing the relationship between them.

6 See section "What is an (algebraic) inequality?" for mathematical details on these inequalities.

Inequalities have been assisting mathematical discoveries from Classical Greek Geometry to Modern Calculus and it took two millennia to change the status of inequalities from mere support for some mathematics to *Inequalities* as a discipline of study (Fink, 2000). Today, there are two journals of inequalities – *The Journal of Inequalities and Applications*[7] and *The Journal of Inequalities in Pure and Applied Mathematics*[8] – as well as many other mathematics publications that print papers with the "sole purpose […] to prove an inequality" (Fink, 2000, p.118). The path that inequalities followed from Antiquity to the end of the second millennium will be investigated in the following sections.

2.2.2.1 Inequalities in Antiquity

> In his *Treatise of Algebra* (1685) Wallis acknowledged his debt to Euclid in relation to limiting arguments. (Burn, 2005, p.291)

> Here lies the key to the relationship between equality and inequality in mathematics, between its poetry and its prose. Mathematics is founded on inequality, the commonest thing in the world. But the kind that mathematicians most pride themselves on, finished mathematics, mathematics for show to the public, is presented as much as possible *in equality form*. (Tanner, 1962, p.164)

Greek mathematicians were profoundly aware of "the power of inequalities to obtain equality" (Burn, 2005, p. 271). Burn employs a metaphor – called "the vice" – to describe the properties of inequalities that Ancient mathematicians used to help produce equality. The vice is similar to what we call the squeeze rule in calculus. It consists of the

7 Journal of Inequalities and Applications (JIA) issued its first volume in 1997. JIA is a multi-disciplinary forum of discussion in mathematics and its applications in which inequalities are highlighted.

8 Journal of Inequalities in Pure and Applied Mathematics (JIPAM), founded in 1999 by the Victoria University in Australia members of the Research Group in Mathematical Inequalities and Applications (RGMIA).

following argument: when a number A is squeezed in between two small quantities, $-\varepsilon < A < \varepsilon$, for all positive numbers ε, then the number $A = 0$. The inequality $-\varepsilon < A < \varepsilon$ works as a carpenter's vice, compressing the inner quantity as much as to leave room for only one number in between $\pm\varepsilon$, and that number is zero. The vice could be used to squeeze a difference of two numbers: $-\varepsilon < A - c < \varepsilon$, and to prove the equality $A = c$ (Burn, 2005).

Greek mathematicians were not aware of the existence of negative numbers. However, they seemed to have used the vice. Here is an account of the vice in Euclid's *Elements*:

> Two unequal magnitudes being set out, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, then there will be left some magnitude less than the lesser magnitude set out. And the theorem can similarly be proven even if the parts subtracted are halves. (Katz, 2009, p.82)

The proposition states that by taking a small quantity, compared to a bigger one $(\varepsilon < B)$, one can get a smaller quantity than the smaller of the two initial quantities by successively subtracting halves from the big one. Thus, the inequality becomes $B/2^n < \varepsilon.$ This can be turned into $B < 2^n\varepsilon$, an inequality which signifies that a multiple of ε exceeds B (Burn, 2005). Classical Greeks as well as Archimedes used the potential of this inequality to calculate the volume of a pyramid as one third of the area of the base times the height (Burn, 2005), and many other similar results.

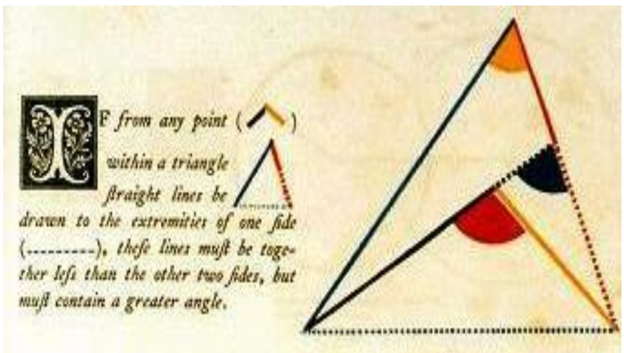
Archimedes also used the method of exhaustion to calculate many important results on areas and volumes. The method of calculating π consists of filling the area of a circle with a polygon of a greater and greater number of sides (see Table 2.1). The ratio of the area of the polygon and the square of the radius can be made arbitrarily close to the actual value of π as the number of the sides of the polygon increases (Smith, 1958). Using a 96-sided polygon, Archimedes could get the value of the ratio between the circumference and the diameter of a circle falling in between these two fractions: $\frac{223}{71} < \pi < \frac{22}{7}$ (Boyer, 1968). Working on π and on calculations for approximating the square roots of numbers, Archimedes was in fact manipulating inequalities arithmetically (Fink, 2000).

Euclid's *Elements* abounds in propositions that express inequality relationships between angles, sides, perimeters, or areas. However, there is no account of using inequalities in arithmetic or numbers manipulation (Fink, 2000). The contemporary translation of Euclid's words uses the inequality symbols to help the reader understand the old text, but those symbols were foreign to Euclid. The modern reader needs the symbols alongside with the text to fully see inequalities in Euclid's work. In the Pickering version of Euclid's *Elements* (Byrne, 1847), for example, the symbols introduced by Oughtred[9] are used to write geometric inequalities. Proposition XXI from Euclid's Book One is chosen to exemplify some of the inequalities well known in Antiquity. Figure 2.7b represents one page from Byrne's (1847) *The First Six Books of the Elements of Euclid*. In this edition of Euclid's works, aside from inequality symbols, Byrne uses colours to make the book attractive and appealing to students. The colours also helped the proofs that were presented as pictures. Figure 2.7a represents Proposition XXI from Euclid's Book One. In plain language, the proposition reads:

[9] See Section 2.2.2.3 for a picture of Oughtred's symbols for inequalities.

> If from the ends of one of the sides of a triangle, two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides (Joyce, 1996-1998).

Figure 2.7b is the pictorial proof of the same proposition.



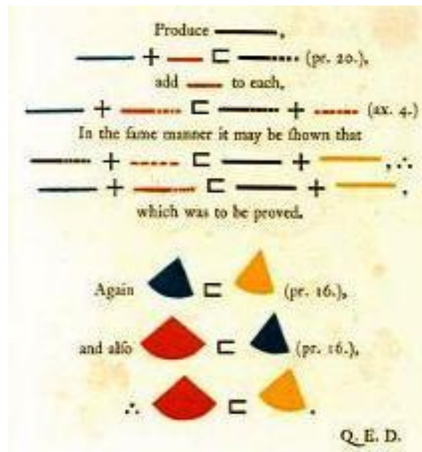
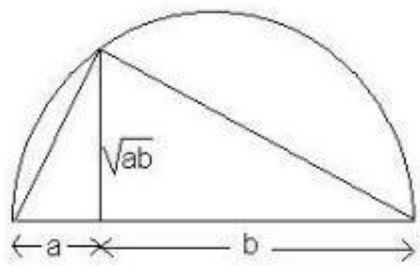


Figure 2.7a: Euclid, Proposition XXI (from Byrne, 1847, p.21)

Figure 2.7b: Proposition XXI, pictorial proof

The means inequality $\sqrt{ab} \le \frac{a+b}{2}$ as well as its proof can be seen in the next figures, 2.8a and 2.8b, respectively. The proof is based on the result that the height of a right triangle is the geometric mean of the segments that it divides the hypotenuse into. This proof of the inequality of the means looks as Euclid might have imagined (Steele, 2004). Figure 2.8a shows that the height of the right triangle is the geometric mean of the projections of the legs over the hypotenuse: $h = \sqrt{ab}$. From Figure 2.8b, the radius can be seen as the highest of the all projections from points on the circle over the diameter, thus proving the inequality $\sqrt{ab} \le \frac{a+b}{2}$.



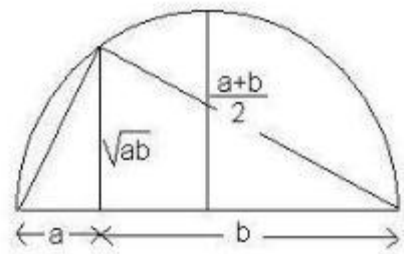


Figure 2.8a: The geometric mean

Figure 2.8b: The arithmetic and the geometric means

Burn's (2005) account summarizes the old history of inequalities and projects the importance of Classical work on inequalities for the further development of mathematics:

> Using inequalities to measure awkward quantities dates back to Euclid and beyond. Archimedes in particular was skilled in using inequalities to deduce equalities, and after translating his method into algebra, such proofs were used by Fermat (1636) and are accessible to undergraduates today. (p.271)

Arabic mathematicians understood the work of the Greeks and proved similar results on the volume of solids (Katz, 2009). They were also skillful in manipulating inequalities in approximations using continued fractions (Fink, 2000). Geometry, Arithmetic, and Number Theory were well established mathematics disciplines in Antiquity. However, inequalities were not recognized as sole mathematics concepts; they were only considered as peculiar tools used to develop other theories.

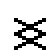
2.2.2.2 Inequalities in the Middle Years or the Development of Algebra

In the history of algebra, three developmental stages are identified: rhetorical algebra, syncopated algebra, and symbolic algebra. This division is due to Nesselmann, based on the notion of mathematical abstraction (Radford, 1997). Rhetorical algebra is the algebra of words. Syncopated algebra, the algebra of the 15th and 16th Centuries, uses a mixture of words and symbols to express generalities. This is the algebra of Pacioli, Cardan, and Diophantus. It is Francois Viete who introduced species and made the distinction between a constant and a variable, both being represented by letters. Viete was the first to solve parametric equations (Bagni, 2005; Sfard, 1995). Before Viete, algebra was at an operational level. As a result of Viete's contributions, equations became the objects of higher-order processes (Sfard, 1995). Viete purified algebra from all the clutter of words and presented it in abstract form, the encapsulation of a pure mathematics idea (Radford, 1997). From Viete onwards, structural algebra got its place in the history of mathematics. The structure in algebra influenced geometry. The works of Descartes and Fermat, on the shoulders of Viete, helped geometry capture generality and express operational ideas. In its early years, algebra needed geometry for reification and verification; now, geometry was using algebra for new reifications and new development (Sfard, 1995). The Middle Ages were a period of great accomplishments for algebra. A reader looking for inequalities under algebra would be surprised to discover that inequalities are missing from the picture. Algebra metamorphosed from words to symbols. Equations or identities were transformed from heavy paragraphs to delicate formulae. Inequalities, however, seem to have been left behind, forgotten, abandoned, and seemingly having no real use in the development of algebra. However, the next section shows that the development of algebra might have influenced the mathematicians to coin a symbol that have permitted inequalities to come along and evolve.

2.2.2.3 The History of the Inequality Symbol(s)

> It may be hard to believe, but for two millennia – up to the sixteenth century – mathematicians got by without a symbol for equality. They had symbols for numbers and operators – just not one for equality. (Lakoff & Núñez, 2000, p.376)

If we can imagine volumes of mathematics developed throughout the centuries without the use of the equal sign, why would it be difficult to think of inequalities being employed or produced without the use of any special symbol? This section attempts an answer to the question: When in the history of inequalities was a symbol for inequality coined, how was the symbol used and how did the symbol influence the evolution of the inequality concept?

This dissertation has already used the inequality symbols – those that are now universally accepted in mathematics literature – and they are: < for *less than*, > for *greater than*, ≤ for *less than or equal to*, ≥ for *greater than or equal to*, and ≠ for *not equal to*. In the previous section, it was also pointed out that, at some point in the history, mathematicians using inequalities in their work adopted the symbols suggested by Oughtred, and which are:  for *greater than* and  for *less than.* It was only in the 17th Century that either one of the two types of symbols for inequality came into being. Tanner (1961) remarked that inequality, one of "the deepest-lying of the basic notions was the last to be symbolized (p.294)." "At the divide between dearth and proliferation [of inequalities] stand Harriot's inequality signs" (Tanner, 1962, p.165). In *An Introduction to the History of Mathematics,* Eves (1969) documents that the symbols < and > were first introduced in mathematics-related texts by Thomas Harriot. Harriot was a mathematician who worked for Sir Walter Raleigh as the cartographer of Virginia, North Carolina today. It is said that Harriot got inspired by the symbol  on the arm of a Native American in coining the symbols for inequalities (Johnson, 1994, p.144). The account states that Harriot decomposed the Native symbol into the two well-known symbols $<$ and $>$. Tanner (1962) argues that the origin of the symbols is less mystical than that. She argues that the inequality symbols are modifications of the equal sign, a symbol which was coined by Recorde as two horizontal, parallel and equal lines, to represent that what is on one side of the sign is exactly the same as what is on the left side of it. Tanner (1962) indicates that, when producing the inequality signs, Harriot "took the equality in Recorde's sign to reside not in the two lengths, but in the unvarying distance between the two parallels" (p.166). According to Tanner (1962), Harriot modified the distance between the two lines of the equal sign, to show that the biggest quantity lies on the side of the biggest distance between the lines.

Harriot used $<$ to represent that the first quantity is *less than* the second quantity and $>$ to represent that the first quantity is *greater than* the second quantity (Johnson, 1994). "The symbol for 'greater than' is $>$ so that $a > b$ will signify that a is greater than b. The symbol for 'less than' is $<$ so that $a < b$ will signify that a is less than b" (Seltman & Goulding, 2007, p.33). Harriot was familiar with the symbolical reasoning introduced by Viete and, moreover, he transformed Viete's algebra into a modern form (Katz, 2009). Harriot simplified Viete's notations to the point at which even a novice in the history of mathematics could understand his formulae, whereas one needs an index of notations to understand Viete's work. Eves (1969) considers Harriot to be the founder of "the English School of Algebraists" (p.249).

Harriot first used the symbol of inequality to transcribe the well-known inequalities of the means and then, he used the inequalities in his work to solve equations.

Here are some excerpts from *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas*, Harriot's posthumously published work.

Lemma 1

> If a quantity be divided into two unequal parts, the square of half the total is greater than the product of the two unequal parts.
>
> If p and q are two unequal parts of the magnitude, then it is true that

$$\left.\begin{array}{c}\dfrac{p+q}{2}\\[4pt] \dfrac{p+q}{2}\end{array}\right| > \; pq$$

(Seltman & Goulding, 2007, p.96)

Transcribed, the above inequality reads: $\left(\frac{p+q}{2}\right)^2 > pq$.

Lemma 2

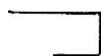
> If three quantities are in continued proportion, the sum of the extremes is greater than twice the middle.
>
> Suppose b, c and d are in continued proportion; then it is true that $b+d > 2c$. (Seltman & Goulding, 2007, p.96)

More lemmas and propositions follow in the text. The propositions use inequalities to solve equations. For example:

Proposition 5

> The ordinary equation $aaa - 3bba = +2ccc$ in which $c > b$, is explicable by a single root. (Seltman & Goulding, 2007, p.100)

The inequality here imposes conditions on the coefficients of the given equation, which in turn helps prove that the cubic equation has only one solution.

The sample of Harriot’s work shown above may lead to a simple conclusion – that once they were coined and it was shown how they work, the inequality symbols became well established and were easily adopted. However, history shows that the mathematics community did not adopt Harriot’s symbols immediately, possibly because Harriot did not publish his work or perhaps because at the same time, in 1631, Oughtred suggested  for *greater than* and  for *less than*. Oughtred's *Clavis Mathematicae* was more popular than *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas* (The Analytical Arts Applied to Solving Algebraic Equations), Harriot’s posthumously published work (Eves, 1969). Therefore, for more than a hundred years, Oughtred's symbols were more often used than Harriot’s symbols for inequalities. Figure 2.7 b shows Oughtred's symbols at work in Pickering’s edition of Euclid’s *Elements* (Byrne, 1847, p.21). Cajori (1928-29) mentions that Oughtred's inequality symbols were hard to remember, prompting many variations of the symbols to be circulated in the literature. Oughtred himself used  for < and  for > in some parts of his work.

Many other derivations of Oughtred's symbols, as well as personal notations or improvised typewriting signs, were used to signal inequalities in the 17th and 18th centuries (Cajori, 1928-29). In the 18th century, the < and > signs finally made their way into Continental Europe (Cajori, 1928-29). Moreover, in 1734, the French geodesist Pierre Bouguer invented the symbols ≤ and ≥, to represent *less than/greater than or equal to*, respectively. These new symbols were used to “represent inequalities on the continent” (Smith, 1958, p.413). More precisely, the < symbol is used to represent quantities that are different, the first one being less than the second one. The $\leq$ symbol incorporates the equality as well; it allows the first magnitude to be equal to the second one.

It is well known that long before the appearance of symbolic algebra, people wrote all arguments in longhand. There were no symbols to represent the unknowns and there were no symbols to represent the relationship between unknowns as well. That was before Diophantus, during the ‘Rhetorical algebra’ stage (Harper, 1987). Writing mathematical statements in plain language is by no means incorrect. However, it may take several pages to describe a statement in plain language, while expressing the same statement in mathematical symbols could even take a single line. It is amazing how much the Greek mathematicians could accomplish by using rhetorical means of expressing inequalities and geometrical embodiments. Symbolic algebra produced the tools for a new embodiment of ideas and for inequalities a representation that is more abstract and specific. Moreover, the use of symbols allows for more work to be performed in a shorter time. Thus, the inequality symbol allowed for compression and aesthetic presentation of many old inequalities and spurred the development of a concept from a mere peculiarity. Radford (2006) argues that algebraic symbolism is “a metaphoric machine itself encompassed by a new general abstract form of representation and by the Renaissance technological concept of efficiency” (p.1). Efficiency helped algebra prosper, while Harriot’s inequality signs stimulated the proliferation of inequalities (Tanner, 1962).

2.2.2.4 A Discipline Named Inequalities

The 18th Century is marked, in addition to a new symbol for inequalities, by the emergence of named inequalities. A new meaning is attached to the name of an inequality: the names are not a description of what the inequalities encompass, such as the ‘inequality of the means’. In contrast, inequalities are assigned the names of the mathematicians who either discovered or proved them for the first time (e.g., ‘Cauchy-Schwarz inequality’). Newton’s name is attached to an inequality. Cauchy’s name is attached to an inequality, as well as to different proofs of other famous inequalities, due to his extensive use of inequalities in his work on limits and series (Fink, 2000). Holder, Minkowski, Hadamard, and Hardy are other mathematicians who were honoured by the use of their names to qualify different inequalities. It is important to mention that Hardy’s work has been much more significant than one inequality: Hardy could be named ‘the father of the Discipline of Inequalities’. He was the founder of the *Journal of the London Mathematical Society,* a proper publication for many papers on inequalities. In addition, together with Littlewood and Polya, Hardy was the editor of the volume *Inequalities,* a book that was the first monograph on inequalities. The work on the book started in 1929 and it was issued in 1934. The authors confessed that the historical and bibliographical accounts are difficult “in a subject like this, which has applications in every part of mathematics but has never been developed systematically” (Hardy, Littlewood, & Polya, 1934, p.v). Their contribution was to track down, document, solve and carefully present a volume comprising of 408 inequalities, and to officially write the first page of the history of inequalities. A question arises here: Why has the concept of inequalities been disregarded for almost two millennia before it was considered worth of special attention and developed systematically? Long before Hardy, mathematicians knew the power and importance of inequalities, since they used inequalities as tools in developing Geometry and Calculus. However, before Hardy, inequalities did not get special attention from mathematicians – nobody took the pains to introduce them to the mathematics community as a mathematical concept rather than as a simple tool used to serve other concepts. Linear or quadratic equations, for example, were studied as independent concepts by Babylonian mathematicians. Seventeen centuries before inequalities were noticed as independent mathematical concepts, Diophantus of Alexandria developed the theory of equations. It is interesting that, before Hardy, no other mathematician devoted a special interest to inequalities throughout the history of mathematics. Hardy himself attested, in his Presidential Address to the London Mathematical Society in 1928, that even though inequalities had been intensively used by analysts, there was no coherent reference to the concept.

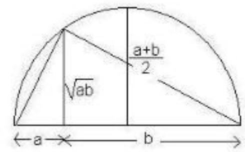
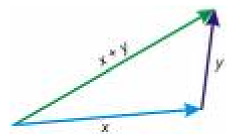
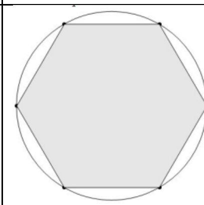
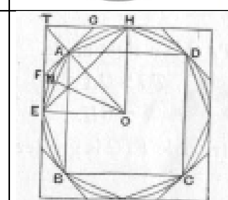
> The elementary inequalities thus form the subject-matter of one of the first fundamental chapters in the theory of functions. But this chapter has never been properly written; the subject is one of which it is impossible to find a really scientific or coherent account. I think that it was Harald Bohr who remarked to me that "all analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove". I will give a few examples. (Hardy, 1929, p.64)

Hardy continues by giving examples of some of the well known and used inequalities at that time, such as the inequality of the means.

Half a century after Hardy, other tenacious mathematicians took the challenge of documenting *all* known inequalities. The product of this new collaborative work, under the supervision of Mitrinović, is a five-volume of inequalities, their proofs, evolution and applications (Fink, 2000). There would be more information to add to this short account of the history of inequalities, especially from this moment in history onwards; however, this would be more than the purpose and the size of this study.

2.2.2.5 Summary in an Unbordered Table

A collage of the historical snapshots containing inequalities is presented in the following Table 2.1. As can be summarized from the table, inequalities were first encountered in Classic Geometry, where they expressed factual relationships between quantities. The Hindu and Chinese mathematicians may have also had knowledge of those inequalities (Fink, 2000). In the big picture of the history of mathematics, Antique inequalities were captured and proved either in longhand expressions or in drawings. The Modern era is represented by the development of Algebra. To the best of my knowledge, there is no reference to any new inequality for almost two millennia, from Antiquity up to the 17th Century. Therefore there is no inequality snapshot in that zone. However, the rise of Algebra and the adoption of mathematical symbols allowed inequalities to become more easily noticed in the big picture of mathematics. With the rise of the theory of functions, inequalities seemed to have gained greater relevancy. Mathematicians began working on proving the famous Antique inequalities (e.g., Cauchy), creating extensions (e.g., Schwarz) or developing new ones (e.g., Newton, Maclaurin, & Bernoulli). Inequalities have been developed inside and through “interactions between different branches of mathematics” (Kjeldsen, 2002, p.2), like the theory of functions, linear algebra, mechanics, calculus, statistics and probability, to name only a few.

Area	Mathematician – Timeline	Inequality	Representation - Purpose
Geometry	Antiquity — **Euclid** 300BC	New, symbolic notation — $\sqrt{ab} \le \frac{a+b}{2}$	 Geometric fact
		$\|x+y\| \le \|x\| + \|y\|$	 Geometric fact
	Archimedes 250BC	$4\pi A \le L^2$	 Optimization
		$\frac{223}{71} < \pi < \frac{22}{7}$ (using a 96-sides polygon)	 Approximating π, using the method of exhaustion
		$\left(-\varepsilon < A < \varepsilon, \forall \varepsilon \ge 0\right)$ $\Downarrow$ $A = 0$	The 'vice' – using *inequalities* to determine *equalities*
Algebra-Analytic Geometry	**Diophantus** 200AD	No inequalities used during this time. However, I assume that inequalities benefited from the development of equations and symbols.	Development of equations, letters to represent *givens* and *unknowns*
	Viete 1540-1603		
	Fermat 1601-1665		Number theory, differential calculus
Inequality Symbols	**Harriot** 1631	< and >	Coined inequality symbols
	Oughtred 1631	and	
	Bouguer 1734	≤ and ≥	

Middle Years – Named Inequalities	Newton	**Newton** 1643-1727	Newton's Inequality	$p_{r-1}p_{r+1} < p_r^2$ where p_r is the average of the elementary symmetric function
	Means Calculus	**Maclaurin** 1729		$\sqrt{ab} \le \frac{a+b}{2}$ in proofs for limits. Proved the inequality of the means
		Bernoulli 1695-1782	Bernoulli's Inequality	$(1+x)^r \le 1+rx$, $r \ge 0$
	Arithmetic-Geometric Means	**Cauchy** 1789-1857	Cauchy's Inequality	$\sqrt[n]{a_1 a_2 ... a_n} \le \frac{a_1 + a_2 + ... a_n}{2}$ Proof by induction and descent
		Schwarz 1884 **Bunyakovsky** 1859	Cauchy-Schwarz Inequality Cauchy–Schwarz-Bunyakovsky Inequality	$\left(\sum a_i b_i\right)^2 \le \left(\sum a_i^2\right)\left(\sum b_i^2\right)$ Useful in vector algebra, in infinite series and integration of products, variances and covariances
	Isoperimetric	**Weierstrass** 1859	Isoperimetric Inequalities	$IQ = \frac{A}{P^2}$ Existence of an extremal - the circle has the highest IQ.
			$\delta - \varepsilon$ definition of limit and continuity	$\lim_{x \to a} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta$ $\forall x \left(0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \varepsilon\right)$
Theory of Functions	More Named Inequalities	1900- The Journal of the London Mathematics Society 1928 Inequalities 1934	**Chebyshev** Holder Minkowski	Names synonymous with famous inequalities.
			Hardy Littlewood Polya	*Inequalities* – a book that organized and gave names to inequalities
			Hadamard "Inequalities became a respectable topic for a paper." (Fink, 2000)	$\det\{x_{ij}\} \le \left(\sum_{ij} (x_{ij})^2\right)^{1/2}$
			Mitrinović *et al.*	*Inequalities* – a five-volume comprehensive book that documented *all* known inequalities

Table 2.1: The development of Inequalities

The history of inequalities table has no final border, since there is more work to be done to the collage. The big production of inequalities started with the appearance of the *Journal of the London Mathematics Society*. Also, the first history of inequalities book was written by Hardy et al. in 1934 when edited the book *Inequalities* (Fink, 2000). Davis and Hersh (1998) argue that the production of mathematics has a rate of two hundred thousand theorems per year. For disseminating the production and proofs of inequalities, there are two journals on the topic. A library search shows 95 papers per year in one of them. It may not be too misleading to assume that more than 200 papers on inequalities are written per year, published in the last 10 years in the two inequalities journals. On top of that, there is the five-volume anthology of inequalities edited by Mitrinović, *et al* (Fink, 2000). It seems that since Hardy, the development of inequalities has been remarkable. Thus, the table without a border at the end infers that the history of inequalities continues and that the production of inequalities is unbounded.

The picture of mathematics is immense. “By multiplying the number of journals by the number of yearly issues, by the number of papers per issue and the average number of theorems per paper, their estimate came to nearly two hundred thousand theorems a year”(Davis & Hersh, 1998, p.21). This calculation takes into account only the new mathematics produced per year. Thus, taking into account all mathematics – the entire history of mathematics – makes the picture quite immense. The goal of this section has been to capture and convey some snippets of information regarding inequalities, extracted from the grand picture of the history of mathematics. This task, although not seemingly ambitious, was not free of surprises. At first, the lens used was not pointed at the region of the picture where inequalities were expected to be. They were eventually found, however, disguised in unexpected forms. Once the ‘eye’ trained enough to notice them in different eras, inequalities were seen in abundance as the table has proved.

The development of a mathematical concept is extremely complex and I will not claim that I have captured all relevant aspects pertaining to inequalities. However, during my research, I have located snapshots of inequalities from the big picture of mathematics and made a collage with the awareness that I was also capturing the periods of challenge in the development of or conflict within the concept. Interestingly enough, looking at the collage now, I do not perceive any problems in the evolution of inequalities. However, someone could legitimately claim that, even though there is no visible epistemological obstacle related to inequalities, the fact that it took almost two millennia for inequalities to become a discipline is itself a signal that learners might have conceptual or psychological difficulties when dealing with them.

Inequalities were at first tools, and when the circumstances became favorable, they flourished into a discipline. Embedded in Geometry, they migrated to Algebra to get the power of symbols from there, and then they settled for good into the Theory of Functions where they were enriched with new structures and philosophy. Embedded in functions, they grew omnipresent in many mathematical areas, from calculus to algebra, to statistics, to numerical analysis, to game theory. Paraphrasing Burn, I conclude this section with a historical account of the concept of inequality: *Inequality* "encapsulates methods of proofs which originated in classical Greek mathematics, developed significantly during the 17^{th} century and reached their modern form with [Hardy]" (Burn, 2005, p.294).

2.2.2.6 Final Remarks

Burn (2005) argues that not only periods of hardship, but also the actual developmental steps of a concept, can inform didactics. My search into the history of inequalities revealed no epistemological obstacles. However, the study brought out that it is recorded and documented that inequalities are not easy concepts to manipulate. Even Hardy, the man who can be called the father of inequalities, confessed:

> There are, however, plenty of inequalities which are hard to prove; Littlewood and I have had any amount of practice during the last few years, and we have found quite a number of which there seems to be no really easy proof. It has been our unvarying experience that the real crux, the real difficulty of idea, is encountered at the very beginning. (Hardy, 1929, p.64)

Thus, the answer to the investigation into the difficulty of understanding inequalities may not reside in history, as expected. However, the answer could be deciphered from the history of inequalities in an unpredicted way. Inequalities are present everywhere in Greek Geometry. The following section reveals that there are not too many inequalities in British Columbia's mathematics curriculum, let alone in the Geometry curriculum within it. Looking back, the definition of inequalities was found under algebra in the mathematics encyclopaedia, but under the algebra branch, there were no references to the old history of inequalities. Old traces of inequalities were detected under geometry, a connection which is scarcely represented in British Columbia's school curriculum. It is possible that the obstacles to understanding inequalities are rooted in the curriculum, rather than in the history of the concept. The next section attempts to address this big puzzle – what role the school curriculum plays in developing students' conception of inequalities.

2.3 Where Are Inequalities Located in the K-12 Curriculum?

It is well known in the community of mathematics educators that, in most countries, inequalities are taught in algebra courses, in conjunction with equations or as a section of the chapter on equations (Bagni, 2005). Moreover, the inequalities are taught algorithmically and disconnected from the study of functions. Research identified a huge gap between inequalities "as a school subject and the mathematicians' professional approach to the subject" (Boero & Bazzini, 2004, p.141). Functions are used in solving equations and inequalities using approximation methods. Nothing of this aspect of inequalities seems visible in school curriculum.

This section attempts to illustrate undergraduate students' background in inequalities, by looking at the primary and secondary mathematics curriculum. In the three sections that follow, some aspects of the inequalities in school curriculum in British Columbia and Romania, the two school systems I have extensive experience with, will be presented. The first section captures inequalities in K to 12 curriculums in BC. Some snapshots of inequalities from grade 5, grade 8, and grade 11curriculum in Romania are given in the subsequent section.

2.3.1 Inequality in British Columbia Curriculum from K to University

In BC schools, inequalities are first met in the elementary school. As early as kindergarten, students have to compare two given sets or quantities and describe the sets using words, such as *more*, *fewer*, as *many as*, or *the same number* (BC Ministry of Education). First grade students are to use the inequality symbols in comparing numbers, for example: $2 < 3$. In grade 4, for example, students are required to order decimal numbers. They also have to fill in the inequality symbols to compare decimal numbers such as $4.96 > 4.90$ or to provide a number to make an inequality statement true: $4.96 < \_\_\_$ (Morrow, 2004, p.300). Later on, in grade 6, they are introduced to the $\leq$ and $\geq$ symbols and they learn to list all positive integers less than or equal to a given number. Linear inequalities of the form $ax < b$ enter the curriculum in grade 9. The Integrated Resource Packages are suggesting the following strategy for teaching linear inequalities at the grade 9 level:

> Display an inequality on the board or overhead (e.g., $3x + 2 > 8$). Ask students to determine a value for x that satisfies this inequality. Plot and compare students' answers. Discuss as a class why there are many solutions. Can students display all solutions efficiently? (BC Ministry of Education)

There is no mention about what is meant by efficiently displaying the solutions of an inequality. I imagine that the teacher has the freedom to direct the learning toward showing the simplest equivalent inequality where the solution is visible (here, $x > 2$) or to list the solution if they work with integers, to show the solution by graphing it on a numbers line, or using interval notation to represent the solution $(2, +\infty)$ if they work with real numbers. A functional approach is used in grade 11 for solving linear and quadratic inequalities. From the graph of a quadratic equation, students learn how to read intervals where the function is greater than zero, for example. Also, from the graph of the equation of a line, they learn to read all pairs of numbers that are solutions of a two variables linear inequality. In grade 12, aside from solving linear and quadratic inequalities, students are expected to formulate and apply strategies to solve polynomial, absolute value, radical, and rational inequalities (BC Ministry of Education). Simple inequalities are also embedded in the domain of logarithmic and radical functions. In grade 12 Calculus, inequalities are tools for the study of applications of the first and second derivatives of a function. A graphing calculator or a sign-chart is used to solve the necessary inequalities.

"One of the ways in which undergraduate analysis differs from school calculus is the use of inequalities. In undergraduate analysis, inequalities are an ubiquitous means of proof. In school calculus they are an occasional misfortune" (Burn, 2005, p.271). In undergraduate calculus, students are expected to use inequalities in proofs for limits using the $\delta-\varepsilon$ (delta-epsilon) definition or the squeeze law. This kind of work with inequalities is quite different from what students educated in BC schools have seen. The delta-epsilon definition or the squeeze law for proving the limit of a functions are new structures for students. Up to this point, students have met what Italian or Romanian mathematics dictionary name *inequations*. With the distinction *inequality-inequation*, up to first year Calculus, BC students have seen only *inequations*, or simple *inequalities* when comparing numbers. The axiomatic approach to proofs is completely new to them. However, manipulating inequalities using the $\delta-\varepsilon$ definition incorporates all those aspects that are foreign to the students. "Thus in relation to … limits, the student transferring from school to university has to make the transition from a pre-Zeno mindset to a Weierstrassian viewpoint" (Burn, 2005, p.271).

In undergraduate mathematics, *inequations* are met explicitly in the algebra review sections. *Inequalities* are used in proving limits and various aspects of the monotonicity of functions. However, more often than explicitly, *inequations* and *inequalities* are implicitly used in the study of the domain of functions, the increasing/decreasing behaviour of a function, in optimization problems, or in many other applications of derivatives. A profound understanding of *inequalities* and good skill in handling *inequations* would be very beneficial to any Calculus student. However, the BC school curriculum does not reveal that students taking Calculus have the experience and skill at manipulating *inequalities*.

The situation described in this sub-section pertains not to the British Columbia's curriculum alone. In some European countries, studies have identified a similar situation; students arrive in university with some experience in computational mathematics and almost no experience in proof, or inquiry about why something is true (Nardi, 2007). The school mathematics curriculum "consists mostly of instrumental elements: there is little effort in these books to try to create … concept images" (Nardi, 2007, p. 96). Moreover, teachers have the right to omit topics of their choice, so even if the curriculum contains some elements of real mathematics, there is no guarantee that the students will be exposed to it. The students are exposed to a "linear and often disconnected view of mathematics" (Nardi, 2007, p.97). From my recollection of school in Romania, the situation seemed to be different there, the preparation for higher-level mathematics being more systematic and meaningfully done. The next section presents the major inequality events in the mathematics curriculum in Romania.

2.3.2 Inequalities in Romanian Curriculum

As mentioned in a previous section, in Romanian, as well as in Italian and French languages (Bagni, 2005) two distinct words are used with respect to inequalities: one is *inequality* and the other one is *inequation*. Prove the *inequality* $|a+b| \le |a| + |b|$ means that for any two real numbers, represented by the variables a and b, the inequality symbol holds. Solving the *inequation* $2x < 4$, means finding the interval $(-\infty, 2)$ of the specific numbers for which the given *inequality* holds. The distinction *inequation* – *inequality* is at least as important as the distinction between *variable* and *unknown,* as explained by Teppo and Bozeman (2001) in their work with young children on algebraic activities. According to Teppo and Bozeman, it is very important to distinguish among *place holders, variables* and *unknown*s, "since each use represents a different type of algebraic entity. Unknowns occur in equations that are essentially numeric in focus. Dummy variables are used as place holders to make general statements about mathematical relations, expresses as a particular sequence of operations" (p.3). It could have been beneficial to English-speaking students to have access to this distinction and to the mathematical experiences each and every one brings into their lives. The final chapter will refer further to this aspect of missed mathematical experiences.

In Romania, as early as grade 5, students are introduced to solving linear inequalities on whole number domain, with whole number solutions.

> Introducerea notiunilor the ecuatie si de inecuatie pornind de la urmatoarele tipuri de relatii: $x \pm a = b; a \pm x = b; xa = b$, cu $a \neq 0$, *a* divizor al lui *b*; $x \div a = b, a \neq 0$; $a \div x = b, x \neq 0$, *b* divisor al lui *a*; $x \pm a \leq b$ (respective $\geq, <, >$); $x \cdot a \leq b$ (respectiv $\geq, <, >$). (Ministerul Educatiei, Cercetarii si Tineretului, 2008).
>
> [Introducing the concepts of equation and inequation starting from the following relations: $x \pm a = b; a \pm x = b; xa = b$, with $a \neq 0$, *a* divisor of *b*; $x \div a = b, a \neq 0$; $a \div x = b, x \neq 0$, *b* divisor of *a*; $x \pm a \leq b$ ($\geq, <, >$, respectively); $x \cdot a \leq b$ ($\geq, <, >$, respectively). (Ministry of Education, Research, and Youth, 2008)]

In grade 7, inequality is presented as an ordering relation, with its three properties (reflexive, antisymmetric and transitive). Inequalities with real numbers coefficients and integer solutions are then solved.

Obtinerea unor inecuatii echivalente prin operare in ambii membri (adunare, scadere, inmultire sau impartire):

$1)\ a \le a, \forall a \in R;\ 2)\ a \le b\ si\ b \le a\ \Rightarrow a = b, \forall a, b \in R;\ 3)\ a \le b\ si\ b \le c\ \Rightarrow a \le c, \forall a, b, c \in R;$

$4)\ a \le b\ si\ c \in R\ \Rightarrow\ a \pm c \le b \pm c;\ 5)\ a \le b\ si\ c > 0 \Rightarrow\ ac \le bc\ \ si\ a \div c \le b \div c, \forall a, b \in R;$

$6)\ a \le b\ si\ c < 0 \Rightarrow\ ac \ge bc\ si\ a \div c \ge b \div c, \forall a, b \in R$

(Ministerul Educatiei, Cercetarii si Tineretului, 2008).

[Obtaining equivalent inequalities by operating in both sides (addition, subtraction, multiplication or division):

$1)\ a \le a, \forall a \in R;\ 2)\ a \le b\ \&\ b \le a\ \Rightarrow a = b, \forall a, b \in R;\ 3)\ a \le b\ \&\ b \le c\ \Rightarrow a \le c, \forall a, b, c \in R;$

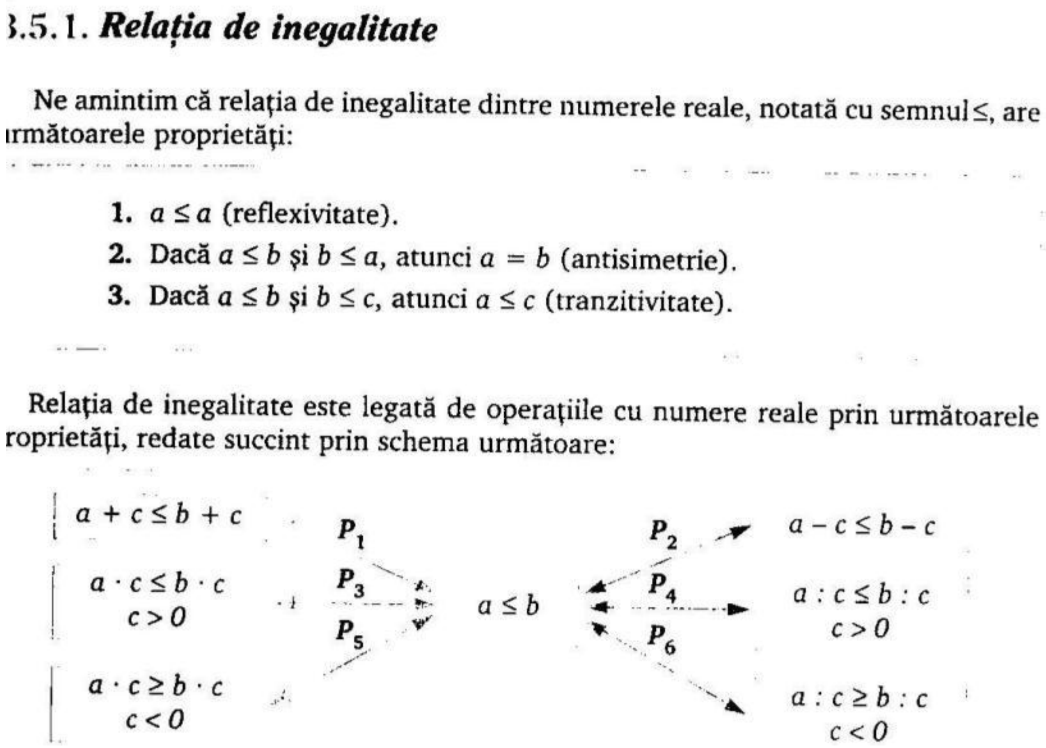
$4)\ a \le b\ \&\ c \in R\ \Rightarrow\ a \pm c \le b \pm c;\ 5)\ a \le b\ \&\ c > 0 \Rightarrow\ ac \le bc\ \ \&\ a \div c \le b \div c, \forall a, b \in R;$

$6)\ a \le b\ \&\ c < 0 \Rightarrow\ ac \ge bc\ \&\ a \div c \ge b \div c, \forall a, b \in R$

(Ministry of Education, Research, and Youth, 2008).]

In grade 8, the $\le$ inequality is introduced as a reflexive, antisymmetric and transitive relationship between real numbers. Linear inequalities with real coefficients and solutions are solved. Also, grade 8 students recognise that the height of a right triangle is the geometric mean of the two legs and identify the inequality of the means $\sqrt{ab} \le \frac{a+b}{2}$.

Figure 2.7 provides a snapshot of the properties of inequality and the properties of inequality in relationship to binary operations from a grade 8 book. This level of abstraction is the object of study in university mathematics in British Columbia and in most North American universities.



3.5.1. ***Relația de inegalitate***

Ne amintim că relația de inegalitate dintre numerele reale, notată cu semnul $\leq$, are ırmătoarele proprietăți:

1. $a \leq a$ (reflexivitate).
2. Dacă $a \leq b$ și $b \leq a$, atunci $a = b$ (antisimetrie).
3. Dacă $a \leq b$ și $b \leq c$, atunci $a \leq c$ (tranzitivitate).

Relația de inegalitate este legată de operațiile cu numere reale prin următoarele roprietăți, redate succint prin schema următoare:

Figure 2.7: Properties of inequalities, grade 8, Romania (from Cosnita & Turtoiu, 1989)

In grade 9, quadratic inequations are solved on a real number domain with solutions represented as intervals. Also, items such as the following are part of the grade 9 curriculum:

1) Prove that the following inequalities are true for all real numbers:

a) $4x^2 - 12x + 10 > 0$ b) $(x-1)^2 + x^2 - 6x + 9 > 0$

c) $(x^2+1)(y^2+1) \geq 4xy$

2) Find the solution of the inequality: $\sqrt{20 - x^2} > |x+2| + |x-2|$.

The solutions to these types of inequalities are presented algebraically as well as visually. Figure 2.8 represents the solution to inequality 2). Algebraically, it is suggested to first explicit the absolute value terms to get three equivalent inequalities:

(1) $\sqrt{20-x^2} > -2x, \ \ when \ \ x \in (-\infty, -2)$
(2) $\sqrt{20-x^2} > 4, \ \ \ \ when \ \ x \in [-2, \ 2)$
(3) $\sqrt{20-x^2} > 2x, \ \ \ when \ \ x \in [2, +\infty)$

By individually solving the three inequalities and then intersecting the solutions with the domain, the final solution is given by the interval $(-2, 2)$. Graphically, the functions $y_1 = \sqrt{20-x^2}$ (a semicircle) and $y_2 = -2x, \ y_3 = 2x, y_4 = 4$ (three lines) are represented and from the graph the solution $(-2, 2)$ is read.

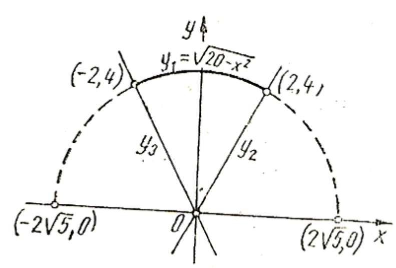


Figure 2.8: Solution to $\sqrt{20-x^2} > |x+2| + |x-2|$
(from Cosnita & Turtoiu, 1989)

It seems that the Romanian curriculum is less disconnected from the higher-level mathematics than is the BC curriculum. To the best of my knowledge, not much research in mathematics education is conducted in Romania. Therefore, I cannot document if students' results there meet expectations. The very few mathematics education studies I could locate mostly contain suggestions for classes and best practices for teaching mathematics concepts.

2.3.3 School Curriculum and Inequalities

Looking in parallel at the curriculum in Romania and in BC, in both countries, in junior high school, inequalities are part of the algebra courses. One big difference between the two curricula is that in Romania, differential calculus is taught in grade 11 and there, at that level, the limit is studied using the $\delta - \varepsilon$ definition. In preparation for using inequalities in the limits context, grade 11 Calculus starts by reviewing algebra, functions and inequalities. At that level, the triangle inequality, the inequality of the arithmetic and geometric means, as well as the Cauchy-Bunyakovsky inequality, which are not new to the students, are proved. Moreover, the proofs stand as model for proving limits. It seems that in Romania, more profound work with inequations and inequalities starts early. A superficial look at some older high school and college algebra books revealed that, at some moment in time, the British Columbian mathematics curriculum contains triangle inequality and the inequality of the means as problems and that inequalities had a more prominent presence in school mathematics (Brumfiel, Eicholz, & Shanks, 1961; Lovaglia, 1966; Davis, 1940).

In conclusion, at least in British Columbia, the school curriculum does not seem to thoroughly prepare students for the upper-level work with inequalities. As mentioned previously, taught in algebra courses, as a subsection of equations, algorithmically and disconnected from the study of functions, inequalities are perceived by students and teachers alike as a misfortune (Burn, 2005), rather than an important brick for a solid foundation in mathematics. The real motivation for studying inequalities seems to be missing. In Chapter 3, which is dedicated to research on inequalities, a list of misconceptions is included. Some of them are explicitly linked to the unfortunate placement of inequalities in school curriculum. Chapter 8 speculates on the long-term effect of lacking meaningful experiences with inequalities in school mathematics.

Chapter 3:

Inequalities in Mathematics Education Research

Boero and Bazzini, in their contribution to the *International Group for the Psychology of Mathematics Education* (PME Group) in 2004, opened with:

> This contribution deals with inequalities: an important subject from the mathematical point of view; a difficult subject for students; a subject scarcely considered till now by researchers in mathematics education. (p.139)

An awareness of the issue that inequalities were not under the microscope of the mathematics educators' community emerged five years prior to this statement, in 1999, when the Project Group at the 23rd Psychology of Mathematics Education Conference called for research on inequalities. This call consequently resulted in a number of research teams that embraced this area of study. The fruits of that call, as well as the research on inequalities conducted prior to the 23rd PME or after the 28th PME are the subjects of this chapter.

The first section presents the above mentioned historical event and gives a summary of the research on inequalities presented at the 28th Psychology of Mathematics Education Conference. The next sections switch from the stream of research presented at the PME to the whole picture of educational research on inequalities. Essentially, the sections will attempt to classify the research into the major themes, introduced by the major questions that triggered the research on inequalities in the first place. Section 3.2 deals with the puzzle over how to teach inequalities. Different methods of solving inequalities are identified, exemplified and compared. Students' error patterns are presented in Section 3.3, a section that seeks to answer the question of how students perform on inequalities. The perplexing question of why inequalities are so hard to master is also addressed. The chapter culminates with Section 3.4, where the research questions guiding this study are presented.

3.1 Inequalities in Mathematics Education

Although inequalities are the building blocks to understanding many aspects of trigonometry, geometry, optimization and linear programming, they are studied in school mathematics mainly in algebra classes, as a subsection of the chapter on equations. Similarly, most published mathematics education studies present inequalities in parallel with equations (e.g., Kieran, 2004; Dreyfus & Hoch, 2004; Vaiyavutjamai & Clements, 2006a&b). There are a few studies where inequalities were connected to the study of functions (e.g., Boero et al., 2001; Garuti et al., 2001; Sackur, 2004).

While research in mathematics education has allotted special consideration to many mathematical concepts, as well as to learning/teaching/understanding mathematics in general, before 1998, not much attention had been devoted to the study of inequalities in particular.

3.1.1 A Call for Research on Inequalities Was Issued

The *International Group for the Psychology of Mathematics Education* (PME Group) "is the most influential forum of research in mathematics education, and the proceedings of the PME conferences are a primary source of information for any researcher in mathematics education, since they summarize the state of the art at that time in the field" (Gutiérrez & Boero, 2006). Over the 34-year history of PME, the teaching, learning, and understanding of algebra have been fundamental streams of research (Kieran, 2006). Even though from the late eighties to the early nineties inequalities were under the lens of some prominent names who were conducting intensive algebraic studies (e.g., Linchevski & Sfard, 1991; Dreyfus & Eisenberg, 1985), they did not become the object of a conference forum until 2004. Before that, the first people to engage in writing about teaching and learning inequalities were schoolteachers reflecting on class work on inequalities and presenting their observations in the *Mathematics Teacher* (McLaurin, 1985; Dobbs & Peterson, 1991; Piez & Voxman, 1997). The main themes of the published papers on inequalities dated before 1999 were related to the difficulties encountered in the algebraic manipulation of inequalities, the methods of teaching inequalities, and the preferred methods of introducing inequalities to students.

The first important date for the history of research on inequalities was 1998, at the 22nd Psychology of Mathematics Education Conference, when the discussion on inequalities was initiated by the presentation of research conducted by Tsamir, Almog and Tirosh (Bazzini & Tsamir, 2004). Consequently, in 1999, a call for research on inequalities was made during the Project Group sessions of the 23rd Psychology of

Mathematics Education Conference. Key research questions were selected at that conference:

> What are students' conceptions of equations / inequalities? What is typical correct and incorrect reasoning? What are common errors? What are possible sources of students' incorrect solutions? What theoretical frameworks could be used for analysing students' reasoning about algebraic equations / inequalities? What is the role of the teacher, the context, different modes of representation, and technology in promoting students' understanding? What are promising ways to teach the topics of equations / inequalities? Is there a global theory that may encompass the local theory of equations and inequalities? (Bazzini & Tsamir, 2004, p.138).

In addition, a number of collaborative research teams were initiated at that conference.

3.1.2 Results Are Presented

Preliminary reports from the studies on inequalities (e.g., Boero, Bazzini, & Garuti, 2001; Garuti, Bazzini, & Boero, 2001; Tsamir & Bazzini, 2001) were presented in 2001, at the 25nd Psychology of Mathematics Education Conference (Bazzini & Tsamir, 2004). The results of the 23rd PME Conference's initiative to conduct research on inequalities were presented, discussed and analyzed at the 28th Psychology of Mathematics Education Conference in 2004. The theme of the 28th PME was Inclusion and Diversity (Bazzini & Tsamir, 2004). There is no doubt that simply by reading the list of presenters, a person with some background in mathematics education research will expect to see diverse approaches to the frameworks used and to the studies themselves. There were five presentations informing the mathematics education community of research conducted on inequalities (Boero & Bazzini, 2004; Kieran, 2004; Sackur, 2004; Dreyfus & Hoch, 2004; Tsamir et al. 2004), and two reactions (Tall, 2004; Radford, 2004) to the papers presented at the 28th PME. The papers covered a wide range of aspects concerning inequalities, from error patterns in students' solutions to didactical perspectives on students' errors; from traditional teaching to teaching with technology; from the didactical aspects of classifying inequalities under algebra to the metaphors of using functions to present inequalities. The frameworks for the studies were diverse as well, including a Vygotskian perspective and Nunez's grounding metaphor construct, Duval's theory on semiotic registers in mathematics and Frege's theory of denotation, as well as Fischbein's theory of intuitive, formal and algorithmic knowledge (Bazzini & Tsamir, 2004). The respondents' interventions were quite insightful. On top of summarizing the findings and giving feedback to the presented studies, they envisioned where and how new studies could answer the question of how students' thinking may be encouraged to evolve from procedures to manipulating mental entities, thus allowing for the acquisition of meaningful higher level mathematics (Tall, 2004).

The following sections incorporate and apply the results reported at the 28th PME into the big picture of research on inequalities.

3.2 How to Teach Inequalities

> Researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities. (Tsamir & Bazzini, 2002, p.2)

The most common methods of solving inequalities have been presented in section 2.1. To summarize, *linear inequalities* are solved using the classical pattern of solving equations, with the amendment that division or multiplication by a negative number results in an inequality with the symbol reversed. The graphic method, the sign-chart (and the improved sign-chart), and the logical connectives method are the most common means for solving *quadratic or polynomial inequalities*. *Rational inequalities* are solved, aside from the methods used to solve polynomial inequalities, by multiplying the inequality by the square of the least common denominator and then using the sign-chart to complete the resulting polynomial inequality.

McLaurin (1985) and Dobbs and Peterson (1991) have suggested that the sign-chart is the best method for teaching quadratic inequalities. They argued that a good understanding of this method will empower students with the needed tool for solving more complicated inequalities for which no other method is available. McLaurin (1985) also pleaded for a unified method to teaching students how to solve absolute value, quadratic, rational and irrational inequalities – by following the sign-chart method. Piez and Voxman (1997), on the other hand, argued that students must become familiar with multiple methods and representations when dealing with inequalities. Flexibility in manipulating algebraic structures would not only allow students to solve other types of inequalities, but would improve students' problem-solving skills as well.

Tsamir and Reshef (2006) have also recommended "present[ing] students with multiple methods when teaching quadratic inequalities" (p.6). The results come from a study conducted on twenty high school students split into three groups. Each group was first introduced to one of the three methods – graphic, sign-chart, logical connectives. After some practice and gaining familiarity with that method, they taught their method to the other students and learned the other two methods as well. At the end of the study, the students were tested on solving quadratic inequalities following a method of their choice. As expected, students mostly used and gave preference to the first method they were introduced to, except those that first learned the logical connectives method (Tsamir &

Reshef, 2006). Linchevski and Sfard (1991) had previously identified the logical connectives method as problematic, since it requires more abstract thinking than the other two methods. In the Tsamir and Reshef (2006) experiment, the graphic method was most frequently employed. However, there were students who used more than one method in solving different inequalities. Interestingly enough, no student used a second method to validate the solution found in the first place. Given that inequalities serve different purposes in higher-level mathematics and there are special situations when only a special representation might be applicable, researchers recommend introducing students to multiple representations of inequalities (Tsamir & Reshef, 2006).

Linchevski and Sfard (1991) have also critiqued the logical approach to introducing inequalities for the first time. At the time of their research, inequalities were taught as special propositional formulae, for which students must find the set of values that would make the proposition true. This was identified as "a structural approach: a mathematical notion … explained in terms of abstract objects (truth-sets)" (p.320). The study confirmed that by introducing inequalities with abstract objects that may be later built upon, instead of using the natural sequence of operation-structure, the students would develop pseudo-structural conceptions. The unfortunate conclusion is that, in this abstract teaching framework, students identified inequalities with a "string of symbols which can be manipulated according to some arbitrary rules" (p.323). With this approach, the understanding of inequalities became merely instrumental. The authors concluded that by replacing the structural approach with an operational one, when first introducing a new concept, could produce a relational understanding of the concept (1991).

Kieran (2004) observed that problem solving activities directed toward generating the symbolic form of inequalities are absent from the current research on inequalities. Thus, for Kieran, inequalities, as well as algebraic equations, could be meaningfully introduced to young students through contextual problems. She observed that students enjoy the process of coming to a generalization by working on recursive particular aspects of a concept. However, she has no answer to the question: What instructional support do students need in order to grow from the context to meaningful symbolic manipulations of inequalities? (Kieran, 2004).

A number of researchers (e.g., Boero et al., 2001; Garuti et al., 2001; Boero & Bazzini, 2004) have criticized the fact that inequalities are included in the curriculum in algebra, when it has been openly recognized that they belong to the study of many aspects of mathematics (Burn, 2005; Tall, 2004). The placement of inequalities under the study of algebra invites a style of learning inequalities through memorized, routine procedures. In this context, students fail to make important connections and to solve inequalities that look different from the model they have commonly encountered. There is a didactical challenge to develop activities that will help students benefit from the connection between equation and inequalities, while making them aware of the pitfalls of applying the transformational techniques used in solving equations to solving inequalities (Kieran, 2004, p.146). Competence in manipulating inequalities in connection with equations means being able to replicate what can be repeated while changing the parts that need to be changed (Sfard, 1998). In addition, Boero and Bazzini (2004) have identified a huge gap between inequalities as "a school subject and the mathematicians' professional approach to the subject" (p.141). They claimed that a function approach to the study of inequalities would be closer to the mathematicians' work with inequalities. They argued that a functional approach to teaching inequalities will also benefit students' use of metaphors in understanding inequalities, as well as their capacity to make connections far beyond mathematics. Working with the hypothesis that "different tools belong to different disciplines" (p.139), Boero and Bazzini (2004) identified a question that remains unanswered in the literature: "Can the study of teaching and learning inequalities be reduced to the study of teaching and learning functions?" (p.142).

Tsamir et al. (2004) also suggested that teaching inequalities through functions or technology could minimize the construction of misconceptions. A function approach to teaching inequalities means using the graphing calculator to graph functions and then reading the solution of an inequality associated with the graph. The "visual enactive activity" with dynamic entities – the functions – "can give a powerful embodied sense of global relationship" between functions and inequalities (Boero & Bazzini, 2004, p.141). Sackur (2004), however, has shown that this approach to introducing inequalities is not free from students' errors and misconceptions. In her study, Sackur pointed out some of the difficulties that can arise when using calculus tools to address algebraic concepts. First, she writes that it is not easy at all for students to shift from the dynamism of a graph where y is the moving entity to reading the solution on the x-axis. Also, it is not easy for students to switch from seeing the dynamic solution as a point moving along the x-axis between some boundaries to giving the solution as a static entity – an interval (Sackur, 2004).

Abramovich and Ehrlich (2007), on the other hand, were in favour of using technology in the study of inequalities. They argued that graphing calculators could be successfully used in solving all types of inequalities, and they believe that students should experience graphing the given inequality as well as the equivalent forms they get by algebraic transformations. The solutions of the new inequalities can inform students about the correctness of their algebraic approach. In other words, they view technology as a means for solving inequalities and, at the same time, a tool for validating the algebraic manipulation of inequalities.

Tsamir and Bazzini (2001) presented the results of a study on students' reactions to traditional methods when facing non-standard learning tasks. The traditional method tested was a teacher demonstrating in small steps how to solve a certain type of problem and then students were given a lot of practice to master the procedure. The subjects of the study incorrectly solved linear inequalities or were unable to explain their work. The study suggests that the root of students' misconception in regard to inequalities is the way they were previously instructed in this topic. The traditional process of solving linear inequalities is very similar to solving linear equations, except for one detail: you reverse the inequality sign whenever you multiply or divide the inequality by a negative number. This common and handy way to introduce inequalities in an algebra course may be the root of many misconceptions students have when solving inequalities (Bazzini & Tsamir, 2001). Teaching inequalities in a traditional manner, such as a teacher presenting a "sequence of topics, theorems and rules, which are demonstrated in suitable sets of examples," (p.61) followed by home assignments that are usually a "repetition of tasks similar to those experienced in class" (p.61), does not leave room for students' creative participation in the process of learning. Bazzini and Tsamir concluded that doing algebra should involve *understanding*, rather than "just formal manipulation" (p.67).

Similar conclusions were derived from an experimental study on linear inequalities in Thailand by Vaiyavutjamai and Clements (2006a). The study measured whether there was lasting improvement in the performance and understanding of inequalities after an intense 13-week teaching experiment. Their main finding is that inequalities are hard to master, especially when presented to students in a traditional teaching style. After taking an intensive course on inequalities, many students remained confused about the meaning of an inequality and about what the solutions to inequalities represented. Six months after the study, students in low- and medium-stream classes performed only slightly better than they had performed at the preteaching inequalities stage (Vaiyavutjamai & Clements, 2006a; b).

Vaiyavutjamai and Clements (2006a) based their study's framework on evidence that traditional teaching and assessment in school mathematics isolates skills into small compartments and thus fails to assist learners in making suitable, long-lasting cognitive connections. The results of their study were pretty pessimistic: all students, apart from the best, showed a "rules without reason" (p.20) approach to solving inequalities. These types of results were not singular: Linchevski and Sfard (1991), Tsamir and Almog (2001), and Tsamir and Bazzini (2001) presented similar results.

Abramovich (2006) used a spreadsheet as a modeling tool for working with inequalities in the mathematical reduction context. The study draws from the Standards document (NCTM, 2008) for teaching and recommendations for teachers, which also includes the familiar approach of reducing a difficult problem to a more familiar one. Abramovich recognized that:

> Many problems in geometry can be reduced to the study of algebraic equations and inequalities; by the same token, many problems in algebra can be reduced to the study of their geometric representations. In calculus, the investigation of infinite processes, such as convergence of sequences and series, can be reduced to the study of inequalities between their finite components. (p.527)

He claimed that even though inequalities are recognized as powerful tools in pure mathematics, only a few mathematics education studies – such as Boero and Bazzini's 2004 paper – have revealed this didactical aspect of inequalities. Abramovich's (2006) spreadsheet approach to using inequalities in problem solving and the graphing calculator for solving inequalities (Abramovich & Ehrlich, 2007) are recommended by the authors to prospective teachers of secondary mathematics in preparation for teaching inequalities. Their recommendation followed a study that blames students' misconceptions in regards to inequalities on teachers' inadequate knowledge in this area (Abramovich & Ehrlich, 2007).

3.3 How Students Perform on Inequalities

> The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic which gives us insight, nor is it chance that causes us to make mistakes. (Tall & Vinner, 1981, p.151)

Published studies identified several *common errors* in students' work when solving inequalities (Tsamir, Almog, & Tirosh, 1998; Linchevski & Sfard, 1991; Tsamir et al. 2004). A comprehensive list of common errors associated with solving inequalities includes:

(1) multiplying or dividing the two sides of an inequality by the same number without checking whether the number is positive, negative or zero;

Example: The inequality $\frac{2x-1}{x+1}<1$ becomes $2x-2<x+1$ after multiplying the left and the right side of the inequality by the denominator (Tsamir et al., 1998, p. 134).

(2) dropping the denominator of a fraction when containing a parameter or the variable;

Example: The inequality $\frac{x-5}{x+2}<0$ becomes $x-5<0$, after dropping the denominator of the fractions, following the pattern of reducing rational equations with right side zero to equating the numerator with zero.

(3) erroneously converting the inequality into intervals;

Example: The inequality $x-5<0$ becomes $(5,+\infty)$ or $(-\infty,5]$

(4) declining solutions that do not fit the general pattern (i.e. an interval for inequalities, a unique value for equations);

(5) dealing with a positive/negative product by considering the component factors as separately being all positive/negative (Tsamir et al. 2004);

Example: The inequality $\frac{x-5}{x+2}<0$ becomes $x-5<0$ *and* $x+2<0$ (Tsamir et al., 1998, p.133).

(6) incorrectly reading the solution from a graph when using functions to solve inequalities;

(7) incorrectly converting the graph into a solution (Sackur, 2004);

(8) solving equations instead of inequalities;

Example: The inequality has no solutions since the quadratic equation associated with it has no real roots (Tsamir et al., 1998, p.133).

Inequalities, with their multiple semiotic registers of representation – algebraic, interval, functional, and graphical (Sackur, 2004) – present many ways in which students could make errors. “Equations were found to serve as a prototype in the algorithmic models of solving [linear] inequalities” (Tsamir & Bazzini, 2002, p.7). However, as it can be seen from the above list, this approach to teaching linear inequalities seems to be the one that could generate most of the listed errors. In addition, linear inequalities are not the only type of inequalities affected by that pattern; rational inequalities introduced as special cases of equations are also affected by eliminating the denominator. Students very often eliminate the denominator when solving equations, without thinking that such an action might influence the sign of the entire algebraic expression. For the graphical approach to solving inequalities, the main problem is students’ ability to correctly read the graph and convert what they see into a valid solution.

In conclusion, questions such as: What is typically correct and incorrect reasoning in regards to inequalities? What are common errors when solving inequalities? What are possible sources of students’ incorrect solutions? What are promising ways to teach the topics of equations / inequalities? (Bazzini & Tsamir, 2004, p.138) are questions that were addressed in the literature review in this section. To the best of my knowledge, some of the most important questions pertinent to inequalities, such as why inequalities are so hard to master, have not yet been addressed. This question, therefore, is one of the main points of contention this dissertation seeks to explore and resolve.

Several researchers in mathematics education have addressed the question why students find inequalities hard master. Sackur (2004) concluded that switching between registers when working on inequalities is a source of students' mistakes. Tall (2004) declared that equations are cognitive obstacles to understanding inequalities. Burn (2005) claimed that inequalities are hard to master because students' mind is not sufficiently and properly prepared to capture the complexity of this concept. Moreover, not only students, but mathematicians find many inequalities hard to prove (Hardy, 1929).

The question why are inequalities hard, which captured the attention of researchers in mathematics education and mathematicians alike, is the overarching interest of conducting this research.

3.4 Research Questions

Having studied inequalities as mathematics concepts with a defined place in the history and production of mathematics, as well as in the context of mathematics education where they pose problems to students, it has become clear to me that there are many unanswered questions regarding the meaningful manipulation of inequalities.

Over the years, my initial list of questions has expanded, with additions arising from a deeper examination of the existing literature on inequalities, as well as from my own empirical research on this topic. Many of these questions have to do with the inner structure of the concept of inequalities juxtaposed with the concept image of inequalities, exhibited by the participants in my study.

Finally, the expanded list of questions was narrowed down to my research questions:

(1) What are undergraduate students' conceptions of inequalities?

(2) What influences the construction of the concept of inequalities?

More precisely, the focus is on the actual difference between various conceptions of inequalities and how those particularities inform the experiences that could have influenced each of the conceptions.

(3) How can undergraduate students' conceptions of inequalities expand our insight into students' understanding of, and meaningful engagement with, inequalities?

In other words, I am interested in how undergraduate students' conceptions of inequalities inform the community of mathematics educators about the most efficient pedagogy of introducing inequalities into different levels of the school curriculum.

These questions follow a progression that mirrors my involvement with inequalities over the years; they also foreshadow the development of the present dissertation.

Chapter 4:

In Pursuit of a Theoretical Framework

The theoretical perspectives discussed in this chapter contribute to the interpretation of undergraduate students' conceptions of inequalities that will be presented in Chapters 6 and 7. The pursuit for a theoretical framework for my study is an important aspect of my journey of becoming a researcher.

In the initial stages of data analysis, I considered many lenses for looking at the data. For example, van Hiele's (1999) five levels of geometric understanding were initially considered as a model for presenting the different levels of understanding inequalities. From the van Hiele's model I moved to Tall and Vinner's (1981) concept image and concept definition framework and then to taking into consideration the potential of looking at the data through the lens of APOS theory (Dubinsky & McDonald, 2001) and Tall's (2007) Three Worlds of Mathematics.

Appreciated at the theoretical level for their potential, some of the considered theories revealed their limitations during the stage of data analysis. Although, they did not become the final lens for looking at the data, they influenced the way of looking at the data. Their limitations were very often seen as new directions to follow or elements to focus on during my study. In what follows I introduce the above-enumerated theoretical frameworks, recognize their limitations of becoming the framework for my study and acknowledge how they informed my work.

Thus, Section 4.1 summarizes different definitions of understanding, such as Skemp's (1976), Hiebert and Lefevre's (1986), and van Hiele's (1999) hierarchical levels of understanding. The section concludes with Van de Walle's (2008) "measure of the quality and quantity of connections that [a new] idea has with existing ideas" (p.25) as the over-arching, functional definition of understanding. Section 4.2 is dedicated to the notions of concept image and concept definition (Tall & Vinner, 1981). The description of different conceptions of inequalities makes use of Dubinsky and McDonald's (2001) APOS – Action, Process, Object, Schema – theory, presented in Section 4.3. The last section deals with cognitive growth in mathematics as framed by Tall (2007) in his Three Worlds of Mathematics.

4.1 Understanding Mathematics

> Since it is the understanding that sets man above the rest of sensible beings … it is certainly a subject … worth our labour to inquire into. The understanding, like the eye, while it makes us see and perceive all other things, takes no notice of itself; and it requires art and pains to set it at a distance and make it its own object. (Locke, 1847, p.33)

One could write volumes about human understanding. However, I will only touch on a few aspects of human understanding that will help situate the data presented in the following chapters into a theoretical framework.

Sierpinska (1994) identified understanding with overcoming obstacles. Moreover, she defined understanding as the act of mentally associating the object of understanding to another object, which is the basis of that understanding. She considered the acts of understanding as clustered together in processes of understanding, which involve connections between acts of understanding through various mental processes such as reasoning, explaining, exemplifying, or accommodating the new object to a previously assimilated schema. According to this paradigm, understanding a mathematical concept is achieved if the process of understanding comprises a "certain number of especially significant acts of overcoming obstacles specific to that mathematical situation" (p.*xiv*) and, then, mathematical objects are constructed into acts of understanding.

Van de Walle (2008) defined understanding as "a measure of the quality and quantity of connections that [a new] idea has with existing ideas" (p.25). In the teaching-learning context, the measure of connections produces hierarchical models that are used to examine students' understanding (Sierpinska, 1994). The hierarchy is made up of progressive levels of understanding, such as van Hiele's (1999) five levels of geometric understanding, or the dual aspect of mathematical knowledge, such as procedural or conceptual understanding (Hiebert & Lefevre, 1986), instrumental or relational understanding (Skemp, 1976), or operational or structural understanding (Sfard, 1991). Poincaré distinguished between shallow and deep understanding, as well, by stating that students "imagine that they have understood when they have only seen" (quoted in Sierpinska, 1994, p.10).

Skemp (1976) identified instrumental understanding as "rules without reason" and defined relational understanding as "knowing what to do and why" (p.2). He also described learning leading toward instrumental understanding as memorizing fixed steps, a process which helps move the learner from one point to another, without an awareness of the relationship between the intermediary steps and the final goal. On the other hand, the process of relational learning of mathematics is described as “building up a conceptual structure” (p.14).

Hiebert and Lefevre (1986) defined procedural understanding as knowing the symbols accepted in a special mathematical situation, or recognizing and following the rules or procedures needed to solve mathematical problems . They described conceptual knowledge as “knowledge rich in relationship” (p.3), knowledge that must be learned meaningfully. Procedures, they claimed, can be learned by rote. However, assisted by conceptual knowledge, procedural knowledge can result in meaningful knowledge that can be easily retrieved and used when necessary.

Sfard (1994) argued that the “most natural way to assess one’s understanding of a mathematical idea is to estimate the ease with which one reasons and discovers new facts about it” (p.49). Sfard (1991) also earlier argued that the type of understanding that an individual reaches helps them achieve either a solid, structural knowledge or an operational one. “[T]he structural approach invites contemplation; the operational approach invites action; the structural approach generates insight; the operational approach generates results” (Sfard, 1991, p.28).

The language of understanding and that of knowing alternate in education research, even though understanding and knowing are not exactly synonyms. One can know something without understanding it (Van de Walle, 2008). Without understanding – which is measured in the connections that the known fact or rule has with other ideas – a known fact may be easily forgotten or retrieved with difficulty when necessary (Skemp, 1976).

The language of procedural, contextual, structural or relational understanding describes the different levels of understanding of inequalities. From the van Hiele's (1999) five levels of geometric understanding, the five conceptions of inequalities grounded in this study inherited the initial labelling from 0 to 4. The limitations of this framework consist in the fact that the categories I have identified in the data were not hierarchical. Moreover, there was no visible action on an object at one level of thinking, as in the van Hiele's model, whose performance would have the potential of encapsulating that object into an idea of thought at the next level. Although not chosen as the theoretical framework for the study, the van Hiele's levels of geometric understanding served the search for a workable definition of understanding mathematics. In the context of the empirical investigations on inequalities presented in Chapters 6 and 7, many of the definitions summarized in this section are used in the descriptions of the different categories of conceptions or levels of understanding grounded into data.

4.2 Concept Image – Concept Definition

What causes the human brain to make mistakes or form misconceptions is a problem that many educators, among others, would like to solve. David Tall and Shlomo Vinner are some of the prominent names of the people that have asked such questions. To understand the mental processes involved in manipulating a concept, Vinner and Tall (1981) formulated the idea of *concept image* to "describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p.2). The concept image develops in time, with the help of all the experiences that a person has with that concept, be they images, pictures, diagrams, embodiments, or metaphors.

Vinner (1983) defined the *concept definition* as "a verbal definition that accurately explains the concept in a non-circular way" (p.1). Thus, the *concept definition* is given by the words used to specify a concept. The definition may be ingested by rote learning or assimilated by meaningfully working on tasks related to that concept and by making associations with that concept. The definition may be the same as the formal definition given in a book or could be a verbal description of the mental image that someone has for a concept. How a concept is described in words can inform us on the concept image a person possesses regarding that concept. Graphs or diagrams of a concept can inform about the concept image as well. When engaged in a task, back and forth thinking movement between the concept image and the concept definition of a concept is detected (Vinner, 1983). This movement, as well as the missing links when mapping the concept image with the concept definition, informs us about students' understanding of a concept.

The concept image and concept definition framework provided a lens for looking at the data and a language of talking about the images seen in the data. Promising at first for their power of interpreting what the students 'see' when dealing with inequalities, this framework appeared to be too advanced for interpreting the images captured from working with my students. There was no important formal definition in our work on inequalities and therefore no interplay between concept image and concept definition that needed to be scrutinized to discuss understanding inequalities in terms of conflicts or maladjustments between the concept image and the formal definition of the concept. More precisely, if my participants were calculus students and the concept under observation was 'limits' for example, this framework could have been successfully applied, since the formal definition as well as students' intuition and images of limits and limiting are equally important for the understanding of that concept. Moreover, those detected conflicts are usually used to describe the misconceptions the students might have about the concept under the lens. In this study, aside from the fact that the tasks for collecting data were not designed to address formal reasoning, with the language of concept image and concept definition I could not ignore seeing and discussing misconceptions. This was the main limitation of this framework – it could mostly explain what the participants did wrong and why, while the focus of my investigation was on what they did and what their reasoning was.

4.3 APOS Theory

Actions, *processes*, *objects*, and *schemas* are the 'structures' of APOS theory (Dubinsky & McDonald, 2001). In APOS Theory, the understanding of a mathematical entity begins with an individual explicit *action* applied to that entity. The action is perceived as external and comprises of step by step manipulation, evaluation or operation. An action may be interiorised as a mental *process*. At this level, the individual can think of performing an action, can predict an outcome, or can describe actions verbally without having to directly carry out each and every step of the actions. An *object* is the *encapsulation* of a process. The encapsulation occurs when the subject is able to appreciate the mathematical entity as a complete object on which transformations or operations may be applied. Operations can be performed on the process itself, or the process can be seen as being part of another process of a similar nature. A mental *schema* consists of a structured collection of actions, processes, objects and other schemas connected to a particular mathematical concept. A mental schema is activated when a decision must be taken about the mental structures needed to solve a particular task. Dubinsky and McDonald argue that elaborate descriptions, referred to as *genetic decompositions* of schemas in terms of these mental constructions, can provide a tool for diagnosing students' concept development and help the instructor organize interventions when assisting learners in the development a new concept.

APOS theory provided the language and the description for two of the conceptions of inequalities identified in the study. APOS decomposes mathematical ideas into actions, processes or objects. This framework educated me in seeing how the participants in the study understand inequalities. For example, the action was embedded in the balance metaphor. Having this awareness, I could see how students mentally move things on or off the scale and try to keep it at the same level of unevenness.

4.4 Three Worlds of Mathematics

Three worlds of mathematics is the name of a theoretical framework of long-term learning that presents three ways in which mathematical thinking develops. It incorporates three different, yet connected, worlds of mathematics: *conceptual-embodied, proceptual-symbolic* and *axiomatic-formal* (Tall, 2007). This framework explains an individual's cognitive development of mathematics from childhood, up to the stage of working on and appreciating pure mathematics.

Conceptual embodiment refers to the thinking about objects' properties after an individual's perception of or physical interaction with the objects. Euclidean geometry is one example of a conceptually embodied world of mathematics. Actions on physical objects followed by actions on mental objects, such as counting, sharing, adding, subtracting or multiplying, show another aspect of the conceptual-embodied world (Tall, 2008b). The *proceptual symbolic world* is characterized by the compression of actions on physical objects into procepts such as a number, sum, product, fraction, equation or other concepts to think about. Flexible thinking and fluent work with symbols are expected from someone operating in the proceptual symbolic world. The conceptual embodiment domain and the procedural symbolic world are intertwined throughout school mathematics. The *axiomatic formal world* of mathematics refers to the axiomatic systems, formal definitions and mathematical proof that are the objects of pure mathematics at university (Tall, 2008b).

The diagram in Figure 4.1 shows the three worlds of mathematics with their connections and interactions. The cognitive growth of mathematics starts in childhood, with child's first physical interactions with the real world. It continues through embodied experiences that are transferred symbolically and manipulated abstractly. To reduce abstraction and facilitate understanding, objects from the symbolic world are connected to contexts, manipulatives, diagrams, or models in the embodied world. When learning mathematics, there is a constant back and forth between the symbolic and embodied worlds. The transition from embodiment and symbolism to the formal axiomatic thinking is expected to happen with the transition from school to university, where the student is exposed to pure mathematics (Tall, 2008b).

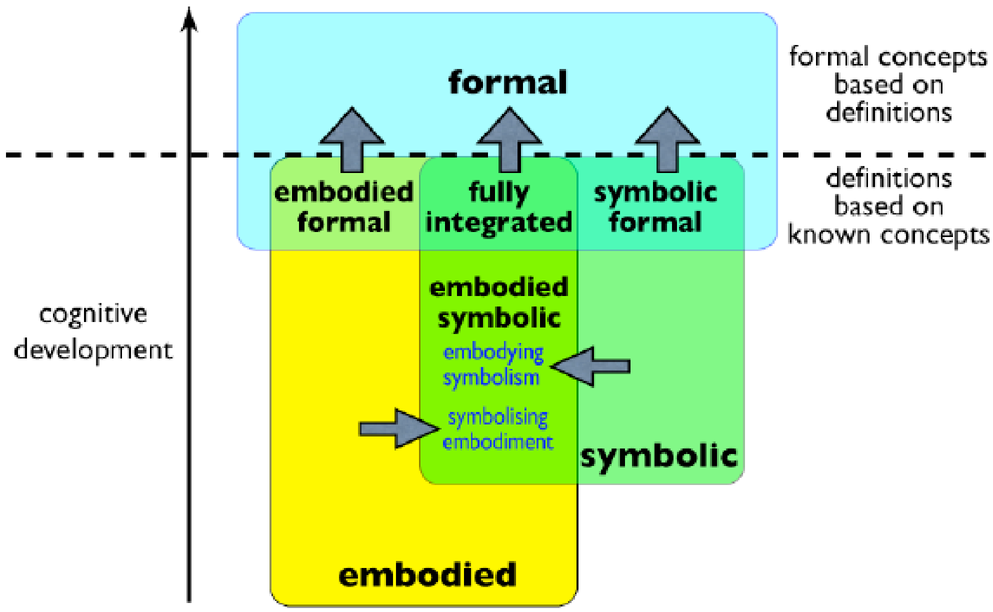


Figure 4.1: The cognitive growth of Three Worlds of Mathematics (from Tall, 2007, p.4)

The theoretical framework of the Three Worlds of Mathematics that incorporates the embodied, symbolic and formal aspects of doing mathematics, together with the overlaps and progression of cognitive development helped communicate the results of the study, by situating them on the big picture of cognitive growth in mathematics. More precisely, applied to the results on inequalities by placing the students' work and thinking into it, this framework gives evidence about the nature of students' thinking on inequalities and about the realms where their work and thinking reside. It also served musing about the regions where undergraduate students' conceptions of inequalities should be located to ensure a performance that meets expectations in undergraduate mathematics. Moreover, the Three Worlds of Mathematics would serve arguing that the earliest experiences with mathematics could either damage or solidify students' mathematics foundation. For that, it facilitates a better understanding of what kinds of early mathematical experiences students need to have in order to perform well on inequalities.

In the following chapter, the methodology for collecting data and the arguments for choosing learner generated examples for producing the data is presented. An account on phenomenography, which is the method for interpreting the data, is also given. The two empirical studies that are the core of this thesis are introduced.

Chapter 5:

Methodology

This research explores undergraduate students' conceptions of inequalities as they emerge from their work on inequalities. In what follows, the methodology employed in conducting the research, as well as the methodology used in preparing the grounds for the two studies, are presented in detail. The first section deals with the discipline of noticing, a framework that helps situate the preliminary empirical involvement with inequalities as understood by the undergraduate students in the literature. Learner-generated examples are chosen as the main source of data for the study. The rationale for choosing learner-generated examples for collecting data is given in section 5.2 of this chapter. Section 5.3 presents the unit of conception from phenomenography as the methodology for analysing the data. Sections 5.4 and 5.5 introduce the participants and describe the two studies contributing to the results of this investigation.

5.1 Situating the Preliminary Work within the Discipline of Noticing

There are so many things that can be noticed when one becomes interested in a subject, as my experience with students' understanding of inequalities has proven.

> To notice is to make a distinction, to create foreground and background, to distinguish some 'thing' from its surroundings. (Mason, 2002, p.33)

I have been an observant educator from the beginning of my teaching career. Working with students, I have noticed problems, applied treatments, introduced cognitive conflicts and observed the results. For most of the time, I have noticed whether students take the challenge or not, how soon they get frustrated by a task, if they expect a hint, or if they shut down when their discovery is spoiled by a given hint. The topics of my observations were very often randomly chosen or engaged in.

My preliminary involvement with inequalities could be situated within the discipline of noticing. "[T]he cornerstone of noticing as a method of inquiry is trying things out for ourselves rather than taking them on trust as a result of some statistical study, logical argument, or authoritative assertion" (Mason, 2002, p.30). As part of my journey, in parallel with reading and accumulating the knowledge that is already out there in the field, I have attempted using many things for myself. Section 5.3 chronologically presents the techniques and frameworks that I have tried out during my study of inequalities.

As I chose inequalities as the focus of my study, I started being more observant of the discourse on inequalities when working with many other mathematical concepts. The result was that the instances where I perceived inequalities to have appeared grew in number. For example, I could see inequality thinking when working on fractions. In other instances, I could see inequalities embedded in working on measurement with teacher candidates. I could provoke discussions about inequalities in various classes. Moreover, I could see people stumbling in their work on other topics and inferred that a bit of training in inequality manipulation and behaviour could have helped their understanding. Nardi (2007) also, analyzing students' mistakes on proofs, claims that, especially if the students got the rest of the question right, their "problem is with dealing with inequalities, not the inductive mechanism" (p.85).

In addition, I have also tried out various methods of teaching mathematics concepts. Thus, in several of my classes, I applied a similar treatment to teaching different concepts, such as transformations of functions, graphing functions, fractions, or inequalities. In other classes, I treated inequalities as a very special concept. In both settings, I collected the results on inequalities items, comparing them with other concepts, or with the remaining part of the exams or surveys. For the majority of the students in either treatment, the results on inequalities were much lower than their results on some of the other tasks. For example, in a Precalculus class, only 4% of the students got a passing mark on the inequality item, while only 24% of the students failed the exam where inequality was one of the items. Moreover, out of 58 Precalculus students, there were only four who correctly solved the inequality $\frac{12}{(x-1)} \leq x$. The large majority of students used a pattern of equations to solve the inequality, multiplying the inequality by $(x-1)$ and completely ignoring the role of the denominator in the solution. Reflecting on those results, the first question I tried to answer was: Is it my teaching that is influencing my students' misunderstanding of inequalities? To answer this question, I planned that the next time I would teach inequalities, I would be more observant of my teaching methods, as well as of my students' response to them. Thus, I had refined my inequalities tasks for a new course I was teaching that contained a section on inequalities. Employing "the discipline of noticing" (Mason, 2002) I paid more attention to students' work and struggles, all the while making my students aware of the pitfalls and problems with the concept of inequalities.

With all the extra caution when teaching a new class, the problem with not understanding inequalities persisted. A question followed: Am I the only mathematics instructor with disastrous results in teaching inequalities to undergraduate students? After asking myself such questions, I inquired into whether my colleague who was teaching the other section had similar results. Had the instructors' teaching the same course last term had the same problems? They had! Who else encountered the same obstacles when teaching inequalities? And that question triggered the search for an indication of the problems with inequalities in mathematics education literature. I was, therefore, not surprised to see many mathematics educators dealing with the same struggle: students' work on inequalities is problematic. Different teaching perspectives were employed and many frameworks for analyzing students' work were used in order to capture the essence of the problem with inequalities. I have noticed again that the studies were mainly reporting on the problematic aspect of teaching and learning inequalities (Bazzini & Tsamir, 2001; Bazzini & Tsamir, 2003; Tall, 2004; Kieran, 2004; Tsamir, Tirosh, & Tiano 2004; Boero & Bazzini, 2004; Sackur, 2004; Vaiyavutjamai & Clements, 2006b). However, not much was written about the essence of what makes this concept so prone to mistakes and misconceptions. Needless to say, there are questions that remain unanswered in the literature. Many of these are related to what makes inequalities one of the hardest topics in Precalculus classes.

5.2 Examples and Learner-generated Examples in Learning Mathematics and Learning about Learning Mathematics

Examples play a key role in both the evolution of mathematics as a discipline and in the teaching and learning of mathematics. There is an abundance of research that acknowledges the pedagogical importance of examples in learning mathematics (e.g., Atkinson, Derry, & Renkl, 2000; Watson & Mason, 2005; Zhu & Simon, 1987; Kirschner, Sweller, & Clark, 2006; Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006; Mason & Pimm, 1984).

It is very well known that not only novices' but also expert mathematicians' self-explanations and understanding of a concept depend on examples. To understand something in general, even experts report that they have to carry along in their mind specific examples and see how they work (Bills et al., 2006). While the experienced mathematician may need one example to promote the understanding of a general method, most novices ask for numerous examples in order to get some intuition about the situation and promote some generalization and reasoning from them. Watson and Mason (2005) claim that learning mathematics, in fact, "consists in exploring, rearranging, and extending example spaces and the relationship between and within them" (p. 6).

In my study, I required my students to generate examples from their present experience on inequalities, as well as extreme examples. In my context, extreme examples, examples that are counterintuitive, would be an inequality resulting in equality (the case of a single number solution to an inequality) or an inequality resulting in all real numbers or all real numbers except for one as a solution. Bills *et al*. (2006) define canonical or conventional example spaces as those collections of examples that are easily available to students, found either in the textbook or in the classroom, presented by their instructors. The examples that could be easily drawn from the conventional example space could contribute to picturing the concept image that the students have for the notion of inequality; extreme examples, on the other hand, could inform about the boundaries of their conceptual understanding (Watson & Mason, 2005). Through examples construction, students "become aware of dimensions of possible variation and corresponding ranges of permissible change within a dimension, with which they can extend their example spaces" (Bills *et al*., 2006, p.18).

Transferring the responsibility of generating examples to students and the responsibility of gaining insight about students' learning and understanding of mathematics to researchers is well represented in literature (e.g., Watson & Mason, 2005; Zazkis & Leikin, 2007). "[T]he examples learners produce arise from a small pool of ideas that just appear in response to particular tasks in particular situations" (Watson & Mason, 2005, p. ix). However, very often, to attend to a task, students have to put pieces together and to make connections. In that case, generating examples seems to have a similar power of discovery. In a reverse task, when students have to mentally reconstruct a backward path to get an example, or when there is no direct path that students could walk to get the answer, there is thinking, creation and verification – it is those processes that can inform researchers about the nature of students' understanding.

To the best of my knowledge, no published study on mathematical inequalities has used learner generated examples as a source of data. Usually, a solve-an-inequality task does not provide much variation in students' solutions. Students' work very often reproduces the procedure learned from the teacher or from the examples offered by the textbook. Research on inequality with data taken from solving inequalities tasks shows some variation in students' errors, but not much variation in the example space that students have to access to solve the task. For example, when given a task which reads "Can $x = 3$ be a solution to an inequality?" (Tsamir & Bazzini, 2001, p.1) the typical answers could be (1) No, the inequality results in inequality, not in equality or (2) Yes, inequalities of the form $a \le b$ can result in equality. Asking them to generate an inequality of some sort and then to work it out so that someone who follows their work would be able to learn how to solve that type of inequality, students are invited to be creative, to search their personal example space of inequalities, to access different registers of presenting inequalities or to connect the different snapshots that create the concept image for that concept – in other words, they are invited to think. Showing a correctly solved solution is not a guarantee of students' understanding of inequalities; they could have memorized procedures or followed step-by-step algorithms.

I argue that by generating an example, solving the example, and explaining the steps, would allow us to see more variation in the data, thus helping us better capture students' understanding of inequalities. Starting from the result often involves undoing, a process which is harder than doing, since it does not start with a memorized set of steps. Starting with the idea of a pilot example in mind – an example that incorporates as many critical aspects as possible – may involve undoing, in order to be able to fit all the properties of inequalities in it. For example, if someone would like to create an example that involves dividing by a negative number, that person truly has to be able to mentally see how the components of the inequality move from one side of the inequality to the other and would thus be able to generate a negative coefficient for the variable. Creating a worked example from which somebody has to learn involves not only one's concern with correctly solving a given inequality, but scaffolding for someone's understanding.

Other studies have also considered student generated examples to be a good tool for looking inside learners' minds and learning about their understanding of mathematics (Zazkis & Leikin, 2007). Moreover, studies found that "[w]hen learners have been asked to create their own examples, they experience the discovery, construction or assembly of a space of objects together with their relationships" (Bills et al., 2006). Appendix 1 contains a participant's reflection on the complexity of a generate-example item in contrast to a solve-inequality task.

In summary, in my study, the roles of producing and learning from examples are slightly shifted: The participants are considered the experts in linear inequalities and they have to produce examples or worked examples for somebody else to learn from them. From the examples provided by the students and their work towards solving the examples as well as their explanations, the researcher learns about participants' understanding and conceptions of inequalities. The experimentation has been possible and unrestricted since I held the advantage of being an undergraduate mathematics instructor and of having potential participants in my classes. For four semesters, I collected data from the undergraduate classes I have been teaching. A full description of the participants in the research is given in section 5.4.

5.3 Conception

'Conception' is the unit of description in phenomenography (Marton & Pong, 2005).

Etymologically, *phenomenography* comes from the Greek word "phainomenon", meaning appearance and "graphein" – description. Therefore, phenomenography can be understood as the "description of things as they appear to us" (Pang, 2003, p.145). Phenomenography is basically the study of *variation* – variation among different ways of seeing, perceiving, experiencing, or apprehending phenomena (Marton, 1981).

Phenomenography "is a relatively distinct field of inquiry … complementary to other kinds of research; it aims at description, analysis and understanding of experiences" (Marton, 1981, p.180). Descriptive and methodologically oriented, phenomenography is concerned with identifying the key dimensions of variation in experiencing phenomena (Pang, 2003). The research questions that come from this strand deal with the different ways of experiencing a phenomenon and how these are related to each other. Recently, by returning to the original question from which phenomenography emerged – the study of students' differences in learning – a qualitative shift occurred in the field of phenomenography; one of the new questions that has emerged deals with what the actual difference is between two ways of experiencing the same thing. The question is directly lined up with the puzzle over why some people are better at learning than others.

Specific to phenomenographic research is the assertion that there are always a limited number of qualitatively different modes in which the students see a particular phenomenon. These different ways of understanding a phenomenon originate in the different critical aspects of the phenomenon that the unique individual discerns (Marton & Tsui, 2004, p.194). When categorizing data with a phenomenographic lens, the focus is on determining the qualitatively different or similar ways in which the participants are experiencing the phenomenon. This method helps derive “the pool of meanings" from all the contributions stemming from the same concept (Åkerlind, 2005). Within the pool of meanings, different categories are formed and “defined in terms of core meanings” (p.4).

For example, in a study conducted on high school students with no previous formal knowledge of economics, the participants produced the data from where four conceptions of price were identified: (A) price as reflecting the value of the object, (B) price as reflecting demand, (C) price as reflecting supply, and (D) price as reflecting both supply and demand. From the same study, two conceptions of trade emerged: (X) trade as a win-win activity and (Y) trade as a zero-sum game (Marton & Pong, 2005). What is worth mentioning here is that, in the case of conceptions of price and trade, a student may have expressed not only one, but different conceptions, depending on the context he has worked on. Since the participants in my study have been supposedly exposed to the concept of inequalities many times in their school years, I expected that the conceptions of inequalities to be less fluid than the conceptions emerged in the study conducted by Marton and Pong.

Marton and Pong (2005) acknowledged that, in the literature, *conception* could be found under various names, such as ‘ways of conceptualizing’, ‘ways of experiencing’, ‘ways of seeing’, ‘ways of apprehending’, or ‘ways of understanding’ (p.336). Although conceptualizing is not similar to experiencing, or to seeing, or to understanding, each one of these labels brings some new light when analyzing learning and concept formation with the lens of phenomenographic conception. In this present work, I focus on finding the various ways undergraduate students ‘see’ inequalities. The different ‘ways of seeing’ inequalities inform the sorting of data in different categories and the derivation of *conceptions of inequalities.*

In summary, phenomenography became the method I used to analyse the data. When sorting the data, I focused on seeing the qualitatively different or similar ways in which the participants were experiencing inequalities. Although appearing to be very similar in aspect or given the same mark on the first sorting, through phenomenographic lens, the data became transparent of qualitative differences. This allowed the sorting into different categories. Then, in each separate category, I identified the core meaning which helped me describe the conception. In the end, using a phenomenography as a method of interpreting data, I began to see the conceptions of inequalities emerging and my research questions answered.

The following sections introduce the participants and foreshadow the studies.

5.4 General Settings

The setting for my study on inequalities is Simon Fraser University and the participants are from seven classes of students that I taught: six ‘Foundation of Analytical and Quantitative Reasoning’ (FAN X99) classes and one ‘Precalculus’ (MATH 100) class. The data collection spanned over four semesters, each semester of 14 weeks of classes. The syllabi of both FAN X99 and MATH 100 contain inequalities. In FAN X99, linear inequalities are studied as an independent topic in the second part of the course. In MATH 100, inequalities are everywhere, beginning with a review of linear inequalities learned in Grade 11, continuing with solving quadratic, polynomial, and fractional inequalities, and culminating in the use of inequalities to find the domain of logarithmic, irrational, or trigonometric functions.

5.4.1 FAN X99

FAN X99 – Foundation of Analytical and Quantitative Reasoning – is a non-credit mathematics course, designed for students who need to upgrade their quantitative background in preparation for quantitative courses. The course was designed to strengthen number sense and mathematical reasoning through problem solving and inquiry. This course is recommended for students who wish to refresh their mathematics skills after several years away from mathematics. The group of students taking FAN X99 varies greatly in skill levels. In the same class, one could find students who enjoy problem solving and exploration, and would immediately get engaged in a task. On the other hand, there are students who will wait for a formula to emerge from someone else's work or from memory, in order to be forced into the problem without a trace of understanding of the concept or the context.

A common attribute for the majority of the students in a FAN X99 class is that they do not want to be in the class. The majority of students have to take this Foundations of Mathematics course to qualify for taking other quantitative courses, such as Statistics for Psychology or Social Sciences or Mathematics for Teachers. Many of them have math anxiety and prefer sitting quietly in the classroom, taking notes, and memorizing something to regurgitate in a test, for a mark that would allow them to finally move away from mathematics. They do not expect to be invited to present their solutions to the class problems or to discuss problems with a partner. In addition, they expect to be provided with rules and algorithms, and not to be encouraged to use their life experience and critical thinking in order to solve a problem. Although their skills are also weak, a small fraction of FAN X99 students takes this course with a declared intention to pursue a business or a computing science degree. This means that they hope to be successful in calculus and discrete mathematics, one day. The class met twice a week for a two-hour seminar for an entire semester. The course mark was determined from performance in weekly quizzes, group assignments, two midterms and a final exam.

5.4.2 MATH 100

MATH 100 – Precalculus – is a course designed to prepare students for first year Calculus. The course is very condensed and includes language and notation of mathematics; problem solving; algebraic, exponential, logarithmic and trigonometric functions and their graphs. The group of students taking MATH 100 is very heterogeneous as well. Students who have taken Principles of Math 11 and 12 and have already met all of the above functions are in the same class with students who will start from scratch with all of the material. While some will tackle the basics of a trigonometric function, for example, the others, who have worked with that function in high school, will only have to focus on encapsulating the function to be able to use it as an object at the next level – calculus. The class met twice a week for a one-hour-and-a-half lecture.

One may wonder if this decision – to involve in this study my classes only, on which there were not too many mathematically inclined students – would not limit the findings. I argue that this decision was not for convenience alone. Having my students under my observation for 13 or 14 weeks in a row and getting to know them, the data was not only some arid mathematical product. They were products of thoughts coming from people I knew and I could relate to their strengths and weaknesses or to their ways of seeing thinks. The data talked to me, as if the participants were talking to me in an interview. Of course, that this memory of who wrote that piece of information that was analysed was vivid at first. However, in the final stages of interpretation, it faded away and the product alone remained on paper to talk to me through the phenomenographic amplifier.

5.5 The Studies

Once the topic of the study had been decided and the participants had been identified, what is the nature of the data is the next major focus and the most tedious part of my work. Given the fact that the participants were my students, I felt confident that I could orchestrate various events for collecting the data: I could create test items, class tasks, and individual or group work. The Task (T) and the Revised Task (RT), which are two of the main sources of data, are presented and described in the next chapter. Other tasks that gave substance to my findings are mentioned in different sections of data analysis, in Chapters 6 and 7. For a complete list of things I tried during my study, I have created Appendix 2, which is a collection of tasks I implemented during my involvement with inequalities. Watson and Mason (2005; 2006) introduced me to examples and learner-generated examples in learning mathematics. From there, I moved on to Marton's theory of variation (Marton & Tsui, 2004). Therefore, I used examples for data collection and the unit of conception from phenomenography for data analyzing and interpretation. However, the data interpretation was tedious and consisted of different stages of looking at the same tasks or at various tasks using similar or different lenses. At some point, the

Conceptions of Inequalities (COIN) emerged, and after that, I decided to start over with a new set of data and a framework to apply to it.

5.5.1 Study 1: Creating an Instrument to Analyse Undergraduate Students' Work on Inequalities

In my first attempt to do an entire cycle of research in mathematics education, I chose two classes of FAN X99 to work on an inequality task. The participants were asked to create a worked example of an inequality. In all, there were 31 participants. The details about how the task was implemented, how students worked on the task and what the results were are given in Chapter 6. It is worth mentioning here that the data revealed that learner-generated worked examples was a good choice for the purpose of the study. However, the task needed some revision, in order to become more inviting in choosing and working the example, action that may produce a more transparent of thinking results in a new iteration. The revised task was implemented during the next semester in two other FAN X99 classes, as well as to a MATH 100 class. The initial task and the revised task are part of Study 1, which was designed to produce an instrument to analyse undergraduate students' understanding of inequalities. The details of the study, the analysis of the data and the results are presented in Chapter 6, where the *Conceptions of Inequalities* (COIN) are named. The framework COIN emerging from Study 1 is carried over in Study 2 and used for analysing the new data. The qualifiers 'preliminary study' and 'Study 1' are used interchangeably in the following two chapters.

5.5.2 Study 2: Using the COIN Instrument to Evaluate Undergraduate Students' *Conceptions of Inequalities*

Study 2 was designed to validate and give more substance to the COIN framework. The participants were again the MATH 100 class and two classes of FAN X99. In MATH 100, the survey was implemented as an item in the final exam. For the FAN X99 cohort, a new task for data collection was designed. The qualifiers 'the major study' or 'Study 2' are used interchangeably in the following chapters.

5.6 Summary

This chapter introduced the methodology of the study. Since the inception of opting for inequalities for my research, the discipline of noticing became the theoretical grounds on which my initial actions on and my awareness about the concept of inequalities were rooted. Learner-generated examples were chosen as the instrument for data collection. The section dedicated to examples presented some theoretical perspectives on the topic and the motivation for using learner-generated examples. The language of conceptions and the insight it brings into discussion being preferred to the jargon of misconceptions, phenomenography became the research specialisation for the final stages of data analysis. With the idea of this research arising in the context of my teaching, the decision to conduct the study on the classes I was teaching during the study seemed necessary and a natural continuation of my work with my students. The rationale for this preference is also part of this chapter.

Chapter 6:

Study 1: The Emergence of *Conceptions of Inequalities*

6.1 Introduction

Before I embarked on the major study for this thesis, I initiated myself into the collection and interpretation of data through a preliminary study. I wanted some real work with real data to experience a workable methodology and framework in empirical research in mathematics education. Having had classroom experience and with the results obtained from the literature described in Chapter 3, I decided to use learner generated examples as the source of my data. Therefore, I formulated a simple *create a worked example of an inequality* task. I designed a questionnaire with that task and gave it to a class of 31 university students enrolled in a Foundation of Analytical and Quantitative Reasoning (FAN X99) course. Analysis of the data revealed that, after two weeks of class work on inequalities, the majority of students were unable to identify a linear inequality or to solve it correctly. These results suggested that my students did not have an adequate understanding of inequalities that would enable them to confidently function in the remainder of the course or even to pass the final exam in that course. These results also motivated me to continue my search into undergraduate students' understanding and conceptions of inequalities.

In Chapter 3, I have reported on a number of different studies have investigated inequalities alone or in conjunction with equations (Linchevski & Sfard, 1991; Bazzini & Tsamir, 2004; Vaiyavutjamai & Clements, 2006a) or in the context of functions (Rivera & Becker, 2004; Sackur, 2004). The results of those studies on inequalities reported mostly on students' misconceptions of inequalities (Linchevski & Sfard, 1991; Tsamir & Almog, 1999; Bazzini & Tsamir, 2004; Tsamir, Tirosh, & Tiano, 2004; Sackur, 2004). There have been indeed some attempts to decipher the nature of those misconceptions (Linchevski & Sfard, 1991; Tall, 2004; Dreyfus & Hoch, 2004). However, there has been a tendency to report mostly on *how students perform on inequalities.* The main objective of this study is to explore *why students perform poorly on inequalities.*

Regarding data collection and analysis, most of the studies based their results on students' work on 'solving inequalities' tasks (Bazzini & Tsamir, 2003; Tsamir & Almog, 2001; Vaiyavutjamai & Clements, 2006a). Some of them based their studies on 'examining if presented work on some inequalities' is correct or not (Bazzini & Tsamir, 2001, 2003; Linchevski & Sfard, 1991) or on 'multiple-choice' items (Tsamir & Bazzini, 2001; Bazzini & Tsamir, 2004). As mentioned in the chapter on methodology, no published research has used learner-generated examples for data collection on inequalities.

The *worked example* of an inequality task was initially given to a class of FAN X99 students. After carefully looking and interpreting preliminary data, the task was refined and implemented again in two other classes of FAN X99 and a MATH 100 class. The following section introduces the initial task, the normative solution and the initial work on sorting, as well as coding and interpreting data. The sections that follow consist of the preliminary results and the emergence of a framework that will become the lens for the major study.

6.2 Setting

In the first iteration of the task, data was collected 8 weeks into a FAN X99 class where the participants had been exposed to problem solving, discovery, and making connections, rather than lecturing. Generating examples, mostly in class, was a daily routine. A document camera was part of the class' equipment. This allowed for students to project notebooks with their work to the entire class. Samples of different normative examples or various solutions to word problems were presented, projected to the class, discussed and interpreted on a daily basis. The extreme or peculiar examples, the counterexamples as well as the examples that did seemed not related to the task were not ignored: these examples usually created some of the most rewarding teachable moments, where the boundaries of a concept were pushed and connections between old and new concepts were made. The course mark was determined by weekly class quizzes on skills, weekly homework assignments, two midterms and a final exam. Homework assignments consisted usually of four rich problems or explorations. They were done with a partner and students were expected to explain all the steps involved in the solution of a problem. The exams comprised equally of skills and word problems and the process of solving a problem was valued more than a correct final answer with no work shown. Thus, the students were prepared to produce examples as well as to explain their work and their thinking.

The model of teaching-learning mathematics in this class did not follow the transmission-assimilation metaphor, where the students are offered examples to learn from and, in case they are asked to construct new objects and understandings, they are expected to match those offered by the instructor (Watson & Mason, 2005). The focus of this class was to use the learner's experience in revealing the general observed in the particular. During the class time and on assignments, students were invited to think and make connections, a strategy that is documented by research in mathematics education to be of great benefit to understanding. With this in mind, the work done with my students prior and during the collection of data was two-fold: one was to collect data that is transparent of students' thinking and the other one was to help students construct their concept of inequalities. The task in Section 6.3 was given as an open book, individual, 30 minutes class work.

The task was refined and implemented again with two FAN X99 classes and a MATH 100 class. For the second iteration of the task, the participants from the FAN X99 classes were taught in a manner similar to the initial class. As mentioned previously, MATH 100 was not a seminar as FAN X99 was; it was a 3-hour per week lecture format. The task was implemented at the end of a lecture and students were given 15 to 25 minutes to individually work on it. For MATH 100, the preparation for the 'create an example task' was minimal – the students were not exposed to generating examples themselves and backing up their work prior to responding to the task. As for inequalities themselves, one class was dedicated to reviewing the concept and the night before the task they handed in an assignment which comprised mostly of linear and rational inequalities. Thus, for the MATH 100 students, the example generation task was novel.

However, the concept given in the task was expected to be well known from previous classes and revised prior to undertaking the task.

6.3 The Task (T)

a) Create a worked example that will show someone how to solve linear inequalities.

b) Is the one example provided in part a) sufficient for someone to learn how to solve inequalities by following your work? Do you think you need to create more examples to demonstrate the full breadth of linear inequalities? If so, how many more examples do you think you need?

The normative (fictitious) solution to item a) – constructing a worked example of an inequality – could be an example that incorporates, if possible, all properties that will help transform a given inequality into an equivalent inequality which are summarised in section 2.1.2, as well as the conventions related to writing the solution in interval form and graphing the solution on a number line. For example, $-2+3x<6x+7$ could be a possible response. The worked example follows:

$-2+3x<6x+7$ $3x-6x<7+2$	- Separate terms containing the variable by adding 2 and subtracting 6x on both sides
$-3x<9$ $x>-3$	- Divide by -3 and reverse the inequality symbol
Solution: $(-3,+\infty)$	- Solution in interval form
	- Graphical representation of the solution

For b) the normative answer could be "I think that a few more examples where *less than* as well as *less than or equal to* are used would benefit exemplifying the different types of intervals necessary for writing the solution." In addition, a few examples where inequalities produce no solution or all real numbers as solution will be expected to be found in some papers.

The responses to the task varied in length from a few simple examples of inequalities (or what they considered to be a linear inequality) to two or three pages of work. Some of them chose to start the work by directly writing down a worked example. Other participants chose to address the task in the context of explaining how to solve an inequality to someone who is new to the subject. More precisely, they introduced first the notations and various representations of solutions, such as intervals or graphs. Others chose to exemplify first what a solution to an inequality is, then to introduce the properties of inequalities and the interval notation, and only after this preparation, to give worked examples of inequalities. Some participants included a word problem as the concluding example of inequalities while others exemplified no inequalities but presented real-life contexts where inequalities were perceived as representing the situation.

6.4 Initial Results

6.4.1 Looking at the Data – Level 1 Analysis

The first sorting of data followed a rubric of anticipated work. The rubric comprises five distinct categories of responses (examples of linear inequalities), labelled with numbers from 0 to 4. The numbering is inspired by the potential marks for the work, given that the task would have been collected with the purpose of testing students' knowledge of inequalities. As the rubric was being created, the focus was mostly on the following questions: (1) Did the respondent attend to the given task? (2) If yes, what types of examples are presented and how is their work accomplished?

After 'marking' the papers and placing them in the five categories of anticipated work, I counted the number of papers falling in each category and recorded them. Table 6.1 shows that 39% of the respondents did not give an example of an inequality. Some of these exemplified a linear equation in one variable, others exemplified linear equations in two variables, and others linear inequalities in two variables. Initially, I had categorized those who gave a different type of inequality as an example in category 0. Although they did not correctly identify the required concept of linear inequalities, after another scanning, some of those papers fell into categories other than 0, such as 1 or 3, considering that they could inform about respondents' understanding of inequalities in general. Table 6.1 comprises the categories of examples and their corresponding percentages.

Category	**Number**	**%**	**Description of anticipated work**
0	12	39%	No inequality or other concept than linear inequality is exemplified.
1	0	0%	A simple inequality is given. No attempt to solve it.
2	5	16%	A simple inequality is given. Solving follows the pattern of equations. Some good steps. The solution is wrong.
3	11	35%	A simple inequality is given. Solving follows the axioms of inequalities. The solution is correct.
4	3	10%	An example that purposely incorporates division by a negative number is given. Solving incorporates maximum variation and aspects related to inequalities.

Table 6.1: The initial results

When working on anticipated solutions to the task, I foresaw that some students will follow the pattern of equations when solving them. Asking the students to be explicit about the steps for solving inequalities, I came across an unexpectedly clear way of explaining such a pattern. Here is the example given by one student with his own explanations on the side:

$2x+8<20+4x$	- Sub ‘<’ with ‘=’ to turn it in a linear equation.
$2x+8=20+4x$	- Solve as you would a linear equation by adding 8 to both sides and subtracting 4*x* from both sides.
$2x-4x=20-8$	
$-2x=12$	
$x=-6$	
$x<-6$	- Resub ‘= with ‘<’ to get possible values for *x*.
$Answer: x<-6$	

The student explicitly turns the inequality into an equation and emphasises that aspect in the explanation of his work “*Sub ‘<’ with ‘=’ to turn it in a linear equation.”* His solution shows this: solving the attached equation and turning it back to an inequality form after isolating *x*. However, no concern is shown for checking the solution to see that by solving an equation instead of an inequality, the student hopelessly divides by a negative number, resulting in exactly the complementary solution. Up to this point, my intuition was that ‘using the pattern from equations to solve inequalities’ means applying the properties of the equal sign instead of the inequality sign to an inequality. However, some students replaced the inequality sign with the equal sign. It is not unusual for students to transform a new or peculiar concept into a concept with which they are more familiar and know how to manipulate; reducing the level of abstraction is one way students cope with hard, abstract algebraic concepts (Hazzan, 1999). Making connections and building on previous concepts is a recommended approach to constructing new knowledge. In this case, it seems that too much emphasis on that aspect of learning could be an obstacle in correctly accomodating the inequality concept. Moreover, in phenomenographic studies, it is claimed that learning occurs when someone working with a new concept that looks similar to a familiar one, discerns what is different between the old and new and focuses on the new aspect that that dimension of variation brings to the concept (Marton & Pong, 2005). Some of my students seemed to eliminate the variation and therefore, the concept lost its specificity.

It is important to mention that I anticipated that some of the participants, in order to capture the division by a negative number in the solving process, will start with a task reversal (Watson & Mason, 2005) – more precisely, that they would start from the solution and go backwards by undoing the steps to solve the inequality. Such a pattern was not visible in any paper. Moreover, when working on the example generation task, the students could just retrieve the structure of an inequality from their memory – which is an algebraic expression using the < symbol – and choose some random numbers to go with that structure. The data exposed simple examples like that in the responses to the task. However, due to their willingness to create exactly one example that will incorporate the complexities they were aware of, some students did not focus on key dimensions of possible variations of inequalities – such as division by a negative number – and instead focused on choosing fractions or big numbers as coefficients. This decision increased their workload and their chances of not getting the right solution when solving.

6.4.2 Looking at the Data – Level 2 Analysis

A second sorting of the data was done with two additional questions guiding the coding: (1) What are the recurrent themes present in the data? and (2) Is the individual's understanding of inequalities visible in the data? Five different categories emerged again, the same number of categories found in the first level of data analysis. Through a deeper analysis of each category, I was able to identify the recurring ideas that helped the classification and thereby decided that 5 is the number of visible variations in the data. Table 6.2 summarizes the five categories and present their frequencies.

My next step was to look at whether there is a one-to-one correspondence between the categories of the first scanning and the classifications that emerged at the second round. For the categories with scores 3 and 4, I could make a one-to-one correspondence between the first and second stage of data interpretation, but some papers from category 0 in the first level analysis moved to category 2, to a pile labelled 'contextual understanding of inequalities'.

Skemp's (1976) and Hiebert and Lefevre's (1986) classification of mathematical understanding provides the language to describe the five levels of understanding emerging from the data. Skemp identified instrumental understanding as "rules without reason" and defined relational understanding as "knowing what to do and why" (p.2). Hiebert and Lefevre defined procedural understanding as knowing the symbols accepted in a special mathematical situation (p.7). The data exposed variation in understanding from completely missing to exemplify the concept of inequalities, to shallow procedural understanding, to procedural understanding and, in a small percentage, to the relational understanding.

Category	Level of understanding	What the example was telling about inequality
0 32%	Confusion - different concept, such as a linear equation with two variables exemplified	Inequalities have foreign representations – images of other concepts, such as linear equations in two variables incorporated in the concept image of inequalities. Inappropriate use of the inequality symbol.
1 16%	Traces of procedural understanding of equations	Inequality is perceived as some sort of equation, thus the $<$ sign is replaced by the = when solving the example.
2 6%	Contextual understanding of inequalities	Inequalities describe real life situations, thus their examples focus more on the contexts using inequalities rather than the concept of inequality itself.
3 36%	Procedural understanding of inequalities – traces of relational understanding of inequalities	Inequalities have special behaviour when multiplied or divided by a negative quantity - focus on different representations of inequalities as well as on the particular aspects that separates equations from inequalities. The properties of transforming inequalities into equivalent one are correctly used.
4 10%	Relational understanding of inequalities	Inequality is a mathematical concept that must be learned in connection with recognizing the symbols, understanding intervals and some axiomatic preparation - focus on pilot examples that will incorporate maximum variation and aspects related to inequalities.

Table 6.2: Levels of understanding

Table 6.2 comprises the emergent themes and qualifies the level of understanding, from misunderstanding to the higher level of relational understanding, as well as the percentages of responses in each category. The differences in percentages are due to the fact that for the second scanning of the data, the focus was not on the linear inequality but on the inequality itself as understood by FAN X99 students. With this lens, I could see that although some of the participants who apparently did not attend to the task still exposed either contextual understanding or procedural understanding of inequalities. However, the majority of the respondents who chose to use graphical representations of lines in order to represent and solve an inequality could not see the inequality on the graph, or were unable to read its solution from the graph. Research claims that the "visual enactive activity" with dynamic entities which are functions "can give a powerful embodied sense of global relationship" between functions and inequalities (Boero & Bazzini, 2004, p.141), an aspect that could benefit students' understanding of inequalities.

Sackur (2004), however, showed that it is not easy at all for students to shift from the dynamism of a graph where *y* is the moving entity to reading the solution on the *x*-axis. Also, it is not easy for students to switch from seeing the dynamic solution as a point moving on the *x*-axis between some boundaries to giving the solution as a static entity – an interval. The data coming from my study seems to emphasize those findings: some of the participants who chose a visual representation of an inequality, failed to make the connection between the graph of a line and the inequality it might represent. For example, in Figure 6.1, the student could not read that $\frac{5}{2}x-2\leq 0$ for all $x\leq\frac{4}{5}$. The student presented the graph as a visual support for the inequality $y\leq\frac{5}{2}x-2$.

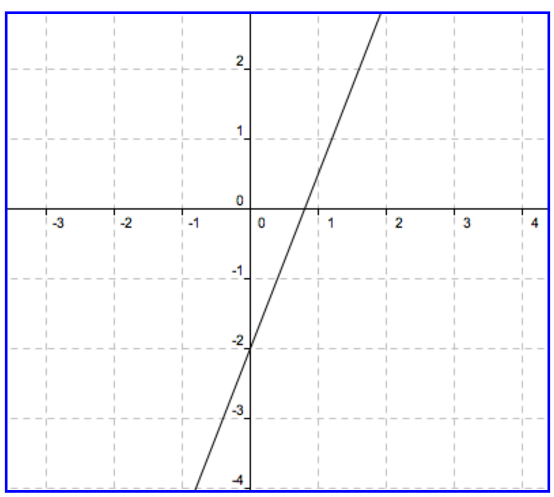


Figure 6.1: The graph of a line

In the first stages of data analysis, the findings were hierarchically presented. As mentioned in Chapter 4, different theoretical frameworks were initially applied to the data. At that moment in time, a shift in my interest in this study occurred; I became more interested in a less hierarchical classification of levels of understanding. ‘Conception’ – the unit of description in phenomenography (Marton & Pong, 2005) – became more appealing than ‘levels of understanding’ for analysing the data. Thus, a third sorting of the data was completed in which the focus of the lens shifted from students’ understanding of inequalities to their conceptions of inequalities. The new scanning of data followed the questions: (1) What are students’ conceptions of inequalities? and (2) What can the conception of inequality tell us about an individual’s understanding of inequalities?

The third scanning of the data validated the five different categories of understanding that emerged previously and attempted to identify the portrayed concept images of inequalities. The framework *Conceptions of Inequalities* began to emerge. The descriptions of concepts were firm. However, the names of the five conceptions of inequalities were not yet selected. Although possible options for the names were listed, the names were chosen only after the second iteration of the task.

6.5 Sanding the Lens - The Second Iteration of the Task

Liljedahl, Chernoff, and Zazkis (2007) observed that very often our own approach to the task obscures other approaches to solutions. My focus on learner generated worked examples influenced the creation of the task. A worked example on inequalities was seen as a step-by-step demonstration of how to solve inequalities. The normative solutions – i.e., my own approach to the task – contained all the steps for solving it, as well as some peculiar examples that showed that inequalities could not be easily put into strict algorithms in order to be solved. However, no respondent addressed this aspect in the initial task. Adjustments to the task seemed to be necessary. Learning from a worked example seems more meaningful than creating a worked example like the one described. As such, in the first part of the second iteration of the task, I introduced a fictional character by the name Jamie, to whom the students could relate and whose mathematics abilities are known to the participants. More explicitly, in the task implemented in MATH 100, Jamie is student's cousin and is supposedly taking Principles of Math 11. Jamie of task implemented in FAN X99 class is participant's homework partner. Jamie missed the class on linear inequalities and needs help with them. He is the one who is going to learn from the generated and solved example. What Jamie knows about inequalities at this moment is not a mystery anymore or an unimportant fact. Jamie could represent the respondent himself a few years ago or at this moment in time. How the respondents scaffold this concept for Jamie could inform me about their concept image of inequality. So, in the first part of the task, Jamie is going to follow the respondent's work.

Part two of the initial task invites the participants to think whether their example covered the whole complexity of linear inequalities and whether they should consider additional examples that serve that purpose. As mentioned previously, nobody attended to the idea that an inequality can produce an empty set as a solution, for example. All the provided examples ended up in intervals and there was no separate mention about that aspect. Therefore, for the next iteration of the survey, I decided to adapt one of Fujii's

(2003) survey items, to create a cognitive conflict in part two of the task. The goal of this cognitive conflict was to force the respondents to rethink their examples so as to incorporate that feature of inequalities in their response.

6.5.1 The Refined Task (RT), Results and Analysis

The refined task was implemented as a closed-book class quiz.

> You know that the best way to learn something is to teach somebody; therefore you have agreed to tutor your cousin Jamie who is taking Principles of Math 11 this year. You are available for him any time and through any means.
>
> a) You've got a text message from Jamie that reads: "*Missed the class on linear inequalities. I have to do my homework. Don't know how to start. Help me with the steps of solving a linear inequality*." E-mail him back the steps for solving linear inequalities. On the space below, show the message as well as your preparation for sending it.
>
> b) Half an hour later an e-mail from Jamie arrives: *"I followed your steps and solved a whole bunch of inequalities. Thanks. Then I attempted this one:* $1-2x>2(6-x)$. *I worked out the algebra and got this* $1-2x>12-2x$ *and then ended up with:* $0>11$. *Here I got stuck. Please help.*" E-mail him back. On the space below show the message you will send to your cousin Jamie. The message should contain your feedback on Jamie's work as well as your input to Jamie's further understanding of inequalities.

Coding the first set of papers from the refined task followed a path similar to the first iteration of the task. However, it was less laborious, given that I possessed the lens to observe the students' work at each level of coding. Marking for completion of the task was again the first sorting. In general, it was easy to fit data into the five categories produced by the first iteration of the task. However, for part a) of the task, in the data coming from MATH 100, 15% of the data contained an aspect completely unanticipated – e-mailing the steps for solving the inequality without accompanying them by an example. After rethinking this new aspect, the issue was easily addressed by correlating respondents' work with another portion of the surveys. For example, some of the responses that had major mistakes in the solutions of the first item or only a vague, unclear, or incomplete list of steps for solving an inequality were placed in category 1. Some of the respondents solved the inequality correctly in the first item and, therefore, those papers were placed into category 3. Table 6.3 shows 34% of the responses that fell in category 3. The 34% is written as 17%+17%, which explains that 17% of the papers come from the beginning in category 3, showing either good examples solved correctly lacking clarity in explanations, or no example of inequality, presenting only a list of steps to follow when solving a hypothetical inequality. For the papers falling in the second 17% category, I looked at the first task for the correct application of the steps in solving the given inequality.

Category		**Description of anticipated work**
0	25%	No example or other explanation given or other concept than a linear inequality (quadratic inequality) is exemplified.
1	29%	A simple inequality is given and the steps to solve it are erroneous. No example of inequality is given; only a list of unclear steps.
2	2%	Example and explanations follow the pattern of solving equations.
3	34% (17%+17%)	No example of inequality, only a list of steps to follow when solving a hypothetical inequality. Good example solved correctly; however, explanations lack clarity.
4	9%	A pilot example is given. Solving incorporates maximum variation and aspects related to inequalities.

Table 6.3: The rubric for MATH 100

Having developed a lens through which to analyse the responses, reading and sorting the data focused on capturing respondents' conceptions of inequalities visible in the way that they presented and explained the example from which someone can learn. For the FAN X99 class, the revised task was revised even further. To avoid a response merely listing the steps for solving an inequality, I asked students to e-mail back the steps, as well as a worked example with the steps. Part a) of the task reads:

> a) You've got a text message from Jamie - your homework partner - that reads: *"Missed the class on inequalities. I have to prepare for the quiz. Don't know how to start. Help me with the steps of solving a linear inequality."* E-mail him back with a worked example of an inequality and the steps for solving inequalities highlighted on your explanations of your work.

Part b) of the task was the same as for the MATH 100 participants for about half of the participants. The other participants worked on:

> b) Half an hour later an e-mail from Jamie arrives: *"I followed your steps and solved a whole bunch of inequalities. Thanks. Then I attempted this one:* $1-2x>2(6-x)$*. I worked out the algebra and got this* $1-2x<12-2x$ *and then ended up with:* $0<11$*. Here I got stuck. Please help.*" E-mail him back. On the space below show the message you will send to your cousin Jamie. The message should contain your feedback on Jamie's work as well as your input to Jamie's further understanding of inequalities.

The idea of producing an example together with the support of their explanations was clearer here. Again, in FAN classes, Jamie was not a hypothetical character: he was respondent's homework partner. Given that in this class, everybody has a homework partner, the student could relate to Jamie and this aspect is visible in the responses. All the examples were accompanied by explanations and the explanations were more generous and personal. Here are some samples of responses which could be evidence for the claim:

S1: $-4+3x \le 2x+6$ first *you* want to isolate *x*

$$3x - 2x \le 6 + 4$$
$$x \le 10$$

Once *you* have *x* by itself *you* want to put *your* answer in interval notation.

S2: To solve a linear inequality *you* must first identify the framework *you* have already given. Whether it's a graph or an equation *you* must isolate the "*x*" into its single form. From there *you* can graph the equation based on what *you* know. For example, " $x < 3$" means that 3 is the biggest number *you* can go up to.

The purpose of these quotations is not to introduce a discussion or interpretation of voices. I claim that this approach to generating and explaining the example to help Jamie understand is more transparent of the respondents' understanding of inequalities than the less personal one.

Category		**Description of anticipated work**
0	11 %	No example or other explanation given.
1	9%	Example and explanations follow the pattern of solving equations.
2	17%	Examples with an incomplete solution and containing mistakes. In some of them, interval or graphical representation is attempted.
3	24%	Example provided and solved correctly. No interval or graphical representation for the solution. No other explanation about other aspects of inequalities.
4	39%	A pilot example is given. Solving incorporates maximum variation and aspects related to inequalities.

Table 6.4: The rubric for FAN X99

Comparing Table 6.3 and Table 6.4, it would seem that a bigger percentage of respondents – 39% compared to 10% – in the second iteration of the task produced what

I call a pilot example, which is an example of an inequality whose manipulation makes use of all properties of inequalities, intentionally including the multiplying or dividing by a negative number property. One could wonder if the higher percentage of producing a pilot example is due to having a different population who solved the task, to a difference in teaching methods, or to the fact that the respondent could relate to Jamie when producing the example. It is not the purpose of this study to look at the nature of these differences, since the study is concerned with the concept formation of inequalities and not a classification of what percentage of the students is at each level of conception.

Part b) of the task for both MATH 100 and FAN X99 classes, contained the cognitive conflict – an inequality with no solution (for MATH 100 and a part of FAN X99 participants) or with the solution as all real numbers (for the other part of FAN X99 students). Exploring the level of understanding beyond the reproduction of rules by inducing cognitive conflict is a "tool to probe and assess the depth and quality of students' understanding" (Fujii, 2003, p.57). Using a similar instrument – an inequality with no solution, partially solved, and presented to the students as someone else's personal puzzlement when confronted with a result such as $0 > 11$ at the end of the algebraic manipulation of a linear inequality – Fujii (2003) finds that only 3.5% of the students were not trapped by the cognitive conflict. All other students (96.5%) were provoked into revealing the missing links in understanding inequalities (Fujii, 2003).

I gathered similar results from the MATH 100 participants, where only 4% of the respondents correctly resolved the conflict and answered that the inequality has no solution. From the remaining respondents, almost half of them completely avoided the item. From those who attempted it, the majority could not go further than checking the algebra work to finally face 0>11, the result of Jamie's work. The work on this task, in many cases, revealed that the respondents' conception of inequality was at a process level. This could be explained by the fact that the majority of students could not follow Jamie's work mentally, or by just reading Jamie's work; they had to perform the action of transforming the inequality. They did not have the ability to mentally carry out Jamie's work and derive the conclusion, and thus, they had to rewrite the inequality and solve it for themselves. Also, after carrying the action on paper, comparing their result with Jamie's, some concluded that the calculations were correctly done, but they could not envision what the final representation of the inequality will say about the inequality. In the FAN X99 classes, where we specifically worked on contradictions and identities presented in the form of linear inequalities, the situation was a bit different: half the students, after performing the manipulations, were able to say that the given item is a contradiction and/or that it has no solution.

Another aspect of the refined task worth mentioning is that the responses were more personalised than in the first iteration of the task.

S1: I did the same thing and ended up with the same answer. … Jamie and I need a tutor for inequalities…

S2: What you have Jamie is a "false" statement, a not possible to solve inequality. … You haven't made any mistake in the work. It's just a … crazy inequality.

S3: Hey Jame, 0 is obviously not greater than 11. So there is no solution and it's a false inequality.

S4: This is not an inequality. Don't worry, the teacher always uses trick questions to see if you understand the concepts.

S5: With the answer 0 >11 you have a contradiction. You solved it as far as you can but you can get no true answer. You have done well and I believe you understand how to do inequalities.

As in the other two sets of data, no participant intentionally presented an inequality ending up in a solution other than an interval as an example for Jamie. However, in this class, where they had access to “crazy inequalities,” they could link the false statement with the contradiction. In this case, it was an inequality with no solution.

In this section, I have presented a task used for collecting data from three undergraduate classes. A story line introduced the task, the different stages of coding, the refining of the task, its reiteration, coding, and the presentation of some particular aspects of the data. Some frequencies of responses from the initial coding were also inserted in the presentation of the results. While storying, I provided evidence from the data and by analysing the common trends, I was able to speak about students’ understanding of inequalities. Moreover, it seemed that all three classes presented a pattern of formation of the concept of inequalities. For example, in all of the above mentioned responses, there were students whose concept image of inequality was very primitive – they seemed to have some vague idea about the symbols used in representing inequalities, but the symbols were incorrectly used. Also, in all three sets of data there were students for whom the inequality was nothing more than a strange relative of an equation. There were students who would correctly identify all the arbitrary aspects of the inequality in part a) of the task and would be able to manipulate their properties.

However, part b) informed that their understanding of the concept seemed superficial. There were students who would have the inequality encapsulated as a dynamic entity, being able to see the dynamism either in the scale metaphor or the interval notation of a solution. And there were respondents who were able to manipulate, without mental effort, all the arbitrary aspects of inequality and to focus on the necessary elements, thus showing the role of the inequality in the mathematical picture. Therefore, the data analysis leads toward a unified question: What are students' conceptions of inequalities? This question will be the focus of the following section.

6.6 Results: The COIN

It is a common practice in research in mathematics education – it may be the case of research in other fields as well – for a renowned framework to be considered and applied in data analysis. The framework may be slightly adapted or revised to cover the novel aspect that the new data may bring to light. While coding and sorting the data from my students, I kept various such frameworks in mind. For example, in the second sorting of the data, an adaptation of Skemp's (1976) and Hiebert and Lefevre's (1986) levels of understanding were used to describe the variety found in the data. Also, the theory of variation (Marton & Ming, 1997; Watson & Mason, 2006) was present in my awareness when grouping the responses.

The unit of conception from phenomenography as well as the research conducted on conceptions of price (Marton & Pong, 2005), which are mentioned in Chapter 4, were taken into account when the conceptions of inequalities were conceived. As other phenomenographic studies claim, the data from my participants revealed a limited number of ways in which undergraduate students perceive inequalities. With careful sorting and coding, I was able to identify five different categories of description of inequalities, labelled conceptions. The five conceptions of inequalities that emerged from the data are presented and described in this section. The framework *Conceptions of Inequalities* (COIN) is used in the analysis of the data collected from the second study.

Data from all the classes that wrote the task (T), the refined task (RT), as well as from the pre-teaching survey (PTS)[10] and solving inequalities and multiple representations (SIMR) were analysed and compared in order to identify the common trends. As a result, five conceptions of inequality were identified:

> **Conception 0**: *Inequality as an amalgam of images or symbols encountered in a mathematics setting*
> **Conception 1**: *Inequality as an instrument for comparing known quantities*
> **Conception 2**: *Inequality as a strange relative of an equation*
> **Conception 3**: *Inequality as an algebraic process*
> **Conception 4**: *Inequality as an object*

Initially, the five categories were labelled with their descriptions: *comparison* (or balance), *equation*, *process*, *amalgam* (or *misconception*, in the initial stages), *inequality* (or exemplary work). Marton and Pong (2005) labelled their conceptions of prices with capital letters A, B, C, and D. Influenced by the initial hierarchical classification of the data given by the marks on work, I ordered the piles from misconceptions to object, and labelled the categories with 0, 1, 2, 3, and 4. However, at that level of classification, the marks were completely ignored. What counted in the sorting was what the researcher saw that the participants were seeing in inequality.

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[10] See Appendix 2 for the content of the PTS and SIMR tasks.

6.6.1 Describing the *Conceptions of Inequalities*

Conception 0: Inequality as an amalgam of images or symbols encountered in a mathematics setting

My description of this conception as ‘an amalgam of images or symbols encountered in a mathematics setting’ comes from the papers that displayed the most confusion and misunderstanding. It aims at capturing all the respondents whose concept image of inequalities is made up of disconnected and meaningless flashes. Groping for symbols or images is a good metaphor to describe the main action of the students with this conception. The symbols were used incorrectly. The respondents revealed no number sense or incorrect use of symbols and conventions. For example, in such papers an interval of the sort $\left(-3, 5\right]$ would transfer into $-3 \geq x > 5$ or in $\geq -3 \ \ x < 5$ and the phrase “all positive real numbers less than 4” into $[1, 4)$ since 0 is not in the required range and the student apparently does not see other numbers from 0 to 1. Excerpts from the pre-teaching survey (PTS) administered in a FAN X99 class are one of the most relevant pieces of data for this conception. For easy reference in the data analysis, the transcripts from FAN X99 are labelled SF1, SF2, and so on. In the same manner, the transcripts from MATH 100 are labelled SM1, SM2, and so on. For each conception, I start the numbering of the transcripts from 1.

SF1: 1. [Inequality] Shows relationship between values. $<$, $>$, $\geq$, $\leq$ that are not equal. Eg. $1 < 4x$

2. Simplest form, isolating variable. Eg. [See Figure 6.2]

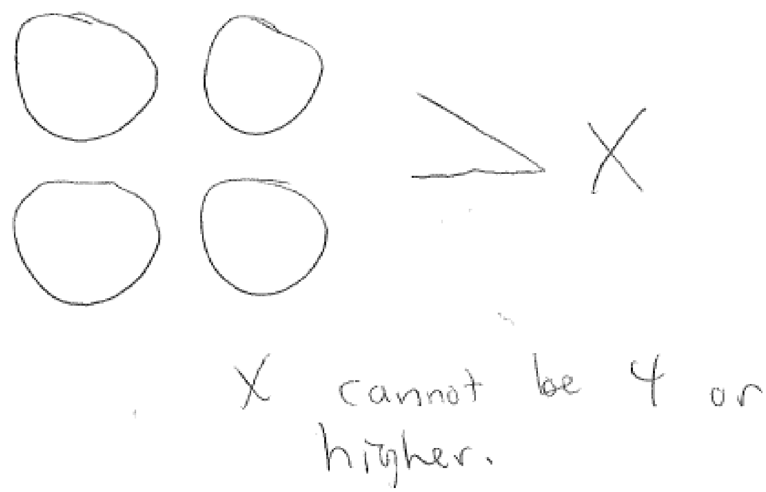


Figure 6.2: Student's representation of inequality

The representation in Figure 6.2 is at the very least peculiar; four circles and the symbol for inequality – greater than *x*. One can wonder whether the dimension of the diagram is important, or the circles being higher than *x*, or the number of circles being greater than the number of *x* signs.

SF2: The solution of an inequality represents if one side is of greater/less value to the other side. Ex. $5 \geq 5$.

SF3: The solution of inequality means the variable that is used can be used to solve both systems of equations. Satisfy both equations.

SF4: The solution of an [in]equality means that in a graph it could never intersect.

The transcripts show that in addition to the inequality not having a workable concept definition, the image of the solution of an inequality is blurry. For example, student SF4 could have thought of a graph of two functions, one of them situated maybe equally distant from the other one. In case the image is of linear functions, the student could “see” that they never intersect. However, for such an image, the solution is the set of real numbers, which is not “seen” in the example.

SF5: There are 6 coins, 1 person has 5 of those coins while the other only has 1. But those 5 coins can be greater or equal to that one coin or less than that one coin.

How can 5 coins be either or equal to or less than one coin? Possibly, the respondent did not have the image of the number of coins in her mind, but the value of the coins. However, this is not explained in the response.

SF6: A mathematical inequality is a solution of an equation that is mathematically incorrect. That means 1=4 is mathematically inequal. The same happens when we use variables if $x = 1$ and we have an equation of $2(x+3) = 8x$

The student does not continue the statement. He may have considered 1 as an example of a solution of the given equation and then another value given to x would generate a false statement which is interpreted as an inequality. A false statement or an inequality may represent the same concept for him.

SF7: Some mathematical questions end up with a[n] un-equal answer such as infinity or undefined.

Here the respondent confuses particular equations that are either contradictions – therefore have no solutions – or identities, whose solutions are the set of all real numbers. Disconnected and apparently meaningless flashes are described by the following two respondents:

SF8: When graphing a line, there are certain ones that are inequal.

SF9: The answer does not match with the question (#). The equation of the question equals not to the answer itself.

What is even more striking in the next transcript is that, even if the student claims that one must have a number sense in order to work with inequalities, her number line has the sequence 1, 2, 3, 0, -1, -2, -3, from the left to the right. Moreover, the inequality symbols are used in the opposite direction, or if in her understanding, 1 is less than 0, then her inequality holds.

SF10:

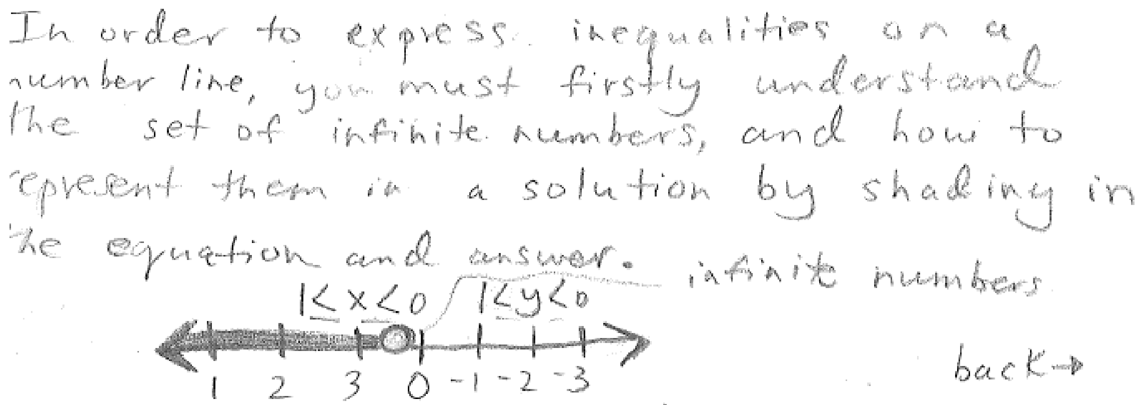


Figure 6.3: Student's representation of inequality

Conception 1: Inequality as an instrument for comparing known quantities

In this conception, the mind frame of the respondents functions in the discrete and non-variable domain where they have to compare two known quantities. If one number is greater than another one, then there is a symbol to mathematically write that relationship. However, for some participants, it is a challenge to understand why it is true that $3 \geq 3$.

SF1: A mathematical inequality uses the greater (>) than and less (<) than symbols. … An inequality is used to compare two pieces of information or to shorten the length of an explanation in words.

There is some confusion here about the concept of inequality versus the symbol used to describe the relationship between two quantities that are not equal to each other.

Conception 2: Inequality as a strange relative of an equation

The title of this conception explicitly captures the referential aspect of inequalities that induced it. In some papers, I could identify that the erroneous solution of an inequality is due to the student following the pattern of solving equations. In the majority of the papers, I did not have to guess the logic behind the mistake because the subjects explicitly stated: "treat the inequality as it would be an equation." The action here consists of the algebraic manipulation of an inequality following the properties of transforming an equation into an equivalent one. Besides the respondent directing the person or Jamie respectively toward solving an equation in the tasks T or RT, the students working on the PTS also defined an inequality as an "equation with unequal sides." In plain language in class, very often, whatever expression with numbers and symbols written on the board or on paper is named an 'equation'. In mathematical conventions, for example, the objects with symbols are either formulae, or polynomials, or algebraic expressions, or numerical expression, or symbolic representation of a computation. The word '*equations'* is saved for two algebraic expressions – in one or more variables – connected by the equal sign.

SF1: $a \neq b$ Equation on two side[s] is not equal.

SF2: Turning the symbol to an equal sign so that the equality can be solved, and therefore the inequality.

SF3:

$$
\begin{aligned}
-3x &< 21 \\
\frac{-3x}{-3} &< \frac{21}{-3} \\
x &> -7
\end{aligned}
$$

Solving linear inequalities is very similar to solving linear equations. The only difference being that the equal sign is substituted for an [in]equality sign. Both concepts are very much alike.

SF3 correctly solves the inequality. However, with this conception, the majority of respondents, such as SF4, not only had the pattern of solving equations in their mind when working on inequalities, but they replaced the inequality symbol with the equal sign. If they unlucky had a negative coefficient for *x* at the end of the solving process, they got the wrong solution for failing to change the sign of the inequality when dividing by a negative number.

SF4:

$2x+8<20+4x$	- Sub '<' with '=' to turn it in a linear equation.
$2x+8=20+4x$ $-2x=12$	- Solve as you would a linear equation by adding 8 to both sides and subtracting 4*x* from both sides.
$x=-6$ $x<-6$	- Resub '= with '<' to get values for *x*. Answer: $x<-6$

The same transcript was used in Subsection 6.4.1 to support the discussion about the anticipated and surprising responses found in the data.

Conception 3: Inequality as an algebraic process

SF1:

$-3x-6 > 4+2x$ $-3x-2x > 4+6$ $-5x > 10$ $x > -2$ $x < -2$	1. First collect the like terms. Put all the numbers with *x* on the left side of inequality sign and the numbers without *x* on the right side of inequality sign. Remember to change the positive and negative sign in front of the numbers. 2. Then do the math. 3. Calculate the *x* value and flip the inequality sign if you get a negative value.

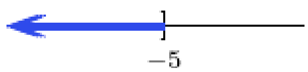
Here, the inequality is seen as a process to be done, with rules to be followed. The rules are usually nicely stated on the right side of the work on the proposed item. Sometimes, "the rules without reason" (Skemp, 1976) are transparent in little details, such as the last two lines in the transcript from SF1: the student divided by -5 to "calculate the *x* value", and then she flipped the inequality sign because she got a negative solution. The transition from inequality notation, to interval notation, and then to graph is usually smooth and there is fluency in the work; however, the fluency seems to be a partial fluency of rules, not of logical connections.

Conception 4: Inequality as an object

This conception is the most advanced one found in the data. The inequality is mentally seen, before applying algebraic transformations, with its power of generating interval solutions. The participants are fluent in manipulating inequalities in all formats. They also accept and understand how inequalities produce equalities, identities, or contradictions and are able to see and describe when such phenomena occur. The answers given in part b) of the task, which were about recognizing inequalities that produce all real numbers as solutions or having no solution at all, gave weight to this conception.

SF1:

$$-2x \geq 10$$
$$\frac{-2x}{-2} \geq \frac{10}{-2}$$
$$x \leq -5$$
$$(-\infty, -5]$$



Divide both sides by negative 2.
Because we divide by a negative the greater than or equal to sign must switch which will enable the inequality to stay true.

Check:
$x = -6$ $\quad -2(-6) \geq 10 \quad 12 \geq 10$ true; 12 is greater than 10.
$x = -4$ $\quad -2(-4) \geq 10 \quad 8 \geq 10$ false; 8 is not greater than 10.

SF2: 0 > 11 This means that this inequality is a contradiction which means there is no solution and not true. 0 cannot be larger than 11. Sol: no solution.

SF3: 0 > 11 The inequality has no solution because it ends in a contradiction. An equation can result with no solution, and so can an inequality.

Inequalities, with their multiple semiotic registers of representation – algebraic, interval, functional, graphical (Sackur, 2004) – show many levels where students could make errors. Aside from indicating were students encounter difficulties when dealing with inequalities, errors can be used as prompts for discussion and clarification of delicate aspects of inequalities. Also, not only errors, but counterintuitive statements, such a false statement 0 < -2 could be used to reveal a relational understanding of the concepts or weakness in dealing with it. For example, in my study, no respondent of task

T produced, intentionally or not, an inequality generating an empty set as a solution. Other studies such as Tsamir and Bazzini (2001) or Fujii (2003) showed that an inequality producing an empty set for the solution is problematic for students. This detail influenced my decision to adapt an example from Fujii (2003, p.57) and to incorporate it in the revised task:

> Solving the inequality $8-2x<2(3-x)$, Carol followed the following steps:
>
> $$8-2x<6-2x$$
> $$0<-2$$
>
> Here she got into difficulty. Comment on Carol's solution.

I decided to give this item in the refined task to see how the students would resolve this type of difficulty, fact that could inform about students' understanding of inequality and solution of inequality. In section 6.5.1, I discussed the results of this item in detail. What is important for this Subsection is that a clear exit out of the cognitive conflict correlated with a pilot example with all the registers correctly expressed helped the argumentation of Conception 4.

6.7 Backwards Forwards

The labels 0, 1, 2, 3, 4 for the *Conceptions of Inequalities* may imply that they are in a strict hierarchy. However, as the descriptions and the samples from the data show, the conceptions are discrete, not in strict hierarchy and the distinction between conceptions is given by the different aspects, inequalities presented in the data exhibit.

Phenomenography sees learning “as a change in learners’ capability of experiencing a phenomenon”, and understanding as the capability of spotting and following a pattern of variation (Pang, 200, p.153). Some subjects were unable to focus on the difference between the equation and inequality. They were aware that an equation and an inequality look very similar except for one little detail: the symbol connecting the two parts of an algebraic expression. However, their focus was not on what differences in understanding and manipulation of the new concept bring this new symbol to an object that looks familiar except for one detail; their concern was to change that little difference, to replace the inequality symbol with the equality symbol. Those subjects were therefore unable to use that pattern of variation to their advantage, to help their conception of inequality. More precisely, the students seem to be missing the mathematical training that would help one build a new concept by making links to an old, similar concept, and by transferring the similarities from old to new with the focus on understanding and learning the differences between the old and the new concept.

In the next chapter, more information on the variation with respect to inequalities is presented, which deals with analysing the data with the new lens: The COIN.

Chapter 7:

Study 2: Undergraduate Students' *Conceptions of Inequalities*

As mentioned previously, research on inequalities has generally tried to answer many different questions such as: What is typically correct and incorrect reasoning? What are common errors? What are possible sources of students' incorrect solutions? What theoretical frameworks could be used for analysing students' reasoning about algebraic inequalities? What is the role of the teacher, the context, different modes of representation, and technology in promoting students' understanding? What are promising ways to teach inequalities (Bazzini & Tsamir, 2004)? Studies reported mostly on students' misconceptions on inequalities or on obstacles in understanding inequalities (Linchevski & Sfard, 1991; Bazzini & Tsamir, 2001, 2003, 2004; Tsamir, Tirosh, & Tiano, 2004; Boero & Bazzini, 2004; Sackur, 2004). One of the main questions proposed by the Discussion Group meeting at the 1998 PME 22 – What are students' conceptions of inequalities? – is still waiting for an answer. A framework that permits the decomposition of the inequality concept into the structural features that the research participants discern and focus on could help a study that aims to inform about what makes some students better at manipulating inequalities than others. Moreover, a methodology of collecting data that would permit the recognition of those features must be implemented. It is the goal of this chapter to portray a study that focused on these aspects.

This chapter opens the door to investigating learners' conceptions of inequalities via the aforementioned and developed COIN framework. The first section provides a succinct portrait of participants in the study. The MATH 100 participants were the same cohort for both studies. For Study 1, data was collected in the second week of classes and for Study 2, data was collected three months after that, in the final exam. The new description of the participants complements the information given in the previous chapter. FAN X99 was a different cohort, however the teaching treatments as well as participants' profile was very similar to the first study. The data from the two different classes and the findings are presented separately, in subsections 7.2.2 and 7.2.3, respectively. The COIN framework is applied to both sets of data. A unified conclusion brings the study to an end.

7.1 Methodology: Portraying the Participants and Presenting the Data

For Study 2, the participants were again undergraduate students whom I taught at Simon Fraser University: a class of MATH 100 students (43 participants) and two classes of FAN X99 students (57 participants – 29 from one class and 28 from the other one).

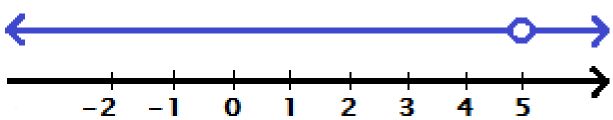
The data from MATH 100 class was collected as an item in the final exam, after an entire semester of intense work on functions with inequalities omnipresent in every topic. As mentioned in Study 1, the first two weeks of classes were devoted to the recollection and improvement of algebra skills from which inequalities were some of the most emphasized concepts. The class started with linear inequalities and the convention of representing the solutions in interval form as well as graphically, and then completed work with rational inequalities, polynomial and absolute value inequalities, as well as inequalities with irrational components. The methods used to solve these types of inequalities varied from reading from the graph the intervals describing the inequality to making a sign chart from where the solution could emerge, or to analysing the effect of each component of an inequality and making logical connections. The algebra refresher was followed by the main part of the course – the study of functions – where inequality was the main tool for working out the domain of a function or describing the behaviour of a function on intervals. I do not exaggerate if I claim that there was at least one inequality to be set up and solved in every lecture, assignment, or term test for a period of thirteen weeks.

The data from FAN X99 class was collected as an item in a quiz – item 1, from VIII. Inequality Survey (IS), introduced in Subsection 7.1.2 – after two weeks of working on inequalities. The main topic was linear inequalities. Transforming English statements into inequalities, setting up boundaries or restrictions for some real life situations in terms of inequalities, and graphical and interval representations of solutions were part of the class work. In summary, the data from the first item of the survey is discussed in this chapter.

7.1.1 Generate Examples Task (GET)

The task for data collection from MATH 100 is comprised of three objects – an interval, an equation, and a graph, with the claim that those objects represent solutions of inequalities. The respondents had to produce inequalities that result in that solution:

> In each case give an example (an equation, a picture or a description) of a mathematical object/concept satisfying the given conditions. Explain briefly why your example meets the conditions. If you can't get an example of one of the items, explain why you think that such an example would be impossible to construct.
>
> a) an inequality with solution $(-3, 5]$;
>
> b) an inequality with solution $x = 2$;
>
> c) an inequality with solution represented by the graph:



Normative examples are:

a) $-3 < x \leq 5$ b) $(x-2)^2 \leq 0$ c) $x \neq 5$

However, given that the work on inequalities was extensive, when the students reached the final exam I felt it was safe to assume that they would have a good knowledge of inequalities and the data will produce maximum variation and room for interpretation. Also, it is important to mention that the students were used to encountering either in quizzes or midterms, extreme (unusual, peculiar, not too common) examples or solutions to the tasks rewarded or at least noticed. Given all these conditions, it was not unrealistic to expect students to give more sophisticated examples, such as:

a) $\dfrac{x-5}{x+3} \leq 0$ b) $-(x-2)^2 \geq 0$ c) $\left(\dfrac{1}{x-5}\right)^2 \geq 0$

or to look in different registers (such as the domain of a function) for an item that will produce the given solution. Section 7.2 on data analysis is abundant of excerpts from the data. As predicted, the data was rich in all sorts of examples, from simple linear inequalities obtained directly by rewriting the interval notation using inequality symbols, to all sorts of rational inequalities, to logarithmic inequalities.

7.1.2 FAN X99 Inequalities Survey (IS)

FAN X99 classes wrote an eight-item survey:

1. What sorts of images or examples come to mind when you consider the concept of inequality?
2. What are the ways in which inequalities and equations are the same and/or different?
3. What does a solution of an inequality mean? Exemplify.
4. Provide (construct) three *different* inequalities that will give the solution $\left(-3, +\infty\right)$:
5. Solve the following inequalities. Give the solutions graphically and in interval form.

a) $4-3x \le 2x+19$ b) $4-3x \le 2x+19$ and $-\frac{1}{3}x > -2$

6. Can you tell me an interesting fact you have learned/discovered lately about inequalities?
7. a) You've got a text message from Jamie - your homework partner - that reads: "Missed the class on inequalities. I have to prepare for the quiz. Don't know how to start. Help me with the steps of solving a linear inequality." E-mail him back the steps for solving linear inequalities.

   b) An hour later an e-mail from Jamie arrives: "I followed your steps and solved a whole bunch of inequalities. Thanks. Then I attempted this one: $1-2x > 2(6-x)$.

I worked out the algebra and got this $1-2x>12-2x$ and then ended up with: $0>11$. Here I got stuck. Please help." E-mail him back. The message should contain your feedback on Jamie's work as well as your input to Jamie's further understanding of inequalities.

8. Try to recall or reconstruct the provenance of your first response.

From the eight items, the task of interest for this part of the study is item 1, the following:

> What sorts of images or examples come to your mind when you consider the concept of inequality?

Item 7 of the survey, which was a repeat of the task from Study 1, was also of interest. The results on part a) of the task being very similar to the results from Study 1, the discussion of this is minimal. Part b) brings a new discussion into the context of the study.

7.2 Data Analysis

7.2.1 Initial Sorting of the Data from MATH 100

The first sorting of the data was again accomplished by marking the papers since the items were part of the final exam. Table 6.5 represents the average mark for the 3 items. Each item was marked with points from 0 to 4. Zero was given to papers where no attempt was made to give any example or to rewrite the solution in a form were inequality is visible. One was assigned for attempts showing some correct transfer from the given format of the solution to another representation. For example when the parenthesis (is associated with the empty point on a graph or the square bracket] with a full point, or < is associated with), a mark is assigned. Two was assigned for correctly representing the solution in a different register – either graph or using inequality symbols. Three was assigned for correctly writing the simplest inequality for each item: a) $-3 < x \leq 5$, b) $2 \leq x \leq 2$ and c) $x \neq 5$. Table 6.6 summarizes all these and provides examples for each category, which was actually students' mark on the papers. As Table 6.5 shows, there was only one paper that got 4 marks on each item. Maximum credit was awarded only to sophisticated solutions, which showed fluency between registers and other examples aside from linear inequalities. Here is part a) of that paper:

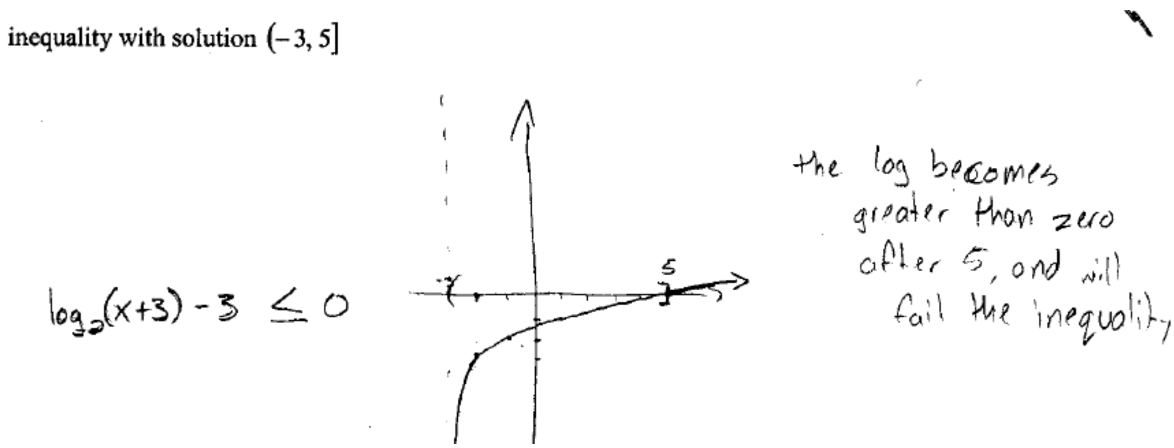


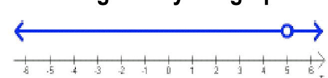
Figure 7.1: A sample of MATH 100 work

Average Mark	Number of respondents	Percentage
0	5	12%
(0, 1]	12	28%
(1, 2]	17	40%
(2, 3]	8	19%
(3, 4]	1	2%

Table 7.1: The marks for MATH 100

Table 7.1 shows that the majority (80%) of the data falls below a mark of 3 for these tasks, which was awarded for writing the simplest inequality representation for each item. These results helped me remember – as if it were necessary – why I chose inequalities for my study in the first place: Because, at the end of a semester with overwork on all aspects of inequalities, students still underperform on inequalities items.

When marking the papers, I noticed that the variation in responses was substantial. Some students took the easy way out, namely the simple normative solution, but the majority of them tried to give examples of different types of inequalities – such as linear, quadratic, rational, logarithmic, as well as different registers for presenting inequalities – from simple notation using the inequality symbols, to notation with connectives, to intervals, to graphing the solution or showing the graph of the function from where the inequality can be read. This variation of responses inspired the new classification of data, which is presented in Table 7.2. The categories are new, as they are not much related to the marks in Table 7.1. In the new sorting of data, a failed attempt at giving a rational inequality as an example for item a) was valued more than the simple inequality $-3 < x \leq 5$ as an example.

	Tasks		
Category	**solution $(-3, 5]$ task**	**solution $x = 2$ task**	**solution given by the graph** 
0	Does not associate the interval $(-3, 5]$ with the simple inequality $-3 < x \leq 5$. Unable to represent the interval on a number line. No attempt to write any inequality that will result in the given solution.	Not aware of the meaning of $x = 2$ as being part of an inequality $x \geq 2$ which could be understood as $x = 2$ or $x > 2$. Does not have number sense: claims that 2 is the only number that results from the inequality $1 < x < 3$.	Does not associate the graph with the union of the two intervals $(-\infty, 5) \cup (5, +\infty)$, or with the inequality $x \neq 5$ or with the simultaneous inequalities $(x > 5 \; or \; x < 5)$. Does not have number sense: claims that $x \neq 5$ is the equivalent to $4 < x < 6$.

1	Has some idea of how to represent the interval $(-3, 5]$ on a number line but the chosen representation is not correct. Does not link $(-3, 5]$ with the simple inequality $-3 < x \leq 5$ and no attempt to provide any other algebraic representation for it.	Fails to recognize the structure of $x = 2$: Confuses the inequality $x = 2$ with the equation of a vertical line in a rectangular system of coordinates and claims that it fails the vertical line test and no example is available for that. Gives examples of equations with solution $x = 2$ or inequalities with solution $x > 2$.	Attempts to switch between registers and writes improper inequalities such as $5 < x < 5$.
2	Correctly represents the interval $(-3, 5]$ on a number line and gives the simple inequality $-3 < x \leq 5$ as an example. No attempt to move to another register.	Correctly represents the solution as a singular point on an axis of numbers. Incorrectly writes $2 \leq x \geq 2$ or $2 < x < 2$	Can identify that a structure with fraction can result in a solution that has only one value missing from the entire real domain, but the example does not result in the given solution (e.g., $\frac{x}{2x-10} \geq 0$ or $\frac{1}{x-5} \geq 0$.)
3	Attempts to move to a different register, such as functions. Attempts to give an example of an inequality other than the simple linear inequality $-3 < x \leq 5$. Identifies the open interval with the structure of a fraction with denominator *x*+3 or a vertical asymptote. Incorrectly uses the inequality symbol or incorrectly transfers the closed/open into numerator/denominator, such as $\frac{x+3}{x-5} \leq 0$.	Correctly represents the solution in the simples inequality form $2 \leq x \leq 2$. Attempts to write simultaneous inequalities that result in the given solution but missing logical connectives. Attempts to move to a different register, such as functions.	Correctly switches between registers and give the inequality embedded as the domain of a rational function with denominator $x - 5$. Correctly associates the graph with the union of the two intervals $(-\infty, 5) \cup (5, +\infty)$ and the written explanation "all real numbers except 5."
4	Correctly moves from the interval to the function register and gives sophisticated example of inequality (e.g., $\log_2(x+3) - 3 \leq 0$). Correctly represents the graph of the function and the solution in the rectangular system of coordinates.	Correctly moves from the interval to the function register and gives sophisticated example of inequality (e.g., $-\frac{1}{3}(x-2)^2 \geq 0$). Correctly represents the graph of the function and the solution in the rectangular system of coordinates.	Correctly moves from the number line to the function register and gives sophisticated example of inequality (e.g., $f(x) = 7(x-5)^2 > 0$). Correctly represents the graph of the function and the solution in the rectangular system of coordinates. Correctly writes an algebraic inequality which results in the given solution (e.g., $\left\|\frac{x}{x-5}\right\| \geq 0$.)

Table 7.2: The variation of examples

The data displays two different categories of variation: variation of good responses and variation of common errors. Section 3.3 presents a comprehensive list of error patterns when manipulating inequalities, errors that were identified by researchers on inequalities: (1) multiplying or dividing the two sides of an inequality by the same number without checking whether the number is positive, negative or zero; (2) dropping the denominator of a fraction when containing a parameter or the variable; (3) erroneously converting the inequality into intervals; (4) declining solutions that do not fit the general pattern (i.e., an interval for inequalities, a unique value for an equation); (5) dealing with a positive/negative product by considering the component factors as separately being all positive/negative; (6) incorrectly reading the solution from a graph when using functions to solve inequalities; and, (7) incorrectly converting the graph into a solution; (8) solving equations instead of inequalities (Tsamir et al., 1998; Tsamir et al., 2004; Sackur, 2004). Two more categories were visible in data from MATH 100: (9) the incorrect appropriation of the inequality symbols and notations and (10) the concepts previously encapsulated became obstacles in solving the inequality with single a solution task.

As examples for (9): a) an interval such as $(-3, 5]$ would transfer into $-3 \geq x > 5$ or in $\geq -3 \;\; x < 5$; c) the use of notation $5 < x < 5$ for 5 being the only missing real value, or b) writing $2 < x < 2$ as being equivalent to $x = 2$. The convention for writing intervals was also not observed as in this example: $(5, -\infty)$ where the negative infinity sits at the right side of 5 or here: $(5, \infty]$ where square bracket was used at nest to the infinity sign. As an example (10), $x = 2$ was encapsulated as the equation of a vertical line which fails the vertical line test whereas the inequality is encapsulated as a function. Thus, no inequality could result in such a solution.

7.2.1.1 The 'solution $(-3, 5]$' Task

Using the symbolic register, a correct and simple example that could influence the placement of a paper at level 2 rather than 0 could have been $-3 < x \leq 5$ or a variation of it, such as $-6 < 2x \leq 10$. However, the students having problems with converting from interval to inequality notation provided various erroneous inequalities, such as $-3 < x < 5$ or $-3 \leq x < 5$. The problem seems to be the incorrect appropriation of the arbitrary components – such as notations and representations – of the concept of inequalities. Some examples that could have had the potential of getting the response on a higher level in the sorting of the data incorporated a good approach resulting in erroneous solutions. Using the register of functions, students identified the open interval with the existence of an asymptote at that point, but not too many of them were able to provide an example for that situation. The graph presented in Figure 7.2 shows the vertical asymptote at $x = 3$ as well as a full point at $x = 5$, which are the key elements of the situation. However, the inequalities that can be spotted on the graph have completely different solutions. If the graph would have used the vertical asymptote $x = 3$ and the graph of the function would have been kept below 0 up to $x = 5$, then a good example of such an inequality could have been seen on the diagram.

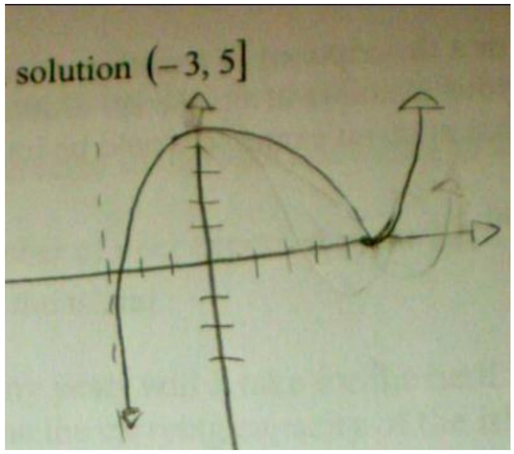


Figure 7.2: A graph with potential

7.2.1.2 The 'solution $x = 2$' Task

The students representing $x = 2$ as a point at 2 on a number line were those who got credit for the item, for representing the given solution in a different register. However, only 5 respondents represented the solution like that. The majority of the responses revealed students' erroneous encapsulation of the $x = 2$ entity as objects that have no inequality embedded in. Many of the attempts produced examples of inequalities that generated intervals as solutions. The following excerpts from students' work exemplify the above perceptions.

SM1: This is not possible because when solving inequalities, the inequality sign must be carried through when solving. An inequality sign also indicate that a solution is greater than, less than or equal to the final number in the inequality.

SM2: Not possible, does not pass the vertical line test, more of an asymptote.

SM3: an inequality cannot be constructed for $x = 2;$ it is a straight vertical line

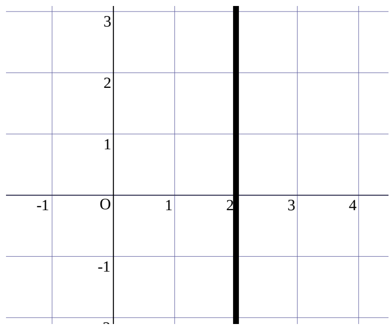


Figure 7.3: A vertical line represented by the equation x=2.

The first of the three quotations reveals that inequality was encapsulated as an object strictly connected with intervals as solutions; the other two quotes present $x = 2$ encapsulated as a vertical line object.

Also, using $1 < x < 3$ as an example of inequality resulting in the unique solution 2 reveals that the conception of inequality is weak because it is based on lower level conceptions of numbers. Here the concept of a number reduced to whole numbers only.

SM4: $2 \le x \ge 2$

shows improper use of simultaneous inequalities and

SM5: $2 < x < 2$

reveals misuse of the inequality notation; the student does not see that there is no value in-between 2 and 2, not included. SM6, whose graph is copied in Figure 7.4, shows a quadratic function, with the vertex and one zero labelled. The inequalities that can be attached to this graph do not produce $x = 2$ as solution.

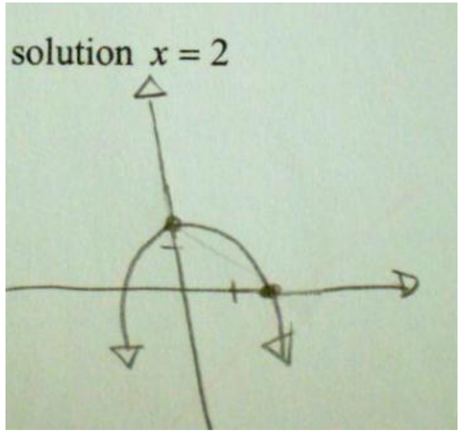
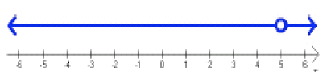


Figure 7.4: A quadratic function

Another interesting example is the failed attempt at using absolute value inequality as an example.

SM6: $|-2| > x$

There is a chance that the student was having the representation of the absolute value as the prototype for this example. Correctly used, the absolute value could have been a good approach to this item: $|x-2| < 0$.

7.2.1.3 The 'solution  ' Task

> Looking for alternative ways of thinking about a problem or concept and developing confidence in multiple representations and multiple perspectives is likely to increase one's effectiveness as a mathematician. (Watson & Mason, 2005, p.42)

If the previously discussed two items could be solved in the algebraic register alone, this task forced students to switch between registers. Surprisingly, less than a quarter of the respondents correctly gave the normative solution: $x \neq 5$ and even less than that gave the interval register of the same idea: $(-\infty, 5) \cup (5, +\infty)$. There were only a few students who correctly converted from the solution represented on a number line to inequality notation, to interval notation, or to function register where the solution could represent the domain or the behaviour of the function. In some responses, writing the solution in interval notation exposed a reversed use of the two logical connectives *and* and *or*, and such the solution became $(-\infty, 5) \cap (5, +\infty)$. A graph like the one inserted below represents a function with the domain all real numbers except 5. It shows no inequality with solution $(-\infty, 5) \cup (5, +\infty)$, however, it exemplifies the perception that, used so much for deriving the domain of a function, inequalities are seen in situations when other concepts would represent them better.

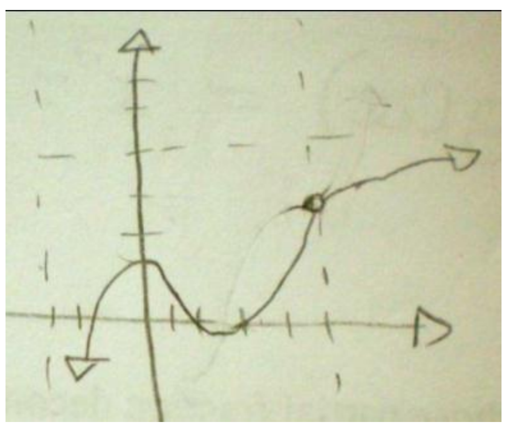


Figure 7.5: The domain of f does not include 5

For the second scanning of data the lens used was the COIN. All five categories were found in the data. From the naming of the conceptions to the Study 2, the descriptions of the five conceptions went through a series of minor edits. In essence they are the same; however the new descriptions seem to better capture what the data exposed. Names are also proposed for the conceptions.

7.2.2 The Data from FAN X99 Seen Through the COIN

The data from the FAN X99 classes was sorted using the COIN framework. Four conceptions emerged from the data. It is not surprising that, given the nature of the task, no student accessed the higher level of conception.

Conception 0 - *Miscellanea*: *Inequality as amalgam of images or symbols encountered in a mathematics setting*

SF1: $\neq$ - when two answers don't agree

SF2: R, ϕ, no solution, <, >.

SF3: Lines on a graph, or line segments on a graph.

SF4: Number lines, triangles, confusion.

SF5: Mathematical inequality is when 2 numbers or variables do not match up as an final answer. You may have equations linking the same system but the product of the equation, rather the solutions do not equal to each other as they are supposed to. For example, the given equation is $A = B$ but the solution for *A* is 5 and the solution *B* is 6 therefore coming and inequality as $A \leq B$.

SF6: Something to do with graphing. Solving an equation with a certain formula. Different formulas are applied to different situations. This can create images on a graph like parabola.

SF7: The sorts of images and examples that come to mind are equations and graphs that are formed from inequality concepts.

SF8: Frustration and confusion and that *x* or *y* must be less than or equal to a number.

SF9:

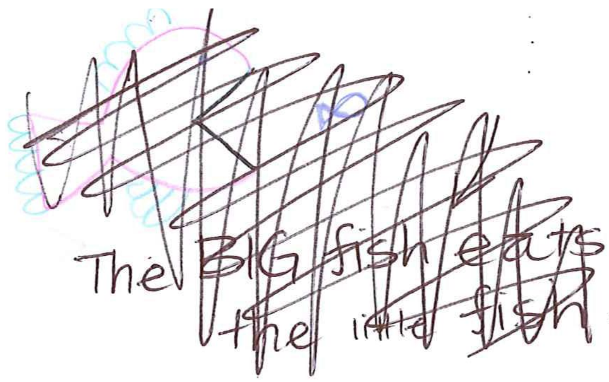


Figure 7.6: The big fish eats the little fish

Again, groping for symbols, images, or words to describe the concept of inequality is visible in all of the above quotes. As in the first iteration of the Task (see section 6.6), **Conception 0** is associated with 'fumbling in the dark mansion'[11] of mathematics and incoherently trying to describe the object one stumbles upon. Graphs with vague or no connection to inequalities are students' responses. Graphs of intervals as well as diagrams used for working with fractions are provided as examples of inequalities. The picture a big fish eating a small one is used as the mnemonic device for the symbol of inequalities, or even a picture of a scale with a different number of objects on each one of the two plates are provided as examples of inequalities.

- **Conception 1 - *Tool***: *Inequality as an instrument for comparing known or imaginary quantities or a tool for expressing restrictions*

A scale for comparing quantities was the main metaphor used for inequalities at this conception. Students exemplified this with numbers, measures, or undefined objects.

[11] The metaphor is adapted from Andrew Wiles' description of the process of doing mathematics – From the movie 'The Proof', produced by Nova and aired on PBS on October 28, 1997 (Nova, 2003).

Associating inequalities with real life situations – such as a power play in hockey where one could see dynamically fewer players on one side and more power on the other side – was also present in the data.

SF1: Mathematical inequalities are equations that do not have a real answer. It is more a comparison rather than an equation. Ex. $2 \neq 1$ but $2 > 1$.

SF2: It is when something compared to another. Then that one thing is either greater than the other one, smaller, greater than or equal and smaller than or equal. Example O > o, the larger circle is greater in size than the smaller circle.

SF3: Larger or smaller numbers being compared to other numbers. I think of a scale some times, one side is lighter, heavier or equal to.

SF4: An image of a scale balancing, making sure that both sides are equal.

SF5: Graphs, number lines, interest problems, balance problems.

SF6: The concept of inequality brings to mind images like unbalanced scale (where one side is heavier/lighter than the other). Another image is of a power play in hockey where one side has less players than the other for a set period of time.

SF7: When I think of inequality, I think of a scale. There is different weights on both sides and their relationship to another changes when the weight on one of the sides of the scale changes. They can be equal. There can also be an infinite amount of weights to use on the scale. [The image of a scale is provided.]

The conception of ‘inequality as a comparison’ is close to the formal definition of inequality. The first part of the quote from SF1 looks more like **Conception 0**; however the examples that accompany the definition use inequality symbols to compare given numbers which can definitely be classified as **Conception 1**. The designation ***Tool*** for this conception is more evident in the responses from the MATH 100 students, who has seen, for example, the inequality at work when establishing the domain of a function. The next section, based mostly on data from MATH 100, will say more about this aspect. SF6 shows a real-life embodiment of inequality, a hockey game where the unequal forces are emphasized and whose effect is seen and emotionally lived. Also, the scale of SF7 is dynamic, the relationship between sides changes when one side is modified. This idea can accommodate the axioms of inequalities. Moreover, the possibility that the scale could be in equilibrium and still represents an inequality shows understanding of the mechanism of an inequality producing a single number as solution, a mechanism that was missed by so many MATH 100 students when dealing with the $x = 2$ item of their task.

- **Conception 2 - *Equality***: *Inequality as a (strange) relative of an equation*

The responses in this category unanimously used the word *equation* or *equal* when describing the concept of inequality. For many of them, inequality is an equation that behaves in a strange manner, either by using a sophisticated symbol (<) for an idea that already has a symbol (=) or by producing a solution that is not a unique number, as normal equations do.

SF1: [Inequality] is an equation where the two sides aren’t equal.

SF2: When one side does not equal the other side of the equation. Does not solve equally.

SF3: Two things that could be equal, but are usually either more than or less than.

SF4: An image of a scale balancing, making sure that both sides are equal.

SF5: An “equation” which does not necessarily provide a solution showing one answer.

The definition of inequality as an “equation with unequal components’ is the main metaphor for **Conception 2**. With this concept image in mind, students often replace the inequality symbol with the equal symbol and solve the equation, which often results in an erroneous solution. What is also interesting about this conception is that it is not derived solely from looking at students’ work and coding as in other groups of papers; it comes directly as students’ declaration, their concept definition of inequality. It was documented that familiar procedures are performed on symbols that do not have natural conceptual embodiments (Tall, 2004). Here, the inequality is not encapsulated yet and the process of solving it is carried in a routinized way based on the procedures known from equations; the familiar look of inequality invited not only the application of the procedure from equations, but a complete substitution of the new symbol with the symbol which was more familiar.

- **Conception 3 - *Process***: *Inequality as a mental or algebraic process*

Either a mental or an algebraic process is visible in this conception of inequalities. For example, the student accompanied the inequality $2 < x < 5$ with the explanation that it comes from an intersection of two inequalities. This is a mental process of seeing the compound inequality '$2 < x$ and $x < 5$' whose solution is given by the intersection of the solutions of the two inequalities connected with the operator 'and'.

SF1: Inequalities are the formula that shows greater than or less than some number. Ex: $a > x > b$ or $-5 < x < 2$.

SF2: I think of greater than or less than. I think of graphing on a number line. I think of $2 < x < 5$ as an equation of intersection. I think of $x > x + 5$ as a contradiction, $x > 5$ as a conditional inequality.

Respondent SF2 also showed the same inequality graphically and correctly used the inequality symbols and the interval notation for representing solutions.

Given the nature of the task – not inviting students to manipulate inequalities – the data for **Conception 3** is not very well represented. Therefore, the excerpts are not as numerous as those for the previous conception. The data from MATH 100, however, is more generous with examples for this conception.

Conception 4 was not recognized in this set of data. This could be because of the nature of the task. When asked about the images or examples that come to mind when considering the concept of inequality, students may not access the higher level of sophistication for that entity. However, the images they presented were rich enough to substantiate the framework for looking at the data.

Aside from the data consolidating the conceptions of inequalities, there is something I consider important to mention here - the chronology of coming up with an example or describing the image of the concept of inequality. One respondent confessed that before having his memory of a concept refreshed, he could not see too much related to that concept: "No images, and at first no examples, it *is foreign* and *sounds weird*, until someone shows me an example like $2x < 7x + 1$ then I will, and until I forget about them again, always associate 'inequality' with $<$, $>$, $\neq$, $\leq$, $\geq$ signs and '$x > y$', where y is a given integer or fraction." Surprisingly enough, a concept may sound weird and look foreign to a student, even if the student has met aspects of that concept many times before.

7.2.3 The Data from MATH 100 Seen Through the COIN

Table 7.2, which was derived when marking the papers, synthesizes five categories of responses to the three tasks the GET questionnaire, presented in Subsection 7.1.1, was comprised off. The categories in the table, as well as a second look at the data through the COIN framework, validated the conclusions presented in this section.

A summary of the most relevant transcripts for each conception follows. The notation SM# is used to introduce different excerpts from the data coming from MATH 100.

- **Conception 0**: *Inequality as an amalgam of images or symbols encountered in a mathematics setting*

Conception 0 is not as abundant in examples from the data from MATH 100 as is the same conception supported by the data from the FAN X99 classes. However, there were students whose conception of inequalities is made of bits and pieces of different images and notations encountered in relation with inequalities. For example, some respondents exemplified with a graph from where somebody could see the given interval emerging as the domain of a function. The lack of explanation of such items makes the researcher assume that the real connection between the inequality and the example is missing because the image of the inequality is blurry.

SM1:

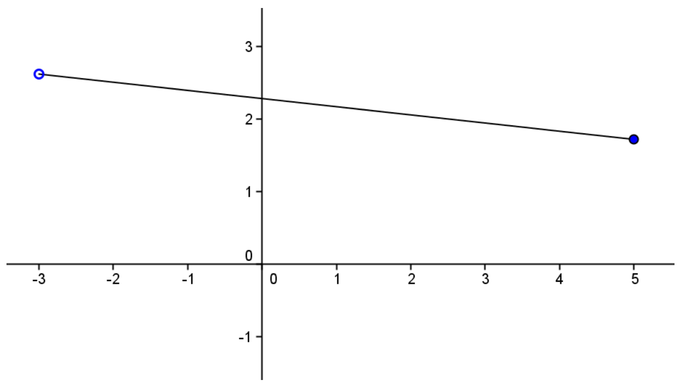


Figure 7.7: A function with domain $\left(-3, 5\right]$

A line segment from -3 to 5 is drawn in Figure 7.7. The line segment has an empty point at the extreme left and a full point on the extreme right. With a bit of experience in graphing functions someone would be able to identify the domain of this function with the interval $\left(-3, 5\right]$. However, the inequality is missing here. The convention when using graphs for solving inequalities is to look at *y* for the condition and read the solution on the *x*-axis. No such connection exits here. Moreover, the same respondent gave the graph of another linear function, as an example for item c). The graph could be associated with the function $f(x) = \frac{x}{2},\ \ x \neq 5$. Again, the domain of this function is all real numbers except for 5; however, that does not explain that there is an inequality there with such a solution. What is surprising here is that the student seems to associate images with elements on which inequalities depend upon in manipulation or representation, while the student's concept image of inequalities is not a solid one. Other participants exemplified the inequality with the graph of a quadratic function defined from -3 to 5 including. The domain of a function and the solution to an inequality share the interval notation as well as the inequality symbols. However, the fact that they are distinct concepts sharing the same representations is not clear in the examples.

Another type of data that placed the work on the **Conception 0** category was the examples from item b) of the questionnaire. For some students, an example of an inequality producing 2 as a solution is $1 < x < 3$. Other respondents clearly stated that since there is an open interval which does not contain its boundaries, this interval contains 2 as the only number between 1 and 3. As explained previously, those students may have only encapsulated integers or whole numbers as numbers. Moreover, some respondents seemed to have the awareness that there are more numbers aside from integers in an interval. However, such awareness did not help them look for another representation for the inequality, but to explain that the given interval contains 2 "along with other numbers."

SM2: $1 < x < 3$ "x is greater than 1 but less than 3; #2 fits the description along with other numbers."

SM3: When factored the solutions for *x* are 2 and -1 satisfying an inequality with solution for *x* as 2.

$$x^2 - x - 2$$
$$(x-2)(x+1)$$
$$x = 2$$

In these excerpts the idea of considering the example to fit the given conditions even when the restrictions do not hold precisely, shows not only a weak conception of inequality and what a solution to an inequality is, but feeble experience with being consistent in mathematics.

SM4:

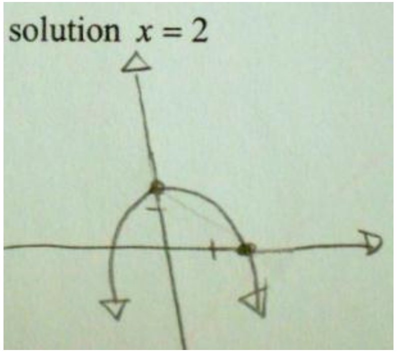


Figure 7.8: Intercepts labelled

The graph of a quadratic function is given as an example for an inequality with solution x = 2. No explanation about the intercepts being visible on the graph is given. It may be the case that the solution of an inequality is mistaken for the zero of a function.

- **Conception 1 - *Tool***: *Inequality as an instrument for comparing known quantities or a tool for expressing restrictions*

In this conception the relationship between a function and the given solutions for inequalities is more evident and more thoroughly explained. Students at **Conception 1** were able to identify that a given solution may represent restrictions imposed on the expression of a function in order to get its domain.

SM1: By having a hole at +5 the graph therefore cannot have $x = 5$ so by making a fraction and having the denominator $(x-5)$ the domain therefore will include $x \neq 5$.

Indeed, a rational function with denominator $x = 5$ and no other restrictions, would produce as domain all real numbers except 5.

- **Conception 2 - *Equality***: *Inequality as a (strange) relative of an equation*

SM1: [x = 2] This is not possible because when solving inequalities, the inequality sign must be carried through when solving. An inequality sign also indicates that a solution is greater than, less than or equal to the final number in the inequality.

SM2: In this case, the $x = 2$ would be a solution that = 0. Solve for x: $-2 + x = 0,\ x = 2.$

The students are convinced that a single number is the sole product of an equation, not an inequality. $y = x - 2$ or $y = (x+3)(x-5)$ are other equations presented as examples of desired inequalities, with no explanations of what aspects of these examples meet the requirements of the task.

- **Conception 3 - *Process***: *Inequality as a mental or algebraic process*

SM1: By having a hole at +5 the graph $\frac{x^2}{x-5}$ therefore cannot have $x = 5$.

SM2: $x \neq 3, 5$ cannot be done since 5 has to be an asymptote in order for it to be an end point but that can't be done since 5] means that there is a value when $x = 5$ and a point on the line contacts it so it's not an asymptote.

SM3: $\left|\frac{1}{x-5}\right| \geq 5$ it works because the domain is all real numbers except for 5.

The thinking process revealed by SM1 – SM3 is the mental association of the domain of a rational function with a hole in the graph. SM2 mentally sees that $x \neq 3$ could come from a function having an asymptote at $x = 3$. SM3 embedded an absolute value inequality with a rational algebraic expression which is defined on a real domain except 5. The mental process was not fully accomplished; the student missed the fact that using 5 on the right side of the inequality will restrict the solution to only $\left[4\frac{4}{5}, 5\right) \cup \left(5, 5\frac{1}{5}\right]$. The inequality should have read: $\left|\frac{1}{x-5}\right| \geq 0$.

SM4: $-3 < x \leq 5 \quad -5 < x - 2 \leq 3$ present a backwards process of generation an inequation from a solution by undoing the steps of solving an inequality.

SM5: $\frac{2x-10}{-x-3} \leq 0$

Here, the fraction is a perfect representation of the brackets chosen to represent the interval. The mental process was carried out correctly when creating the fraction; however, the student did not acknowledge that the negative in front of the denominator gives a reversed pattern in the sign-chart for solving the inequality. The inequality $\frac{2x-10}{-x-3} \geq 0$ would have insured the desired solution.

SM6: $\frac{x+3}{x-5} \leq 0$

This inequality resolves in $-3 \leq x < 5$. The assumption is that the student understands the asymptotic behaviour as well as the importance of the structure of fraction in solving the inequality; unfortunately, he erroneously interpreted the given interval, more precisely the role of square or round brackets. The arbitrary, as Hewitt (2004) calls the mathematical notations and conventions, overwrote the necessary aspect of the structure presented here. “[T]he structural approach invites contemplation; the operational approach invites action; the structural approach generates insight; the operational approach generates result” (Sfard, 1991, p.28). The respondent identified the structure with the potential of producing the given solution. However, there was no visible action after presenting the structure, to verify the results.

SM7: $(x+3)(x-5) \qquad x^2-5x+3x-15 \qquad x^2-2x-15>0$

The quadratic inequality has the potential of producing an interval as a solution. Taking the ends of the given interval, the student partially undid the process of solving the inequality. The missing detail here is that the quadratic inequality with < in the composition gives open interval – such as (-3, 5) – as solutions and those with $\leq$ produce a closed interval [-3, 5].

SM8: $\dfrac{(x+4)(x-5)}{(x-5)(x+3)}$

The respondent then simplifies the (x – 5) common factor of the numerator and denominator. This ensures a hole in the graph at 5. The fraction contains other elements not taken into consideration.

SM9:

1) $x+2\geq 4$ 1) *inequality* $(-\infty < x \leq 2)$

2) $x+4\leq 6$ 2) *inequality* $(\ 2 \leq x < \infty)$

$x+2\geq 4+x$

This example was accompanied by a number line with the two solutions represented graphically. It shows an interesting attempt to generate an inequality producing one single value as a solution. Omitting the graph, the algebraic part of this work shows nothing more than, possibly, an attempt to create a linear inequality with the required solution. The graph containing both solutions meeting at 2 could be interpreted as the graph of a compound inequality. The last line in the algebraic process looks like the attempt to put together the two inequalities. Intersecting two initial solutions or linking the two initial inequalities with the word *and* could have been the grand finale of this work: $x+2 \geq 4 \quad and \quad x+4 \leq 6$. However, student's concept image of inequality fades here.

The excerpts for this section show that the concept of inequalities is developed in the respondents' mind at the level of process. The students could see in their mind the structure that will produce such a solution. The whole process of mentally carrying all the manipulations was however too complex for many of them to allow for a correct outcome. Fractions, absolute value, radical or polynomial functions were used as examples for this conception. As in the data from other tasks, I considered some work to be at this level of conception even though many of the examples students provided did not fully resolve in the given solution.

- **Conception 4 - *Object***: *Inequality as seen by mathematicians – a complex mathematical concept that could be expressed in different registers – symbolic, interval, or graphic; and could perform different functions – compare quantities, express and resolve constraints or deduce equality.*

This conception is the most hypothetical one. There was only one student producing inequalities, logarithmic and quadratic, concluding in the indicated solutions. The most impressive of the three examples is the logarithmic inequality for the $(-3, 5]$ solution, where the open part of the interval was associated with the vertical asymptote and the closed one with the curve crossing the *x*-axis. A search in the repertoire of functions with vertical asymptotes must have been in the mental process of designing the example. The specific elements of the functions are correctly combined with the technicalities of producing an inequality that has an algebraic face not only a graphical one.

SM1: a) $\log_2(x+3)-3 \le 0$ the log becomes greater than zero after 5, and will fail the inequality

The inequality is accompanied by a graph (Figure 7.9) that has a vertical asymptote at -3, stays below the *x*-axis for all values less than 5 and crosses the *x*-axis at 5.

b) $-\frac{1}{3}(x-2)^2 \ge 0$ Parabola opens down, $x = 2$ is the only value where the function is greater or equal to 0.

The graph of a quadratic function (Figure 7.9) that stays below the x-axis and touches it at the only point x = 2 is associated with the inequality.

c) $f(x) = 7(x-5)^2 > 0$ when $x = 5,\ f(x) = 0$, and 0 is not > 0.

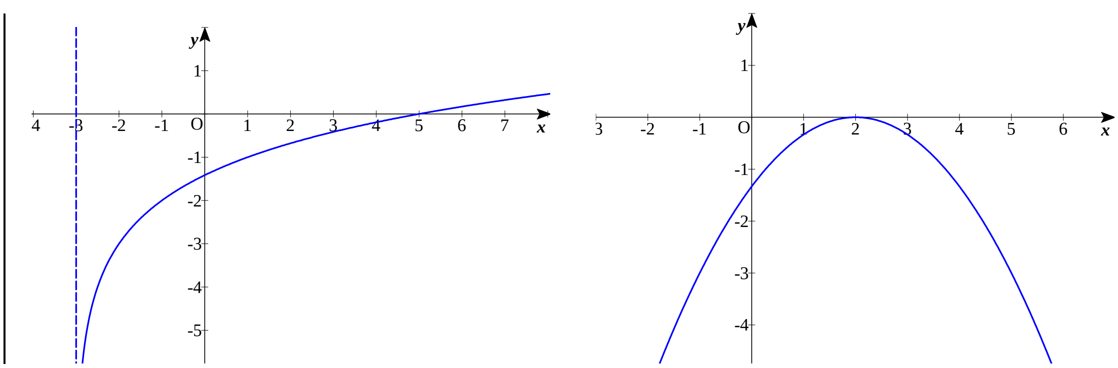


Figure 7.9: Graphs from SM1's work

As could be seen, the examples here, all coming from the same student, are sophisticated, correct and correctly explained. Different registers were used to represent the same inequality.

In this conception, inequality is a complex mathematical object expressed in different registers – symbolic, interval, or graphic. As could be seen in the quotes, the inequality conception at this level is composed of many other entities – such as functions with their domain and graphical representation, it involves several processes – such as mental algebraic transformation of inequalities as well as graphical transformations of functions. There is a fluency of switching not only between different registers of the same inequality, but of choosing from the repertoire of functions some peculiar ones, such as the logarithmic function, to represent the given solution. In summary, this conception shows that the actions, processes, and objects on which the concept of inequality depends or are related to, are coordinated in a mental structure, which is called a *schema* by Dubinsky and McDonald (2001) in APOS theory.

7.3 Summary

A detailed statistical analysis of the undergraduate students' conceptions of inequalities, which the reader might expect to see, has not been prepared as part of this study. However, it seems necessary to mention that the majority of the data falls in the lower levels of conceptions. Moreover, conception 4, or the ideal conception of inequality expected from students studying undergraduate mathematics, is almost imaginary. In fact, there was only one MATH 100 student whit work on inequalities qualified at this level.

As for both categories of participants, MATH 100 and FAN X99, there was a small number of students with conceptions identified at level 3. Needless to say, a performance that could meet expectations in Calculus correlates with a conception of inequality situated at least at a *Process* level.

In summary, this study identified, described, and presented five conceptions of inequalities: (0) Miscellanea, (1) Tool, (2) Equality, (3) Process, and (4) Object. As mentioned in Chapter 6, the numerical labels of the conceptions have been chosen for convenience, not for the purpose of showing a hierarchy. However, Figure 7.10 gives a visual correspondence of the conceptions of inequalities in the big schema of the Three Worlds of Mathematics.

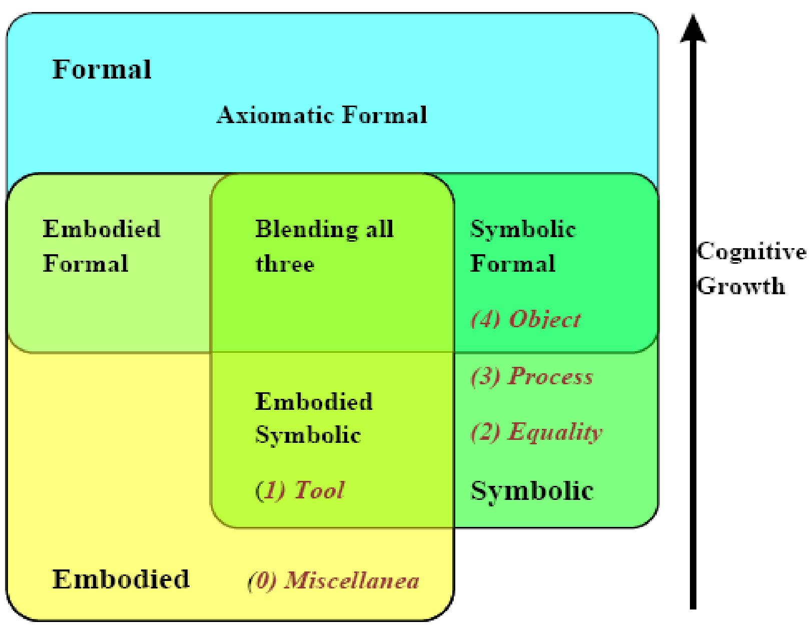


Figure 7.10: The conceptions of inequalities on the three mental worlds of mathematics

The decision to project the conceptions of inequalities on the Tall's Three Worlds of Mathematics is twofold: (1) to connect the conceptions of inequalities that emerged from my studies in a schema of cognitive growth in mathematics and (2) to facilitate the discussion about the importance of coining the COIN.

As stated before, the concentration of students is in (0) Miscellanea, (1) Tool and (2) Equality, conceptions that could be placed in the lower regions of the map representing the cognitive growth and accumulation of mathematics. Conception (3) Process could be placed higher in the Symbolic region and Conception (4) Object at the blending of the Symbolic and Formal worlds. It is well known that qualitative work in undergraduate mathematics requires a solid conception of inequalities, situated in the Symbolic Formal region or at the blending of all three worlds. Tall (2007) argues that, to be ready to understand limits, one's cognitive development must have been reached the Symbolic-Formal level on the cognitive growth (see Figure 7.11). Data from the two studies conducted on undergraduate students the majority of whom would like to take upper-level mathematics courses, shows that they are not mentally prepared for such an endeavour. It is well documented in the literature that, the shift from process to object in mathematics is accomplished with difficulty by students (Sfard, 1991; Tall, 2008b). Developing sophistication in mathematical thinking through the blending of embodiment and symbolism necessitates years of meaningful mathematical experiences. Unfortunately, there are no leaps from (2) Equality in one semester to Limits in the next semester.

Figure 7.11 shows the sophistication of the accumulation of mathematical knowledge and skills. The studies presented here show that, with the exception of a few participants, students could not capture a net of connections of the concept of inequality. Moreover, the conceptions look static and atemporal; discrete fragments that seemed not to have grown meaningfully from embedded experiences through the procedural-symbolic manipulation toward the axiomatic-formal realm but have been grabbed like an umbrella from behind the door when leaving on a bad weather and discarded the moment the rain stopped.

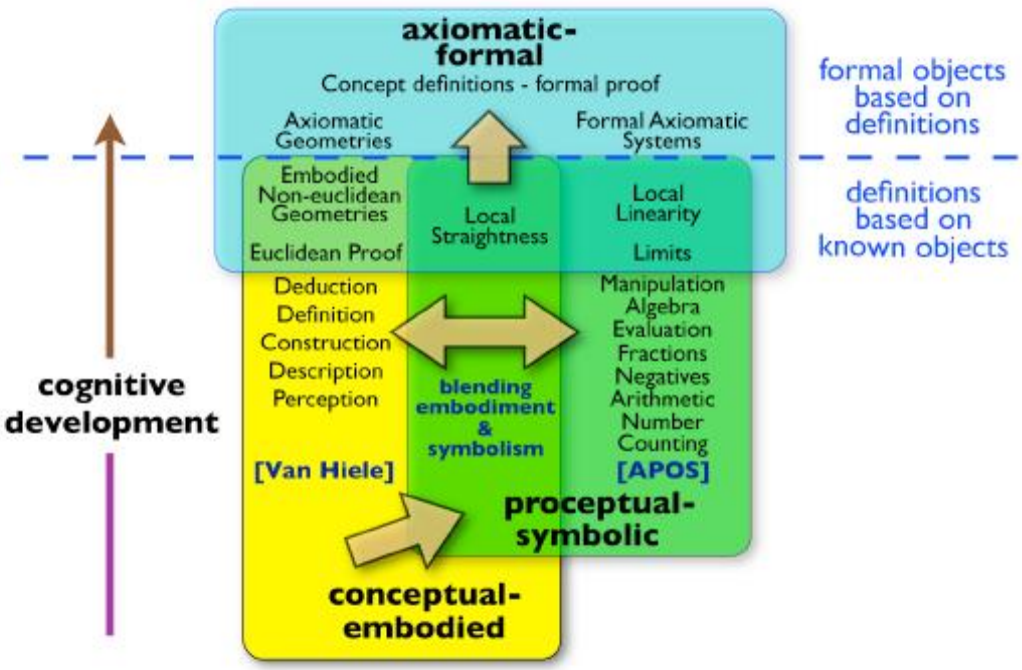


Figure 7.11: The Three Mental Worlds of Mathematics

In conclusion, undergraduate students’ conceptions of inequalities show limited knowledge about inequalities. In the case of *Conception 0*, the knowledge is reduced to fragmented images of contexts where the inequality symbols were used. An attempt to reproduce such experiences shows mental confusion and ambiguity. *Conception 1* is displayed by students who may not have encapsulated meaningful mathematics experiences, but have life experience and are able to see inequality in real life contexts.

Conception 2 is the most troublesome for students working on inequalities; they may think they know how to solve inequalities since they can solve equations. Participants with inequality at the level of *Conception 3* can follow, either mentally or on paper, a process of passing a given inequality through different equivalent stages toward a solution. *Conception 4* is what an instructor teaching calculus would like to recognize when a student is dealing with limits or functions – meaningful manipulation of *inequalities* and correctness in solving *inequations*.

Chapter 8:

Discussions and Conclusions

> In the popular presentation of mathematics, the question whether inequality theory is fundamental to mathematics seems to fade in comparison with its indispensability as a modern workshop tool. There is surely no understanding mathematicians unless you see something of them in undress. (Tanner, 1961, p.294)

In what follows, I bring a journey to an end. I summarize the findings, discuss the contributions of this study to the field of mathematics education, and indicate the limitations of this study.

Three research questions were posed at the beginning of the study:

(1) What are undergraduate students' conceptions of inequalities?
(2) What influences the construction of the concept of inequalities?
(3) How can undergraduate students' conceptions of inequalities expand our insight into students' understanding of and meaningful engagement with inequalities?

The two studies presented in Chapters 6 and 7 thoroughly answered the first question by presenting evidence for five conceptions of inequalities: (0) Miscellanea, (1) Tool, (2) Equality, (3) Process, and (4) Object. Question 2 was implicitly touched on when analysing the data from the studies. Summarizing the findings and compounding the insights from this journey, the following sections shed some more light on question 2 and answer question 3.

8.1 The *met-befores*

> Frequently, from the educational point of view, algebraic inequalities are introduced to pupils after algebraic equations, and the solving techniques are strictly compared; nevertheless, in classroom practice, techniques for equation solving, when applied to inequalities, lead sometimes to wrong results. (Bagni, 2005, p. 1)

It is well known that learning is about building new experiences on prior experiences. Recalling prior knowledge before introducing new concepts is a didactic principle. It is vital for students' engagement in the task and discovery of new material to begin the exploration or the new problem from a place students have already been. There is an abundance of evidence confirming that learning occurs more efficiently when the teacher makes instructional decisions based on what students know about the topic prior to being presented with the new material (Skemp, 1976; Fischbein & Barash, 1993; Tall, 2004; Van de Walle, 2008). Tall (2008a) uses the terms *met-befores* or *set-befores* when referring to students' prior experiences. He defines *met-before* as "a personal mental structure in our brain *now* as a result of experiences met before" (p.4). The met-befores, when used for instructional decisions, have the power of situating the teaching-learning activity where the students are, a technique which is evidenced that will increase students' learning. On the other hand, there are met-befores that can potentially do more damage than good to students' meaningful involvement with a new concept. Reflecting on research conducted on equations and inequalities, Tall (2004) argues that many of the ‘cognitive obstacles’ that students face when solving inequalities are linked to the individual’s subconscious connections with equations.

> It is my belief that the phenomenon of ‘cognitive obstacles’ arises precisely because the individual’s subconscious links to incidental properties in earlier experiences are no longer appropriate in a new context. … I hypothesize that it is precisely the met-befores in solving linear equations that causes problems in inequalities researched by [TTT]. Students taught to manipulate symbols in equations, will build personal constructions that work in their (possibly procedural) solutions of linear equations but operate as sub-conscious met-befores that cause misconceptions when applied to inequalities. (p.160)

Tall is not the only mathematics education researcher to point to that direction; Bagni (2005) also noted that a “forced analogy between equations and inequalities, in procedural sense, would cause some dangerous phenomena” (p. 1). Instead of using the algebraic analogy between equations and inequalities as met-befores, he proposed taking into consideration the historical differences between the two concepts.

Aside from acknowledging their importance in education, it is also important to mention here that the framework of the ‘met-befores,’ at the level of research in education, explains especially the misconceptions or the incorrect associations of a new concept with previously studied concepts. Many studies on inequalities have identified misconceptions and many of them were explained by the met-befores framework from either algebra or equations (e.g., Tsamir et al., 1998; Linchevski & Sfard, 1991; Tsamir et al., 2004). From the classical misconceptions presented in section 3.3 and new ones identified in my study, I would like to mention two met-befores that interfered with students’ abilities to solidify the concept of inequalities:

First, the mnemonic of remembering on which side of the inequality symbol to place the big number and on which one the small one. In their early years, many children are taught to represent the inequality as a “big mouth eating the big number.” In my study, there were students interpreting this met-before the other way around: “the big fish eats the small fish.” Schoenfeld and Arcavi (1988) argue that mathematical notation is a subtle, difficult to learn, and powerful tool if students posses it. Sometimes, teachers, for the sake of helping students adopt a new notation, employ mnemonic devices that can block the process of thinking and creating a personal, powerful tool for manipulating mathematics. For example, the student with the crossed-out diagram seen in Figure 7.6, represented the “7 is greater than 3” this way: $7 < 3$. The metaphor the student tried to remember reads: “someone in my class told me about >. < means that the sign should be looked at as a mouth eating the larger number.” A mouth eating a larger number became the mouth of a big fish, which, in their other met-befores, was eating the small fish (see Figure 7.6).

Second, using the balance model of equations, students intuitively accept that if you make the same change on both sides of an equation, the result will be an equivalent structure. Overgeneralizing the intuitive balance model, students do the same on both sides of an inequality – including eliminating the denominator of a fraction without concern, and expecting to produce good results (Bazzini & Tsamir, 2003). Thus, the intuitive knowledge from equations is inadequately used when solving inequalities. Students use the rules they have in mind, which could be “more acceptable intuitively and visually” (Fischbein & Barash, 1993, p.163) than the correct procedure.

Literature thoroughly documents that the analogy with equations gets in the way of understanding inequalities. It is possible that students replace the inequality symbol < with = because there seems to be no invitation to action in the inequality symbol, but there is the 'do something' embodiment in the equal sign (Kieran, 2006). It could be the case that students are missing embodied experiences and have no access to manageable metaphors for inequalities. The action that they met before related to the inequality symbol was merely 'compare' and that comparison was used mostly in a static setting of comparing given numbers. The embodied 'compare' from geometry seems to be missing from their experience, and, together with that, the whole dynamic aspect of inequalities. Also, at the axiomatic-formal level, students were completely lost since they missed initial training in the game of proving something mathematically.

The examples presented here, as well as the literature review on inequalities, seem to converge to a common conclusion: some met-befores, especially inequality's analogy with equations could hinder the formation of the conception of inequalities.

A question arises: how far should we go into investigating the 'met-befores' if we want to know which of them hinder the conception of inequalities? Is it enough to inquire about the previous chapter on equations? Is it sufficient to inquire about the previous algebra course? Or, do we have to look at the entire algebraic experience of a student? Is algebra the only place to look for the 'met-befores'? Figure 8.1 shows that it is not sufficient to consider students' experiences with algebra as a place for problematic met-befores - one must focus on the earliest encounter with the inequality symbol as a source of the long term damage to the concept that we have observed.

The following section first focuses on students' early encounters with mathematics, and then shifts attention from the *met-befores* to the *missed-befores*.

8.2 The *missed-befores*

Lima and Tall (2008) find that students' difficulties in solving equations are not in the transition from arithmetic to algebra, but in seeking meaning for the symbolic objects in the embodied world rather than the symbolic one. It seems that students, unable to find connections in the world of mathematics in which they perform a task, search for meaning in neighbouring worlds. This could explain why undergraduate students performing on inequalities seek meaning for the symbolic manipulation of inequalities in their resemblance to equations. Without doubt, this act is very ineffective, as we have seen. Students search for props in the realm of *met-befores*. They are, however, unaware that the reinforcement they are looking to recall or resurrect has never been there in their mind. They have completely *missed* the opportunity to learn it.

I define the *missed-befores* as "all experiences and embodiments that were not met before and, if met, they could have had the potential of helping *now* a relational understanding of a concept." Paraphrasing Tall (2004), I hypothesize that it is precisely the *missed-befores* that prevent so many undergraduate students from acquiring conceptions of inequalities situated higher on the Three Worlds of Mathematics diagram. Moreover, I hypothesize that if the *missed-befores* had not been missed, the positions of the COIN on the Three Worlds of Mathematics diagram would look more like Figure 8.1.

When comparing Figure 7.10 with Figure 8.1, it can be seen that all five conceptions of inequalities are moved up and spread over the three regions that support the production of pure mathematics, research on inequalities, and applied mathematics.

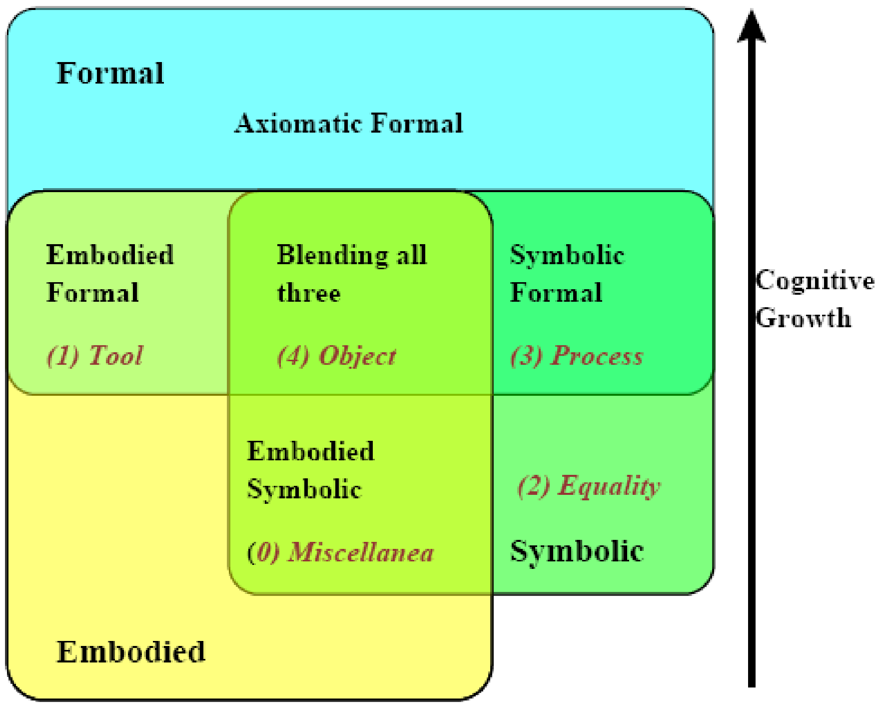


Figure 8.1: The ideal placement of COIN on the Three Mental Worlds of Mathematics

(0) Miscellanea moved from embodied to embodied-symbolic region. Research claims that algebraic training starts long before the introduction of letters into calculations; it can be performed with numbers and word problems (Kieran, 2004; Tall, 2001; Linchevski, 1995). Similarly, training in inequalities can be introduced long before algebra, by comparing numbers, or by exposing students to embodied geometrical inequalities, for example. The production of an equality by means of inequalities, one of

the major aspects of inequalities, is completely missing from the school curriculum. Experiences, such as ‘playing like Archimedes’ – deducing the circumference and the area of a circle using successive approximations – would introduce young learners of mathematics to the power of an inequality to produce an equality.

(1) Tool now sits in the embodied-formal region of Figure 8.1, where embodied knowledge alone is not sufficient and the mind prepares to answer the question: “why does this work?” In regards to students being unable to answer such a question, Nardi (2007) remarked that “we are losing them somehow in the early days and the repercussions are felt later and severely” (p.123). Those students, it seems, have been inappropriately exposed to mathematics in their early years of schooling. I suggest that ‘playing like Archimedes’ experiences, as well as knowledge of formal proofs from geometry can help the ascent of this conception of inequalities.

(2) Equality, I believe, would remain in the same region. However, this would be a conception with a smaller representation than what it has been given in this study. Equality is a conception that is influenced more by the met-befores than by the missed-befores. Training in noticing the aesthetic differences between two concepts and focusing students’ attention on learning what new knowledge these differences bring would reduce the percentage of students that find themselves at this level.

(3) Process moved from symbolic to symbolic-formal. The possessors of such a conception would not only be able to follow rules and algorithms, but would be able to see the versatility of inequality in proving limits.

(4) Object would be a blend of all three worlds – embodied-symbolic-formal. This conception is the inequality that Tanner (1961) has seen as the ‘mathematics in the

undress,' the tool at work in producing pure mathematics. Tanner observed that mathematicians do not publicly display the rough work done intimately with inequalities to produce results for the mathematics' presentation. However, mathematicians at this level of conception will use inequality not solely as a tool, but as the final product for the show.

All in all, there are many missed-befores that can be mentioned in support for the production of each individual conception. To the above list, I would like to add a few more: mathematical language consistency and transition from algebra to calculus.

As documented in Chapter 6, students use the word *equation* loosely. Equation is not the only word loosely used to describe objects that are not equations. Unknown or variable, for example, are used interchangeably in places where only one of them would do justice to the context. According to Teppo and Bozeman (2001), in Carraher, *et al.* (2001) it all starts in childhood, when the concepts of unknown, variable, or place holders seem to be incorrectly or carelessly used in algebraic in nature activities. Teppo and Bozeman argue that it is very important to distinguish between *place holders*, *variables* and *unknown*s, "since each use represents a different type of algebraic entity. Unknowns occur in equations that are essentially numeric in focus. Dummy variables are used as place holders to make general statements about mathematical relations, expresses as a particular sequence of operations" (p.3).

Working on their assignment for a Discrete Mathematics course, my son and his study friend asked for help to prove an identity. To prove that $a = b$, I proposed to work on proving that $a \geq b$ and then $b \geq a$. First, they enquired why that would work and then, my son replied: "Mom, how are we supposed to know how to do this since nobody ever showed us how to work with inequalities?" They have missed that! They, and many other students, have missed experiencing much of the 'mathematics in the making'. They have not even visited the workshop of rough work, where inequalities are extensively used, let alone using the workshop's tools. How could they have first-hand experience with inequalities, when "inequalities have been erased from many algebra courses that used to contain them[?] Is it perhaps that inequalities belong to the "rough work "that so many like to hide? A trade secret not to be too widely divulged?" (Tanner, 1961, p.294). Burn (2005) sees a huge gap between calculus students' mind and the mind ready to understand limits. As a solution to this, he proposes unconventional work with inequalities, for students to experience "the power of inequalities to obtain equality, when no direct path from equality to equality is available" (p.271).

My study seems to be converging toward a unifying conclusion: Revisiting the *met-befores* as well as the *missed-befores* in the framework of cognitive growth could inform curriculum designers about the embodiment, symbolism and formalism students have met before and can build upon or have missed before and must attain, in order to encapsulate the concept of inequality as a solid mental object. It seems that the *met-befores*, which are personal mental structures, as well as the *missed-befores*, which are symbolic experiences and embodiments, should be taken into consideration in preparation for exposing students to inequalities. Instead of losing them in the early days (Nardi, 2007), we should empower our students with experiences that will help the long term development of their mathematics concepts.

In summary, the *met-befores*, as well as the *missed-before,* influence the production of the concept of inequalities. One cannot have a construction stronger than its foundation.

8.3 Contributions

In Chapter 3, I referred to inequalities in mathematics education research. This study contributes to that category of research. It looks mostly at learner-generated examples of inequalities and answers one of the major questions the community of mathematics educators has proposed: What are students' conceptions of inequalities? (Bazzini & Tsamir, 2004). Five conceptions of inequalities are identified in Study 1 and validated by Study 2. Moreover, a framework for analysing students' work on inequalities is presented: the COIN.

In Chapter 5, I referred to the role of examples, worked examples, and learner generated example (LGEs) in mathematics as well as in mathematics education. The use of learner-generated examples is novel for studying inequalities. Although extensively recognized as a powerful pedagogical strategy that promotes active participation of students in the process of learning (Watson & Mason, 2005), the use of learner-generated examples is novel for studying inequalities. The use of LGEs for collecting data appears to be powerful in revealing aspects of inequalities that were not transparent for interpretation in studies where the tasks were more about solving given inequalities than constructing something. In working with LGEs, students had no access to a memorized procedure to start with. Moreover, LGEs served to decompose the concept in order to get access to the inner structure of an inequality. Undoing, making connections, starting work in unexpected places, searching for inequalities in the realm of functions or graphs were part of the participants' work for producing the data. All these mental actions were transparent and helped the coinage of the five conceptions of inequalities.

8.4 Limitations of the Study and Suggestions for Further Exploration

Further to this end, I would like to acknowledge the limitations of my study and to make suggestions for further exploration.

8.4.1 Limitations of the Study

A limitation of my study stems from choosing convenience samples of participants, mostly my students and not strong ones either. Moreover, this study considered first-year mathematics students alone. Additional calculus or other upper-level mathematics classes of students could have been selected for the research. Such a broader selection could have potentially answered question 2 more accurately, namely *what influences the production of the concept of inequalities*.

In addition, a limitation regarding the methodology of the study can be mentioned. Data was collected as written work alone. When administering the surveys, time was sometimes an issue; some participants claimed that given more time, they would have accomplished more. Also, participants were not interviewed, which would have provided further evidence for the reasoning behind some procedures or could have elicited more information about past involvement with inequalities aspects.

8.4.2 Suggestions for Further Exploration

There are several directions that can be suggested for further investigation on conceptions of inequalities. Regarding the sample, a study using senior undergraduates, master students, school teachers, and professors of mathematics as participants could validate, complete, or revise the conceptions of inequalities. Moreover, if the data were to be collected through interviews, a more traditional phenomenographic analysis could be conducted for possibly more solid results on conceptions of inequalities.

Another possible direction could be to look into connections among students' conceptions of inequalities and their success in learning mathematics. This could be realised by shadowing calculus students while they are working on limits or functions and by noticing if their major frustrations for not 'getting it' come from their weak background on inequalities. Further developing the theme of connections, an investigation of the links and the interplay between preservice teachers' knowledge of different mathematics concepts and their readiness to teach mathematics could be a natural continuation of my interest in conceptions of inequalities as well as my long-lasting interest – preparing elementary teachers for teaching mathematics.

8.5 Last Words

Although there seems to be more to say on this topic, I must bring the study to an end.

Along with the five conceptions of inequalities, this thesis has evidenced that the majority of undergraduate students participating in the study possess weak conceptions of inequalities. It has also shown that the school curriculum is far from being adequate to help students acquire better mathematical foundations on which the concept of inequalities may grow and flourish. Moreover, the study has documented that the curriculum is disconnected from the historical development of inequalities and their important role in the production of mathematics. The study has suggested some curriculum amendments. Some of the suggestions for early mathematics could be interpreted as being at a too higher level for the young minds. Bruner (1960) claims that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p.33). I am optimistic that, honestly prepared for real mathematics since childhood, the undergraduate student can be more successful in manipulating inequalities.

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Appendices

Appendix 1: Examples and Learner-generated Examples in Learning Mathematics and Learning about Learning Mathematics

Anne Watson and John Mason use the word *example* for “anything from which a learner might generalize” (p. 3). For example, discussing the results of the task that reads: “Give an example of a number that lies in between 1/5 and 1/7,” the majority of my students produced 1/6 as an example. When asked orally to give another example, some said that there is none. Then, somebody answered with 6/35. Asked for another example, somebody gave us three more examples: 11/70, 12/70, 13/70. From the production of these examples, the students generalized that there are plenty of numbers between 1/5 and 1/7. Moreover, a workable way of producing as many examples as you please is to take a multiple of the least common multiple of the denominators of the two initially given fractions and choose the numerator within the boundaries of their new numerators.

Sfard (1991) claims that “a person must be quite skilful at performing algorithms in order to attain a good idea of the “objects” involved in these algorithms; on the other hand, to gain full technical mastery, one must already have these objects encapsulated in his brain, since without them the process would seem meaningless and thus difficult to perform and remember” (p.32). This insight, therefore, is fundamental to the understanding of inequalities. In order to solve inequalities, one must master and understand the properties and axioms related to equivalent inequalities. However, without a profound understanding of the object of inequality itself, performing transformations on inequalities looks meaningless and error-prone. Using a metaphor, I would call Sfard's argument a dance between operational and structural aspects of a concept. She argues:

> Thus, almost any mathematical activity may be seen as an intricate interplay between the operational and structural versions of the same mathematical ideas: when a complex problem is being tackled, the solver would repeatedly switch from one approach to the other in order to use his knowledge as proficiently as possible. (p.28)

The task of generating examples of inequalities serves as a good example of the complex interplay between operational and structural conceptions of the same notion. To be able to construct an example of an inequality starting from its solution, one should know the structure that will result in such a solution. Moreover, after attempting to create an example, one must solve it, meaning that the person moves from the structure to the procedure in order to check if it results in the desired solution.

Other studies have also considered student-generated examples to be a good tool for looking inside learners' minds and learning about their understanding of mathematics In their reflections on the tasks, the students themselves noticed the complexity of a generate-example item compared to a solve-inequality task. Here is an excerpt from one participant's notes acknowledging the difficulty of a construct-an-example task:

> For me question number 1 [the "solve the following inequalities" item] was more clear in the sense that [an inequality] is there & we have to [solve it]. I got kind of lost with questions 2 & 3 because we're not used to coming up with questions [examples] so question 3's instruction wasn't something we've seen in the past. I was

> more successful in answering question 1 because I was used to seeing that kind of question…

The student acknowledged the fact that generating examples requires more effort than solving a problem by making an analogy with a previous work. Retrieving information from studied examples in the past makes a task more approachable as opposed to constructing one's own examples. Therefore, the student measured her success by being able to apply the algorithm and solve the given equation. For the example generation item, however, she was unsure about her success in searching for connections and constructing the required structure. This could be evidence that the task under discussion here is complex and the results will give enough variation to capture respondents' thinking.

In conclusion, in my study, the participants were considered to be the experts in linear inequalities and they produced a worked example for somebody else to learn from it. From the examples provided by the students and their work towards solving the examples, the researcher learned about participants' understanding of inequalities.

Appendix 2: The Instruments for Data Collection

Experimenting with examples (EE)

1. Create a worked example that will help another student from this class learn how to solve order of operations exercises.

2. Connect the numbers 2, 5 and 3 with operation signs to generate examples that will capture the essence of order of operations.

3. Give an example of a number that lies in between 1/5 and 1/7. How many numbers are in between 1/5 and 1/7?

4. Write a system of two linear equations with a solution (3, −2).

5. Look at your worked example. Reflect on your work. Were you on task? Why? Why not? Compare questions 1 & 2[12] in your quiz with question 3 and reflect on your performance on both tasks. Were you more successful in one of them compared to the other one? Why?

Pre-teaching survey (PTS)

1. Explain in short terms the concept of *mathematical inequality*. Please use symbols, pictures and words. Even if your image/belief about inequality is vague, I will appreciate it if you as clear as possible will try to give an explanation.

2. What does a solution of an inequality mean? Exemplify.

Solving inequalities and multiple representations of inequalities tasks *(SIMR)*

$4-3x \le 2x+19$ for FAN X99

$\frac{x-2}{x+3} \le 2$ for MATH 100

[12] Questions 1 & 2 with reference in this item are not items 1 and 2 in the list of tasks. They are ‘solve equations’ questions, in contrast to ‘give an example of an equation or a system of equations with given solutions’.

Exam Items (EI)

a) Sketch the graph of $f(x) = x^2(x+1)(x+2)$. On your graph, clearly label all the intercepts and show the behaviour of the function at the *x*-intercepts.

b) From the graph (or by other means) find the solution to the inequality $f(x) \le 0$.

c) Find the domain of $f(x) = \sqrt{\dfrac{15-5x}{x^2-3x-10}}$.

The task (T)

a) Create a worked example that will show someone how to solve linear inequalities.

b) Is the one example provided in part a) sufficient for someone to learn how to solve inequalities by following your work? Do you think you need to create more examples to demonstrate the full breadth of linear inequalities? If so, how many more examples do you think you need?

The refined task (RT)

You know that the best way to learn something is to teach somebody; therefore, you have agreed to tutor your cousin Jamie who is taking Principles of Math 11 this year. You are available for him any time and through any means.

a) You've got a text message from Jamie that reads: "*Missed the class on linear inequalities. I have to do my homework. Don't know how to start. Help me with the steps of solving a linear inequality*." E-mail him back the steps for solving linear inequalities. On the space below show the message as well as your preparation for sending it.

b) Half an hour later an e-mail from Jamie arrives: *"I followed your steps and solved a whole bunch of inequalities. Thanks. Then I attempted this one:* $1-2x > 2(6-x)$. *I worked out the algebra and got this* $1-2x > 12-2x$ *and then ended up with:* $0 > 11$. *Here I got stuck. Please help.*" E-mail him back. On the space below show the message you will send to your cousin Jamie. The message should contain your feedback on Jamie's work as well as your input to Jamie's further understanding of inequalities.

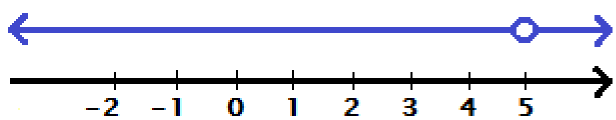
Generate examples task (GET)

In each case, give an example (an equation, a picture or a description) of a mathematical object/concept satisfying the given conditions. Explain briefly why your example meets the conditions. If you can't get an example of one of the items, explain why you think that such an example would be impossible to construct.

a) an inequality with solution $(-3, 5]$

b) an inequality with solution $x = 2$

c) an inequality with solution represented by the graph



Generate more complex examples task (GMCET)

You sit down with Jamie and work on inequalities. He got pretty skilful at solving all sorts of inequalities. However, you are not sure yet if his work is just procedural or he really understands inequalities. To check for his understanding in a pleasant way, you decided to play the *give me another example* game. The rules of the game are the following: *You ask him to give you an example of an inequality with a given solution. And then another one. And then another one, completely different then the other ones.*

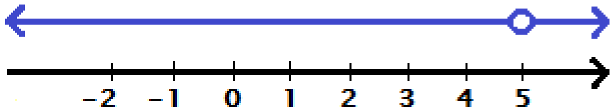
For the solution $(-3, +\infty)$ Jamie's three examples were (1) $x > -3$ and (2) $-2x < 6$ and (3) $-4x < 12$. In the space below, comment if Jamie's examples satisfy your requirements. Provide your examples, in case Jamie's examples are not different enough.

You continue playing the game of giving examples on three other potential solutions of inequalities. For each of them you and Jamie were able to come up with three different examples of inequalities that will result in the given solution. In the space below, provide the examples you and Jamie produced for:

a) Solution $(-3, 5]$:

b) Solution 2:

c) Solution represented by the graph:



Inequalities survey (IS)

1. What sorts of images or examples come to mind when you consider the concept of inequality?

2. What are the ways in which inequalities and equations are the same and/or different?

3. What does a solution of an inequality mean? Exemplify.

4. Provide (construct) three *different* inequalities that will give the solution $\left(-3, +\infty\right)$:

5. Solve the following inequalities. Give the solutions graphically and in interval form.

   a) $4-3x \le 2x+19$ b) $4-3x \le 2x+19$ and $-\frac{1}{3}x > -2$

6. Can you tell me an interesting fact you have learned/discovered lately about inequalities?

7. a) You've got a text message from Jamie - your homework partner - that reads: "Missed the class on inequalities. I have to prepare for the quiz. Don't know how to start. Help me with the steps of solving a linear inequality." E-mail him back the steps for solving linear inequalities.

   b) An hour later an e-mail from Jamie arrives: "I followed your steps and solved a whole bunch of inequalities. Thanks. Then I attempted this one: $1-2x > 2(6-x)$. I worked out the algebra and got this $1-2x > 12-2x$ and then ended up with: $0 > 11$. Here I got stuck. Please help." E-mail him back. The message should contain your feedback on Jamie's work as well as your input to Jamie's further understanding of inequalities.

8. Try to recall or reconstruct the provenance of your first response.