Initializing Thinking in the Mathematics Classroom

By

Kevin John Wells
B.Sc (Hons), University of Leeds, 1978
P. G.C.E. University of Keele, 1979

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

In the
School
Of
Education

Mathematics for Secondary Education

SIMON FRASER UNIVERSITY
Summer 2009

Approval

Abstract

This thesis examines the issues around initializing thinking in mathematics. In particular, non-start and quick-stop behaviour of high-school students between grade 10 and 12 during problem-solving is investigated. Factors which determine the approach of students during this crucial entry stage to problems are studied, along with the anxiety levels and the belief systems which shape their actions. Students' responses are studied through their personal reflections and their attempts at initializing problem-solving. Using metacognitive and reflective approaches to help the students, along with establishing a classroom atmosphere which supports and encourages thinking, the causes of non-start behaviour are examined. The Mathematics Anxiety Scale and the Mathematics Related Beliefs Questionnaire are used at the beginning and end of the study to indicate any changes that have occurred. The aim is to provide teachers with a sense of how to foster this important area of mathematics and help alleviate non-start and quick-stop behaviour.

Keywords:

Mathematics;
Thinking;
Non-start behaviour;
Problem-solving;
Affect;
Teaching environment.

Dedication

To my wife, Karen,

- as a thank you for the support you gave me during this time.

Acknowledgements

I would like to acknowledge the invaluable assistance of Dr. Peter Liljedahl as senior supervisor during the development of this thesis, and for his thought-provoking course teaching.

I would also like to thank Dr. Nathalie Sinclair for her contributions to in helping shape the final document, as well as her enlightening course teaching.

In addition, I would like to acknowledge the other Mathematics Education professors at Simon Fraser University for helping to develop my thinking during the Master's course: Dr. Rina Zazkis, Dr. Steven Campbell, Dr. David Pimm, Dr. Tom Archibald, and Dr. Peter Borwein.

Contents

List of Tables

List of Figures

Chapter 1. Introduction

> "*Most people would rather die than think; in fact, most do*"
>
> *Bertrand Russell - The ABC of Relativity*, 1925, (p. 166[1])

It's a Monday morning, 8:45am, early September. A class of grade 10 students await the introduction phase of their mathematics class. Some are paying close attention, perhaps many listen half-heartedly, while others have their attention elsewhere. The teacher scans the faces with an experienced eye and allows a silence to hold its presence and draw attention closer before speaking.

"I have a problem." He states, then "And I want you to help. " Again a pause, but this time to add a level of anticipation; "I have 12 balls which look exactly the same. Eleven of them have the same weight, but one of them is either lighter or heavier than the others (you don't know which). Using a balance scale you can determine which ball is the 'odd ball' and whether it is too light or too heavy. "

A few hands go up, but the teacher raises his hands in anticipation of the obvious questions. "Here's the thing though - you have to do it in *three* weighings. How do you do that?" How do the students respond to this problem? Specifically, how do they *initially* approach the problem? How do the teacher's actions over the next few minutes affect the way the students approach the problem? If you place yourself in the position of the teacher at this point in the lesson, how would *you* visualize the next stage of the lesson going? Perhaps you visualize an idealistic scenario where the students form small groups and actively engage in the problem, suggesting and discarding ideas until an optimum solution is found. Perhaps you see a classroom where all students are contributing to the process and where real learning is going on for each

[1] Variations of this quote exist, such as "*Most people would rather die than think, many do", but I choose to use the version a colleague of mine, Daniel Stewart, had on his classroom wall and which I have on mine today.*

person. More likely though, you will envisage a room where many students are having difficulty in starting the problem. You will know that some students are finding creative ways to avoid starting the problem. You will see others whose first response is to look to others to get started. Any method that seems to have possibility may quickly spread to the other students in the room, removing the problematic aspect of the problem for the majority. What determines this 'non-start' and 'quick-stop' behaviour in our students? Can the teacher affect these processes in a positive or negative way? Is promoting genuine thinking for our students attainable, or should we accept Bertrand Russell's suggestion (quoted above) as normality?

The aim of this study is to investigate the nature of the initial thinking undertaken by students in a mathematics classroom. With a greater understanding of the way students approach a problem, what techniques they employ, and the manner in which they avoid thinking about a problem, the teacher is better able to establish a classroom that promotes mathematical thinking in as many students as possible. Further, the teacher is better positioned to push those students to continue thinking beyond their desire to 'give up'.

'Thinking', however, is an elusive quality to measure and almost impossible to quantify in a classroom setting. There are so many factors which can affect the way a student performs on any given day. This 'affective domain' is an important element in determining how the student will respond to the question posed, and how much time that student will persevere with the problem. Factors which may determine students' actions are often set before a teacher first meets a class. The craft of the teacher must then be to create a suitable environment that will nurture the thinking process regardless of the students' emotional states on entering the room. To foster an expectation of thinking and to reward the presence of such thinking can only be done if the teacher has some sense of how their students respond to questions such as the one posed above. The objective of many new mathematics curricula is to get students to be able to creatively solve problems, but how do we prepare our students to develop these thinking skills?

When I first started this research I recall a student saying to me: *"I never know how to start, that's my problem always. Then I ask you and it seems so obvious. It makes me feel dumb."* As a new teacher I remember feeling I was doing a good job in being able to answer all of my students' questions, perhaps even a little smug to know the 'clever answers' and 'tricks' to some of the more obscure problems. The more I taught, however, the more I began to realize that I was not helping my students by telling them the answer, and I began to *ask* them instead. The more I did this the more I realized that they often knew the answer all along, they just didn't know how to ask themselves the right questions. On many occasions they did know the questions and did know the answer, they just didn't believe it.

Even before thinking about the direction of this research I had been greatly interested in the difficulties students face in starting a problem. I was intrigued by how much effort some students seem to put into avoiding starting problems and how quickly some students gave up. I had been pushing problem-solving techniques at my students for several years and I would say that the results were often disappointing – students who seemed good at problem-solving stayed that way and those who weren't didn't seem to get much better when faced with a problem they were not familiar with. The mechanical steps of solving a problem seemed teachable but not lasting. Even more disturbing was how often the fundamental errors a student was making at the start of the previous unit were once again rearing their heads.

My challenge in this thesis is primarily to develop a better sense of the first faltering steps a student takes towards a solution, but in doing so to also learn how best to stimulate this. As J. Bay-Williams and M. Meyer (2005) observe: "*the goal is not to frustrate your student but* *to get the student to learn a strategy that makes sense so it will be remembered and used in future situations (p. 341) ."*

In the following chapters I seek to investigate the behaviours that determine how students initialize their thinking. In chapter 2 the research questions are presented which drive the process, followed by a review of the literature that helped situate the research in chapter 3. Chapter 4 outlines the methodology used with the participants of this study, students in a Canadian high-school setting between grades 10 to 12 (typically 16-18 years old). In chapter 5 data collected for grade 11 students are presented. Data are collected to examine the affective domain around problem-solving, and the way in which students initialize their thinking when presented with mathematical problems. Chapter 6 presents data collected for grade 12 students, with a focus on student reflection portfolios, over the same period. In chapter 7 the data collected for grade 10 students in the first semester is presented. While the grade 11 and 12 classes were semester based, meaning they are taught every day for half the school year, the grade 10 classes were taught linearly, meaning they are taught every-other day for the full school year. Chapter 8 examines the data collected for all grades over the first semester and established some ideas for further investigation into the second semester. In chapter 9 the data collected for the grade 10 classes during the second semester is presented. Chapter 10 offers a discussion of the themes which emerge from the study, placed in perspective with the data collected on the students. Finally, chapter 11 summarises the results from the study and offers conclusions based on those results.

Chapter 2. Research Questions

"If in other sciences we should arrive at certainty without doubt and truth without error, it behoves us to place the foundations of knowledge in mathematics."

Roger Bacon Opus Majus, bk.1, ch4.

The area of interest examined in this research, the initialization of thinking in mathematics, lies between two other areas for which there exists a number of studies: the anxiety of students in taking a mathematics course – or in their negative beliefs and attitudes around mathematics in general; and ways to promote thinking in a mathematics classroom. Specifically, the objective is to look at the initial response students have when faced with a problem given to them in a mathematics class. If there is a better understanding of how students deal with problems at the initial stage then teachers may have more success in helping their students to become more successful in problem-solving in general. Research does not seem to focus on this specific area, although several texts offer advice for students and teachers on how to start problems. Problem-solving advice typically ranges from a list of strategies to heuristics based predominantly on Pólya's work, and is included in most textbooks in varying degrees of formality. It is argued that many students are familiar with these methods by the time they get to senior grades, and are exposed to them frequently during, yet still find difficulty initializing their thoughts when first given a mathematical problem. The result can be a 'non-start' behaviour, whereby the student does not even try to start a problem, which becomes ingrained in many students by fairly early stages in their development. It may also be manifested in 'quick-stop' behaviour whereby the student makes a cursory start to a problem only to give up at a very early stage.

A study in this area would not be complete without having an understanding of the conditions that build towards anxiety in mathematics. At issue here however is, given that the student is subject to these conditions - and despite this, can we still look for ways to help a student initialize their thinking and persevere with a problem after reaching the first hurdle? It is argued that in order to do so we must first understand how a student initially responds to a question. It would be a topic for a further study to determine if this information could be used to then improve a student's *actual* ability to fully solve a problem. An initial hurdle to overcome**,** however**,** would be to improve a student's *perceived* ability to solve a problem. If the student is more comfortable in their environment in the mathematics classroom then it will be argued that they are more likely to at least start a problem and less likely to simply dismiss the problem without giving it any thought whatsoever.

Research Question One

What is the basis of 'non-start behaviour' in students?

My aim here is to investigate some of the reasons why students fail to engage with a problem when it is first presented. This is termed 'non-start' behaviour in this research. In particular, I am interested in what aspects of this behaviour can be addressed in the regular classroom. I wish to examine how a student's ability to start a problem is influenced by both anxiety and their beliefs about the subject and their own ability to do mathematics. By taking observations of the students on an ongoing basis, rather than in specific 'staged' lessons, events which may alter a student's willingness to work on a particular day will not be significant compared to underlying influences which remain constant. In some cases, however, this may not be true as the student can be in a situation where their work is constantly affected by external factors. It may also be that the belief system of a student is set so strongly by the time they reach the classes in this study that they are unable to change, at least within the timeframe of this study. Perhaps as important in this study is an examination of how the emotional element can be changed so that the student is willing to suspend their belief-systems long enough to engage in the problems posed. As Malmivuori (2001) points out, this can be done by stressing learning processes instead of stable constructs or learning outcomes. The B.C. Ministry of Education IRP's for Mathematics (2008) state that a positive attitude is an important aspect which has a profound effect on learning. The ministry references research by Nardi and Steward, (2003) which indicates that by encouraging risk-taking and developing self-confidence students are more engaged and likely to be more successful.

RESEARCH QUESTIONS TWO

What is the cause of 'quick-stop behaviour' in students?

In promoting active problem-solving in the curriculum the BC Ministry of Education (through the WNCP) is emphasizing constructivism (Steffe, Cobb, and von Glassersfeld, 1988) and discovery learning (Dewey, 1916; Bruner, 1961). In this regard my question looks to how well students actually engage in the problem and gain benefit from it as opposed to merely substituting a peer for the teacher in the learning process. I am interested in trying to determine if students stop by habit, lack of self-belief, or recognition of 'the simplest route'. My interest in this particular question was peeked several years back at a professional development conference where several people (non-mathematics graduates teaching occasional math courses) considered that using other peoples' solutions was a legitimate way of problem-solving. By this I do not mean direct plagiarism but the process of waiting for other people to do the thinking and then adopting their ideas. This includes the teacher doing the thinking for the student. My aim is to gain more insight into this process.

RESEARCH QUESTION THREE

C*an 'non-start' and 'quick-stop' behaviour be alleviated in students?*

This question naturally lies at the heart of this research. If it is possible to modify a student's mode of thinking in regard to starting problems then the ability to develop this area would have, I believe, real implications for teachers and the teaching process. Where the first two questions help to understand the nature of these behaviours it is natural to look for ways to help students in this area. I recognize, however, that making improvements in the starting process of a problem may not necessarily translate to a measurable level of success in completing problems, or in the overall performance level of a student. The entry phase of a problem is one of four stages identified in most problem-solving heuristics (Pólya, 1955; Mason et al., 1982; Schoenfeld, 1985; Perkins, 2000) but it would seem to be a vital stage in developing the skills of the student further. By creating a classroom environment which promotes a positive emotional response by students, encourages risk-taking, and promotes and rewards thinking, can non-start and/or quick-stop behaviour be affected?

Chapter 3. Literature Review

"There is no need to shy away from a mathematical question, and no reason to stare at a blank piece of paper feeling hopeless. Driving straight down the first path that appears hoping brute force will succeed is not a good tactic either."

Mason, Burton and Stacey, 1982. (p. 1)

In this chapter the research is positioned within the body of work related to this topic. I begin with an analysis of research on the affective domain, which is summarised under the heading *Affect and Belief.* This is followed by an examination of research in the area of *Problem-Solving*, organised according to the authors who have most profoundly impacted my thinking in this area. Finally I look at studies on the thinking process under the heading *Developing Thinking.* This section examines the work of Robert Fisher, followed by a number of authors who write on promoting thinking in the classroom. This section concludes with a brief examination of the concept of Learned Helplessness.

Affect and Belief

Stemming from the pioneer work of McLeod (1989), there is a growing body of research related to the affective domain and its influence on learning in the mathematics classroom, (Goldin, 1993, 1997; DeBellis, 1996, 1998; Hannula, Evans, Philippou, and Zan, 2004; Cobb, Yackel, and Wood; 1989, Op't Eynde, De Corte, and Verschaffel; 2001, Malmivuori, 2001 and 2006). Such research points to the many factors which contribute to the dynamics of a student's involvement in the classroom. Malmivuori (2001) suggests that: '*How pupils view and approach mathematics and mathematics learning situations will determine their goals and modes of understanding, responding, and behaviour in doing and learning mathematics' (p. 8).*

What comprises the realm of the 'affective domain' is, however, not clearly defined. McLeod (1992) identified beliefs, attitudes, and emotions as key elements of the domain, to which DeBellis and Goldin (1997) added a fourth element, values. Hannula (2004) would extend the elements further to include motivation, feeling, mood, conception, interest, and anxiety. DeBellis and Goldin (1999) regard affect as an internal system of representation which interacts with cognitive representation and meaningfully encodes information, influencing problem-solving. The affective domain is seen as going beyond attitudes, beliefs, and self-concepts to becoming central to the cognitive process.

McLeod (1992) noted that of the three elements, beliefs were the most stable and, as Malmivuori (2001) notes, the hardest for a teacher to change. Goldin (2002) views beliefs as containing prepositional encoding to which the holder attributes some kind of truth. Emotions, the most intense of the elements, are considered to be least stable and therefore easier for a teacher to change. Malmivuori (2001) also notes that self-beliefs mediate socio-cultural mathematics beliefs and affect the way students learn or problem-solve. Gómez-Chacón (2000) considers beliefs around students' self-concept as mathematics learners to be one of the variables with most influence on mathematics learning. In addition, self-concept is important to the way a student sees the world of mathematics in general. Hannula (2004) adds that negative affect (e.g. mathematics anxiety) can seriously undermine academic outcomes and those beliefs, such as mathematical talent being innate, may be harmful for learning.

Goldin (2000) calls for more research into developing teaching strategies in which affect is more than incidental to the learning in the classroom. In particular, he refers to local affective goals which he suggests lead to rapidly changing states of emotion during problem solving activities - This is contrasted with global affect, which can be deep-set and slow to change. Attitudes and beliefs are considered to be aspects of global affect. DeBellis and Golding (1997,

1999) suggest a model for the stages of local affect which lead to an eventual development of global affect. In this model two pathways may be followed by the student, both of which start with the common local affects of ‘curiosity’, ‘puzzlement’, and ‘bewilderment’. Bewilderment is thus seen as an important stage in the problem solving process, which may branch to either ‘encouragement’ or ‘frustration’ as the next step. DeBellis and Goldin suggest that students on the negative pathway search for safe procedures, whereas students on the positive pathway tend to seek exploratory opportunities. This is also seen as an area where teacher intervention frequently occurs, and may lead to unintended results such as an acceptance of authority-based problem-solving. If feelings of frustration are left unresolved they lead to anxiety and, eventually, a global structure of fear or hatred of mathematics. As a consequence the student may construct powerful defensive mechanisms in order to avoid the situation. DeBellis and Goldin consider that affect may disempower students and hamper performance, even leaving understanding unrecognizable when it occurs.

DeBellis and Goldin (1999) also introduce the idea of *meta-affect,* or the affect about affect. They give an example of fear as being either pleasurable or horrifying, determined by the meta-affect around it. In some cases this fear can enhance the experience, while in other cases it can be stymieing. During problem solving in particular the experience of frustration, or being ‘stuck’, may be treated differently by different students. DeBellis and Goldin point out that advanced graduate students in particular – they emphasize the term *especially* , but there seems no reason to suspect that this is not an important aspect at all levels– may fear exposure of self-perceived inadequacies when presented with a problem.

Of particular interest is the pattern of behaviours described by researchers when students are given a problem in a mathematics class. DeBellis and Golding (1997) discuss the concept of denial and distinguish between what might be called ‘self-denial’, where the student does not recognize their lack of understanding, and an external form of denial where the student is attempting to mask their lack of understanding. Guerrero, Blanco and Vicente (2002) observed avoidance as one of the attitudinal and behavioural signs that many students show when they are faced with mathematical problems. Goldin (2000) points out that a student may try to imitate an intended procedure without any real understanding, or guess an answer without basis in an attempt to disguise their uncertainty. Goldin notes that: *"The 'slow learner' delays commencement, fidgets, needs a glass of water first, or has to visit the toilet. The 'impulsive child' learns to reach out and grab an arithmetic operation like a lifeline." (p. 216)*

Goldin also refers to the importance of encouragement from the teacher following partial success. He notes that even a simple 'You're on the right track," can help to conquer doubt and enable the student to persist when they may have otherwise given up. Hannula (2004) observes that a word problem may induce fear even before the student has *read* the task – a direct suggestion of non-start behaviour. Malmivuori (2001) asserts that: '*Mathematics represents a subject towards which pupils´ attitudes are developed very early in the school learning context' (p. 8).* This implies that many of the beliefs a student has upon entering the higher grades are already set. As noted earlier, such beliefs lie in what Goldin calls *global affect* and are, as such, hard to change. Dweck and Elliot (1983), Kloosterman (1996), and Op 't Eynde and De Corte (2003) suggest that a student's beliefs about the nature of mathematics may influence their motivation and performances. Malmivuori (2001) adds that many students hold the belief that mathematics is only of use for mathematicians, scientists, engineers, or teachers. Malmivuori adds that mathematics is generally seen as a 'difficult' subject, perhaps even as a filter. In addition, she also points to the students' belief about the role of the teacher as the central figure in the mathematics classroom, assessing their work for correctness rather than as a support figure. A combination of these beliefs speaks to the *motivational* aspect of affect, as introduced by Hannula (2004), and the student's desire to engage in the problems given. Motivation may be extrinsic, intrinsic, or a combination of both and Hannula considers it to have the potential to direct behaviour. Extrinsic factors in this case refer to the problem, the classroom environment, and the actions of the teacher. One aspect of this research is to examine the effect these extrinsic factors have on motivation, in particular in the initial stages of problem-solving. In order to do this the objective is to examine the local affective goals and the changes in emotion around problem solving and, in particular, non-start and quick-stop behaviour.

Problem-Solving

Pólya: How to Solve It

The seminal work on problem-solving was written by George Pólya in his book '*How to Solve It*' (1957). Variations of Pólya's heuristics are commonly found in mathematics textbooks where they are presented in specific problem-solving chapters. The MathPower series (McGraw-Hill, 1994 (Gr.7); 1995 (Gr.8)) is a good example of this approach. As the initial step of a four-part process Pólya requires that the student should 'Understand the Problem.' He specifies: '*What is the unknown? What are the data? What is the condition? Draw a figure. Introduce suitable notation. Is it possible to satisfy the condition?* (p. 7) Pólya then splits his initial phase into two, consisting of '*getting acquainted',* and '*working for better understanding*'. Under the second phase '*Devising a Plan'* Pólya gives a specific list of questions the student should be asking themselves, key to which is that the student call upon their own bank of related knowledge. In short, the student needs to have some sense of how the problem fits into their existing schema of knowledge and be able to build on this in order to find a solution. Goswami and Brown (1989) refer to this process as *analogical reasoning* and point out that Piaget argued such skills did not develop until early adolescence. Seeing relations between relations was a hallmark of the final stage of logical reasoning according to Piaget, but Goswami points to research which showed this idea was flawed and that such reasoning was seen at an earlier age. Novick (2006), however**,** describes analogical transfer as a 'weak' method often used by novices as its success depends on 'domain specific knowledge' which only more skilled problem solvers are likely to have. Novick and Holyoak (1991) and Novick (1995) show that there is a crucial *adaptation* process that solvers must execute in order to modify the example problem's solution procedure. Novick writes:

> *A common complaint of educators is that students often know how to solve problems if they are told what procedure to use, but they have difficulty determining on their own the best procedure to apply. This is because their knowledge is inert - i.e., it does not include information about the conditions under which the procedure can be applied. (p. 2)*

In his preface to the first printing (1944), Pólya writes that he was driven to write the book by his own puzzlement of how mathematicians came to initially solve the problems described in lectures he attended. In his preface to the second edition he writes that he wishes to address the situation reported in a study done by the Educational Testing Service published in *Time* magazine (1956):

> *... mathematics has the dubious honour of being the least popular subject in the curriculum ... Future teachers pass through elementary schools learning to detest mathematics ... They return to elementary schools to teach a new generation to detest it. (p. ix)*

In his forward to Pólya's 'How to Solve it', John H. Conway writes:

> *Perhaps his most important point is that learning must be active. As he said in a lecture on teaching, 'Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics. And what does it mean to do mathematics? In the first place it means to be able to solve mathematical problems. (p. xx)*

Pólya wrote that the questions a teacher asks are very important to the process and that if the student is 'unable to do much' the teacher should ask questions unobtrusively so that the student is "*left with the illusion of independent work*". Pólya suggests that most people can only remember 3-4 steps of a plan and so must write it down and discuss it.

Mason, Burton and Stacey: Thinking Mathematically (1982)

The ideas expressed in this book formed a foundation for the approach used in this research. The book is a very rich source of material for teachers wishing to use problem-solving and problem-solving techniques in their classroom. The authors present a heuristic which is based on the processes of *specializing* and *generalizing.* The specializing process is built around the ideas of *entry* and *attack,* and the entry phase itself is further divided into *getting started* and *getting involved.* The authors note that "*Many people read a question once or twice and then expect to jump straight ahead to a final solution.* ***Rarely is that possible."*** *(p28)* Mason et al. **s**uggest that students start by asking themselves the key questions: *What do I know? What do I want? What can I introduce?* They recognize that there is more to getting started than first appears and that emotions can block recognition of a starting point (cf. Goldin, 2000). The state of mind of the student towards starting greatly affects whether or not there is movement from getting started to getting *involved* with the problem. Getting involved refers to the process of actually working on the problem rather than just writing down the information. The student must become engaged with the problem in order to move between these two stages and it is this step which is the basis of this research. Like any good beginning however the process is intricately tied up with the end. In this case the end is perceived by most people to be *the* answer. The goal of most students is to find *the* answer as quickly as possible, and if they cannot see a route to this end very quickly then anxiety starts to creep in. That a student should think this way is hardly a surprise given that most of mathematics is taught this way, so an important aspect of this research was in looking at ways to help shift that mindset within students.

Fundamental to Mason et al.'s process in developing problem-solving skills is that the student recognizes that being 'stuck' in a problem is a natural state of affairs. Being stuck is referred to as '*an honourable and positive state, from which much can be learned' (p. 49)*. They suggest a writing rubric where the student is encouraged to write down their ideas as they tackle a problem, using a page divided into two, and write down the word 'STUCK!' if they reach a barrier to their thinking. This, they suggest, helps to distance an incapacitating emotion from their actions and frees them to take an alternative approach. The two-column approach promoted by the authors, along with the metacognitive nature of expressing ideas, was used and adapted extensively in this research. The idea of emotion as a changeable local affect ties in with the ideas of DeBellis and Goldin (1997, 1999, 2000).

Leone Burton: Thinking Things Through (1984)

Leone Burton has researched extensively on the area of problem-solving and her work was used as a guide to setting up the classroom process. She writes:

> *The greatest value of this approach* (problem-solving) *is in the effect it has in the classroom. Hesitancy and dependency in the pupils is replaced by confidence and autonomy. Dislike of mathematics turns to enjoyment, indeed enthusiasm. Low self-images are replaced by expressions of authority* (p. 9).

Burton notes the keys to these desired changes as:

1. Challenging problems and challenged thinking. Burton notes the '*overwhelming importance'* is to foster a spirit of enquiry. She claims that '*problem-solving cannot be taught. It happens in an environment where skills have already been acquired and exercised.'* (p. 10) Burton stresses that the problems given should start from where the students and gives examples of many problems which require little in the way of mathematical pre-requisites.

2. A questioning approach. Burton makes the point that the spirit of enquiry is lost when students continually answer other peoples' questions and that this can make problem-solving a '*sterile activity'* (p. 10).
3. Reflection – in particular she specifies time to reflect on '*what was constructive, and what was not, in the attack on a problem*' (p. 17). Burton suggests that teachers take an active role in the class during problems solving, much as suggested by Schoenfeld (1982), with the aim being to ask reflective questions such as '*why do you think that?, 'What difference will that make? Is it always so? How did you arrive at that conclusion? Is there another way of doing that?* A key question the teacher must always be asking them self is '*Who is doing the thinking?'* (p. 20) In addition, the teacher needs to be aware that questioning does not happen automatically.

Burton writes that problem-solving behaviour is positively affected if the emphasis is on the process rather than the outcome: '*They need to feel free of pressure to finish' (p. 21).* She also points out that the students do not use mathematics to the level they have been taught when problem-solving, but instead adopt a simpler approach. This makes it important to provide opportunities for students to select and use knowledge from their experience and to develop that. In doing so, the student is shifting the focus of their attention from learning to using that learning.

Burton also stresses, in the choice of problem, that it '*introduces no additional burdens and does affect attitudes to problem-solving positively'* (p. 19). Note the emphasis she places on positive here, as opposed writing that it does not affect the student *negatively*. An important point Burton makes is that the students must develop good habits as problem solvers. Using diagrams, models, notations such as 'stuck', keeping records, being systematic, and communicating the process are all ways to develop such habits. Keeping a problem-solving journal is a good example of this, as are problem-solving boards and class folders. Burton also stresses that teachers must model the role of the problem solver and that the students must see them engaged in this process too. In terms of initializing problem-solving Burton uses the idea of the ‘Entry’ and ‘Attack’ and notes that ‘*A feature of the behaviour of experience problem solvers is their possession of a repertory of useful strategies, general and specific, which guide their problem-solving behaviour at any moment’* (p. 25). She has developed a series of questions and procedures to help the process, which I summarize below in Fig. 3.1, as they were presented to students in this research:

Table 3-1 Burton’s Entry procedures

Entry Phase		
Organising Questions	**Useful procedures**	**Procedures**
What does the problem tell me?	1,3,4,5	1. Explore the problem
What does the problem ask me?	1,2,3,4	2. Make and test guesses
What can I introduce to get started?	3,5,6,7	3. Define terms and relationships
		4. Extract Information
		5. Organise the information
		6. Introduce a representation
		7. Introduce a form of recording

These suggestions are a development of the problem-solving heuristics of Pólya but she does make useful linkages to help develop strategies. She also goes into each procedure in more detail. One procedure of interest is numbered 16 in table 3.2: ‘Upset Set’. This is similar to Perkins’ (2000) ideas around being stuck in a thinking rut where it is necessary to change your view of a problem in order to succeed. Students are very familiar with the idea of a memory returning ‘out of the blue’ when they have seemingly stopped thinking about something. (Hadamard, 1945; Poincaré, 1914) They are able to relate to this idea and this was something which was stressed throughout this research. Burton also suggests helping to prompt students in their thinking, and to develop their intuition, by asking students to complete the sentence “It feels as though, etc." so as to get at some of their ideas that they are not yet confident about or that they' can't quite articulate.

Table 3-2: Burton's Attack Procedures

Attack Phase	
Organising Questions	**Useful procedures**
Can I make connections?	1,2,7,19
Is there a result which will help?	1,3,4,5,6
Is there a pattern?	0,1,5,7,8,9,10,17
Can I discover how or why?	0,5,8,10,11
Can I break down the problem?	0,9,12,13,14
Can I change my view of the problem?	15,16,17,18
Procedures	
0.Be systematic	11.Work backwards
1.Search for a relationship	12.Focus on one aspect of the problem
2.Analyse relationships	13.Eliminate paths
3.Make simplifying assumptions	14.Partition the problem into cases
4.Find properties the answer will have	15.Reformulate the problem
5.Try particular cases	16.Upset set
6.Adjust guesses	17.Develop the recording system
7.Formulate and test hypothesis	18.Change the representation
8.Try related problems	19.Make a generalization
9.Control variables systematically	
10.Use one solution to find others	

Alan Schoenfeld

Schoenfeld also refined Pólya's ideas after research on the strategies students used to attempt problem-solving (Schoenfeld, 1982, 1994). Schoenfeld's work provides further insight into the area of initializing thinking and how to support the crucial 'entry' stage of a problem. Schoenfeld considers a good problem solver one who can solve a problem for which he does not have access to a solution schema. Schoenfeld made a collection of over one hundred videotapes of students working on unfamiliar problems and found roughly 60% of the students made a quick decision on the method of attack (to use Mason et al.'s terminology) and then pursued that method regardless of results. The charts below (as presented to students in this research) show the difference Schoenfeld found between a typical student (Fig.3-1) and a mathematician (Fig. 3.2).

Figure 3-1: Schoenfeld – Students thinking

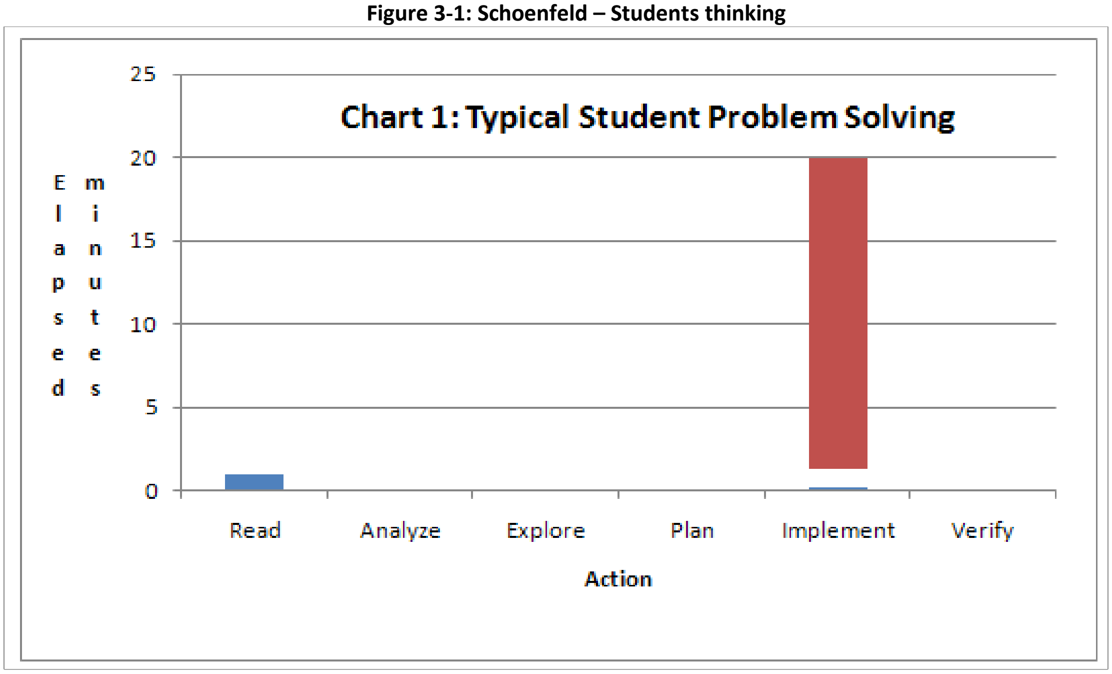
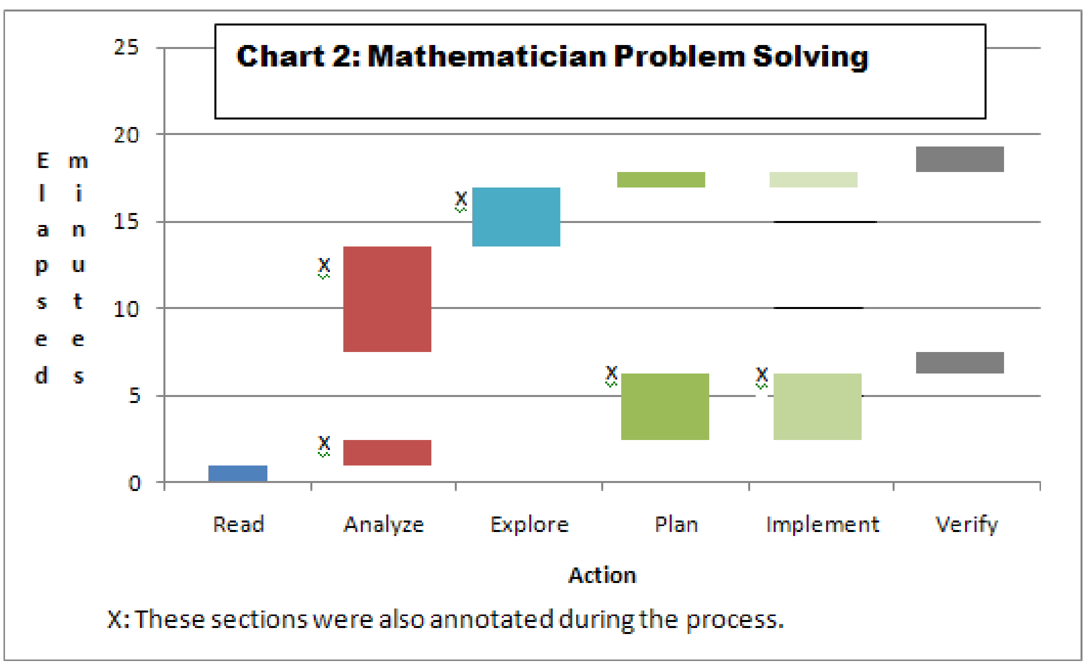


Figure 3-2: Schoenfeld – Mathematicians thinking



Perkins (2000) uses the analogy of being stuck in a *boxed canyon*, and of lingering at an *oasis of false promise* in his book *Archimedes Bathtub: The Art and Logic of Breakthrough*

Thinking. An 'x' in chart 2 represent an explicit comment by the mathematician on the state of the problem solution, similar to the approach suggested by Mason et al. that the student write down their thinking in their problem-solving rubric. In addition, the mathematicians wrote down a great many ideas that were not followed up on, which helped to avoid getting stuck in Perkins canyons and oases. This was a point stressed to students in this research as being an important part of the problem-solving process.

Schoenfeld divided his classes into groups of three or four and circulated the room, stopping to ask the following questions at any time: (p. 356)

- *What (exactly) are you doing? (Can you describe it precisely?)*
- *Why are you doing it? (How does it fit into the solution?)*
- *How does it help you? (What will you do with the outcome when you obtain it?)*

After explicit training Schoenfeld was able to show that the students were much better in approaching their problems, and adopted methodologies similar to the professional mathematicians. In particular, they were much better at the planning stage.

In **'***Reflections on Doing and Teaching Mathematics* (1994)', Schoenfeld looks at what is meant by *doing* mathematics is concerned about a list of misconceptions, or beliefs (c1f. McLeod, 1992) students have, even through university: (p. 53)

- Problems have only one right answer
- There is only one correct way to solve a problem – the teachers, as demonstrated
- Ordinary students can't be expected to understand mathematics; they expect simply to memorize it and apply what they have learned without understanding.
- Mathematics is a solitary activity.
- Students who understand mathematics will be able to solve any assigned problem in 5 minutes or less.
- The mathematics learned has little or nothing to do with the real world

He suggests that the roots of these misconceptions lie in the classroom experience. While doing mathematics is usually described as pushing the boundaries of knowledge in mathematics, Schoenfeld argues for the case that is should be extended to include pushing the boundaries for a particular group, even if the knowledge is already widespread outside of the group. I found these observations to be very fundamental to the way I wished to approach this research, and the above list became points to stress whenever I gave problems to the students, or discussed the mathematical process in general. In addition I tried to use the principles outlined in Schoenfeld's book to guide my choice of problems in attempting to stimulate student thinking.

Developing Thinking

Robert Fisher: Teaching Children to Think (1990)

To further support the notion of developing thinking skills in students and establishing a classroom atmosphere to support the process, I found the work of Robert Fisher instructive. Fisher writes from a more general point of view but includes a section on mathematical thinking, which he considers involves:

1. Creative thinking (Creating hypothesis, insight)
2. Critical thinking (logical chains of reasoning)
3. Problem-solving.

Fisher describes creative thinking as "*a collection of attitudes and abilities leading to creative thoughts; part of this is intuition or mental leaps*" (p. 31). He also asserts that creativity levels show little correlation to IQ scores and adds that children need the right conditions for creative thinking to flourish. He claims that a damaging change in the learning process can occur when a child is around 3-4 years old and stops guessing and inventing answers because his efforts are rejected. Instead, he asks 'What is that?' and learns that the answer is not what he thinks but what the adult thinks. Fisher observes that the paradox of creativity is that we need to be stimulated by the thinking of others in order to start thinking ourselves. This notion may be seen in the need to scaffold the start of a problem to prevent children getting bogged down because they are not clear about the task. It is also seen in the benefits of group work as discussed by Burton (1984). This was an important idea which was kept in mind as this research was carried out.

Fisher cites Carl Rogers (1961) observing that children require psychological safety and freedom in order to develop their thinking. Teachers must:

- Accept the students as individuals of unconditional worth and have faith in them.
- Avoid external evaluation and encourage self-evaluation.
- Empathise to see the world from students' eyes.

Fisher also lists some comments to encourage creative thinking when students are working, similar to the ideas of Pólya (1957) and Burton (1984) above:

- That's an interesting idea.
- Tell me about it.
- How did you reach that conclusion?
- Have you thought of alternatives?
- Try it yourself first, then if you need help ask.
- That's an imaginative idea.
- That's a good question.
- I'm sure you can get it right.

Problem-solving is described as applied thinking and is contrasted with creative (Divergent) thinking and critical (analytical) thinking, which are forms of investigative thinking and may entail thinking for its own sake, or be applied. Fisher considers that there is no better way to see if a child understands a process or body of knowledge that to see if he can use it to solve a problem. Fisher uses the analogy that working on common problems can be a way to "*get the ferries moving between the islands of experience, linking and extending the network of thinkin*g".

(p. 98) Like Burton, Fisher promotes the idea of children working in groups or pairs, and cites Doise and Mugny (1984) in finding that they produce more effective solutions than working alone.

Fisher notes that: “*Research shows that boosting confidence increases the ability to solve problems and prevents the students from giving up too soon, before the right and bright ideas come up. ”* (p. 113) and echoes the ideas of Burton (1984) and Pólya (1957) in suggesting strategies for supporting problem-solving:

- Describing with interest what the child is doing
- Asking the child what he is doing
- Supporting the process when needed
- Learn alongside
- Admit you don’t know and make mistakes
- Trust the child to make decisions
- Intervene only when appropriate
- Encourage collaboration
- Allow time to think things through
- Reward risk taking
- Accept a range of results
- Praise and motivate

All these are key ideas used to help establish a classroom which supports thinking and the student’s willingness to take risks in attempting to solve problems.

Fisher cites Chase and Simon (1973) in finding that one key element in successful problem solvers was that they knew more. He contends that memory is important in all thinking and is involved in every step of problem-solving. Half-remembered facts and misunderstandings are described as “*pernicious, for they produce the same satisfactions as understandings”* (p. 121). Fisher looks to metacognitive skills to help acquire and control knowledge and cites Kahney (1984) in writing that the presentation of a problem has a ‘*powerful effect on ability to* *relate to previous knowledge'. (p. 128)* Fisher adds that: "*In encouraging a child to think mathematically we must engage all aspects of a child's intelligence*" (p. 208) i.e. not just numbers and algebra but verbally and in writing, observing and discussing, asking questions, physically and visually. Mathematical thinking can also be stimulated by asking children to think up their own problems.

Another important aspect of the problem-solving process used in this research was based on the idea that success is rarely a continuous path but comes by returning to original ideas or stepping back to examine a fresh approach before persevering with a problem. Fisher cites L.L. White observing that "*success is born of Failure*" and includes the examples of the board game *Scrabble* and the *Dr. Seuss* books which were both originally rejected. Fisher also notes self-esteem to be an important factor in the promotion of a child's academic achievement and ability, citing the work of Bloom (1976), Purkey (1970), and Lawrence (1988). Purkey (1978) found that having High expectations of success from a student actually produces higher achievement.

Fisher also considers factors that hinder thinking, with one of the most pervasive being fear of failure. Students draw little comfort from past success, tending to regard it as accidental, and failure is regarded as conforming incompetence. Gordon (1961) is quoted as writing: "*All problems present themselves to the mind as threats of failure*." Fisher cites Nisbet and Shucksmith (1986) in finding that a key problem with children with learning disabilities is that they fail to plan. This also echoes the work of Schoenfeld discussed above. Fisher also considers an overload of concepts, such as having to worry about handwriting, spelling, presentation etc., in a problem can impair thinking. Ambiguity can also be stressful, where the child frets over the question "*What am I supposed to do?*" Fisher writes that children respond to anxiety with various coping mechanisms:

- Avoidance
- Blame others
- Denying reality
- Insulting Self (I don’t care)

This is similar to DeBellis and Goldin (1997) referral to *defence mechanisms*.

An Analysis of Research in Establishing a Pattern of Thinking in the Classroom

The above authors formed the basis for developing the research philosophy and methodology used in this research. It is based upon challenging thinking, questioning and reflection, in which metacognition plays a large part. Each promotes the idea of placing an emphasis on questioning the student as they work on problem-solving, and helping them to develop strategies to both make an entry into a problem and to persevere when stuck. The sources examined below outline research on two particular areas of interest: the heuristics and suggestions to improve problem-solving techniques, and the conditions required to stimulate students into adopting these techniques. In addition, I also examine the notion of ‘learned helplessness’ further. The ideas around learned helplessness were something I felt important to the development of this research.

My contention is that in order to foster good mathematical thinking in a classroom the teacher must be able to do much more than present problem-solving heuristics. Such heuristics, as presented in most school mathematics textbooks, are usually adaptations of Pólya’s work and there is little doubt that these heuristics *describe* well the process used to solve problems. However, while often recited by teachers and students alike, such heuristics are often viewed as an inert set of procedures (Novick, 2006) which rarely help the average student engage in a problem. The presentation of a problem can be overwhelming to a student and give rise to a level of anxiety which often prevents the student from even trying to start a solution. Tobias (1993) clearly sets out a case for the paralyzing nature of such anxieties, which stem from the typical practices of mathematics classrooms, and looks to the movement towards the ‘standards’ based teaching as rectifying the source of some of these anxieties.

Lester et al. (1989) further investigated the role of metacognition in problem-solving and found that teacher modeling of the process was a key component of the process. Lester found that developing the necessary skills involved behaviour modification in the ‘unlearning’ of behaviours developed through prior instruction. These changes can be made but require a long period of time with careful focus on both the cognitive and metacognitive process.

Thomas Romberg (1994) points out that ‘*Mathematics is not a fixed set of concepts and skills to be mastered, but an empirical science. If students work on such problem situations, such work will enhance learning; and second will connect ideas and procedures among different branches of mathematics*.’ Romberg notes “*that such connections in memory are organized in Schema. These develop over long periods in time and by continual related contextual events (p. 294)”*. This was an important statement for me in the recognition that the goal I was attempting would not be something that occurred all of a sudden, but rather would be the initial (or in some student’s case a continuing of) stages of a much longer process. It was important to keep this in mind when students expressed reluctance to initially buy into a particular process. Overall, the foundation of this research was to establish the expectation by students that when they came to class they were expected to think about what they were doing, and not be ‘spectators’ to the process.

Promoting Thinking

In *Principles and Standards for School Mathematics (NCTM, 2000)*, an emphasis is placed on problem-solving as a means to ‘build new knowledge’. This document has become a foundation for the recent curricular changes, but as Nicole Rigelman (2007) points out the questions posed by the teacher in the classroom need to promote deep student thinking. Questions such as

“What is the question asking us to do” tend to funnel towards a particular solution while questions such as “Can anyone see it in a different way?” show students that the teacher values the process as well as the product. Rigelman’s article suggests that the way you orient your classroom towards better thinking can be quite subtle in the form of how you phrase your questioning, as well as more overt in simply doing more problem-solving. Students need to be encouraged to build on ideas, both theirs and others, and to consider multiple ideas and methods.

Judith Fraivillig (1999) cites research by Cobb and his colleagues which emphasize the need for teachers to establish the social norms that develop mathematical thinking. Fraivillig argues that, while it is important for teachers to build thinking practices into their classrooms, teachers need to intervene to advance children’s thinking, rather than merely elicit such thinking. Fraivillig’s study is instructive in noting that an important part of a teacher’s development is the ability to leave behind traditional ways of teaching in favour of the problem-solving approach. In essence, the research suggests that guiding students to a particular method of solving a problem is a typical (didactic) form of teaching, but in order to really foster mathematical thinking our intervention should be to stimulate the ideas of the student rather than a particular solution to the problem. The answer is not the focus of the problem, but rather the student’s personal route in attempting it, and what they learn from that route.

Sorenson, Buckmaster, Francis, and Knauf, (1996) refer to Krulik and Rudnick (1993) to specify the characteristics of a good problem solver as having: *the desire to solve problems*, *being willing to take risks*, *not being afraid to make educated guesses*, and *being willing to skip steps*. Meiring , in Hatfield, (1993) adds *the ability to note likenesses and differences and discard irrelevant detail, identifying critical elements, evaluating alternative routes, estimating, generalizing, learning from mistakes, having less anxiety, understanding, concepts,* and *transferring learning*. Hatfield (1993) specifies *metacognition, planning*, and *breaking the problem down to smaller units*. Good teachers of problem-solving are deemed to *have confidence, be enthusiastic*, and *have a positive attitude towards the process*. They should act as a model for the students and think with them. They should not focus on the answer but on the different routes to get there. Meiring (1993) suggests:

- Teach strategies and planning, and help select appropriate approaches.
- Have students work in groups to share ideas.
- Encourage various forms of visualization.
- Present varied problems.
- Allow students enough time to solve problems and reflect on their solutions.

Each of these readings helped support the notion that a ‘thinking’ classroom needs to be one where the teacher is active in supporting the process and cannot simply start the process and stand back. Modelling and encouraging are as important as providing a rich source of material. In addition to these practices however, it must be recognised that a student can come to a new class with a set of barriers which can block thinking. One of these is the concept of learned helplessness.

Learned Helplessness

“*Learned helplessness is a psychological condition in which a human being or an animal has learned to act or behave helpless in a particular situation, even when it has the power to change its unpleasant or even harmful circumstance.”* (Seligman, 1975). Pourdavood et.al (2005) cite McLeod and Ortega (1993) as defining **learned helplessness** in the mathematics education context as "*a pattern of behavior whereby students attribute failure to lack of ability*". They link the notion of learned helplessness to the persistence a student demonstrates in problem-solving. McLeod and Ortega found a student's self-concept could be modified by the way the teacher interacts with the classroom and the general classroom atmosphere, what they refer to as the ‘social context’. In particular they describe how *‘Classroom conversation, such as a teacher's characterization of a problem as "easy" can profoundly demoralize students.’* McLeod and Ortega use the term *mastery orientation* to indicate where students have confidence in their ability to solve challenging problems. Wieschenberg (1994) offers the use of the term ‘Conditioned Helplessness’, reasoning that the term ‘learned helplessness’ is specific to a situation where the individual has given up due to punishment regardless of effort. The concept of ‘conditioned helplessness’ should apply to cases where the individual has been overly supported and thus has never developed independence. These forms of helplessness can be manifested in the behaviour, noted by Fisher (1990) above, in which students cover anxiety with self-blame. Frequently, as Wieschenberg notes, a comment heard from students, parents, and fellow teachers is “I was never very good at mathematics.” Wieschenberg points out the damage that this casually made statement can have on students, by validating the emotion.

Wieschenberg cites Weiner’s Attribution Theory (1973) which examines how the behaviour of people is determined to how they attribute their success and failure. Wieschenberg points to Weiner’s ideas of a "schedule of reinforcement" in the environment and quotes Weiner as writing "*Individuals highly motivated to achieve success assume personal responsibility for success and attribute failure to lack of effort. Persons low in achievement needs do not take credit for success and ascribe failure to a lack of ability*" (Weiner, 1973). Wieschenberg offers steps to *motivate the students and change pessimistic explanatory styles to optimistic ones.* These steps include encouraging group learning, using brain teasers to create interest, and offering different paths to solve problems. In particular Wieschenburg contends that there should be a focus on helping students understand the process of problem-solving and of learning concepts if progress is to be made in this area.

Summary

The articles cited above, along with the established research on learned – or conditioned – helplessness, point to the type of approach needed to stimulate thinking in the classroom. A focus on process, teaching strategies, and encouraging students to develop their own thinking process through metacognition, questioning and reflection is required. In addition, however, the actions of the teacher in the classroom are also highly significant in the process. Enthusiasm is essential so that the students do not see the subject as ‘dry’, but the manner in which the teacher supports the learning is also of great importance. The research suggests an approach where the teacher is part of the solution process and not a separate entity. Even the language (both verbal and physical) used by the teacher can affect the emotional state of the students and the level of their anxiety over the process. Jackson and Leffingwell (1999) suggest that covert behaviours, veiled or implied, can have the same negative effect as overt behaviours and instructors need to be very aware of the impact they have on their students.

The questions prompted by this research are then ‘how do the students respond to these ideas, and how do they typically respond when given genuine mathematical problems?’ Schoenfeld provides research on how students’ processes can be shaped by the instructor’s awareness of this process. Pólya, Mason et al., and Burton offer concrete suggestions to help students in this area, while Fisher provides additional insights into the nature of students’ thinking. Research ideas covered in the last section help to cement the ideas and offer further suggestions to help promote thinking through classroom practice. This thesis is intended to help examine some of the gaps between the heuristics and classroom practice. In particular, the question of why students fail to initialize their thinking when posed a question leads to an exploration into the area of ‘non-start’ thinking. The question of why so many students give up very quickly after they start is examined as the area of ‘quick-stop’ behaviour. Finally, I ask the question of how these two behaviours can be alleviated.

Chapter 4. Methodology

"A good teacher does not teach facts; he or she teaches enthusiasm, open-mindedness and values. Young people need encouragement. Left to themselves they may not know how to decide what is worthwhile. They may drop an original idea because they thought someone else must have thought of it already. Students need to be taught to believe in themselves and not give up."

Gian-Carlo Rota (*1985*) (p. 96)

Data Collection

There were four co-educational classes involved in this study; two grade 10 classes of mixed ability, who were preparing for a compulsory provincial standardized exam; a grade 11 class of mixed ability; and an elective grade 12 class preparing for an optional provincial standardized exam. In this study the grade 11 and 12 classes were taught on a semester system (one class every day September to January), while the grade 10 classes were taught on a linear system (one class every other day September to June) In this way I was able to be flexible in looking at the way students were initializing their thinking at the end of the first semester and adapt to new ideas which emerged from the data during the remainder of the year. My aim was to develop a theory in response to the research questions at the end of the first semester, which I could then test further into the second semester.

The initial step was to collect data to examine the local affective emotions and the beliefs in place around problem-solving. Two anonymous surveys and a diagnostic test were given at the start of the year in order to establish a baseline for further study. This first of these, the Mathematics Anxiety Rating Scale (MARS-R) was used to give a sense of the level of anxiety of the students as they started the year. The second survey, the Mathematics Related Beliefs Questionnaire (MRBQ), was given to gain a sense of the students' beliefs about mathematics, with a view to assessing how this might affect their learning. Both surveys were given anonymously and both were modified slightly from their original, as indicated below. The aim was to repeat the surveys at the end of the year for comparison. The grade 10 and 11 students were then given a diagnostic test developed by the Vancouver Island School Board (VIDMA) in order to give an indication of students' abilities to match their anxieties and beliefs. In this case I wanted to gather a sense of how any information which stood out from the surveys matched the achievement level of the student. The diagnostic test was given without prior warning and it was made clear that it was a diagnostic and not part of their grade.

The students were required to keep two journals which were also used as part of the data collection process. The first was a problem-solving-journal in which students were asked to solve problems assigned in class or for homework. The aim was to monitor the students' progress in expressing their ideas and developing their thinking. Students were asked to pay particular attention to recording their thoughts at four specific moments: a) as they were first assigned the problem - in particular, how they thought they might solve the assigned problem; b) if they became stuck in a problem and how they thought they might become unstuck; c) If they solicited help from a classmate; d) at the end of the problem where they were asked to reflect on their learning. In addition, they were encouraged to write down their thoughts at any time they had an idea concerning the problem. Feedback from the journal was used to modify the process as the course progressed. The second journal was for reflections based on prompts given in class or as part of the homework assignments. Only a few of the assigned prompts had relevance to this line of research, but these were focused on looking for changing emotions around initializing a solution, and beliefs about the nature of mathematics. It was stressed that assessment of the journals would be based on their engagement with the questions or prompts rather than a specific correct answer.

Data were also collected from problems assigned as part of unit tests and exams which were assessed using a rubric favouring the process rather than the result. The aim here was to examine how the students used any of the suggested strategies for problem-solving in a test situation, and to look for any differences of approach compared to how they approached more typical test questions.

In addition to these methods of data collection I circulated the room with a clipboard while the class was working in groups and recorded conversations related to their thinking process. I tried to do this discretely without a group's awareness. At times I would join a group and ask alternate questions if I felt the progress on the problem was off-track. In these cases I tried to accurately record relevant extracts immediately afterwards. I avoided relying on memory and did not record data if I was unable to do so immediately. As the course progressed I tried to focus attention on certain members of the class who were demonstrating non-start or quick-stop behaviour, in an attempt to monitor their progress. None of this data were consciously used to grade the class in any way. I also recorded field notes during and after individual tutorials with individual or groups of students. These sessions were often held at lunch or after school and gave a more direct opportunity to study the habits of students who were having difficulties with the material

The Surveys: Establishing a baseline for Anxiety and Beliefs

The Mathematics Anxiety Rating Scale

Suinn and Winston (2003) indicate that high scores on mathematics anxiety were significantly negatively correlated with performance in school. The purpose of administering this survey therefore was to gain a sense of the students' level of anxiety as an indicator of how they might be expected to respond when asked to think mathematically. Is a high level of anxiety a predicate of non-start thinking (where the student does not start to think about the problem), or quick-stop thinking (where the student makes a start but stops at the first sign of difficulty)?

Hopko and Suinn and Winston (2003) provide reliability and validity information to indicate that the shortened version of the original MARS-R gives the same information as the original 98 item inventory from 1972. I chose to use this shorter version of the original MARS-R by Hopko (2003) as I felt the length of the full version might be off-putting to students and be more appropriate for individuals of concern rather than a whole class survey. I also included some of my own questions specific to this area of interest (see Appendix 1 for the full survey).The questions were scored using the original MARS-R scale of 1-5, with 5 being most anxious:

When I am … I feel ANXIOUS	Not at all	A little	A fair amount	Much	Very much

Analyzing the MARS-R data

The data collected from the MARS-R was collated for individual students and collectively for the class as a mean score for each question. The aim was to look at both a trend for each class and to highlight individuals exhibiting high levels of anxiety. In terms of the class mean it is recognised that polar views in response to a question can balance out, meaning that the mean may not be an indication of a homogeneous viewpoint, but a mean skewed towards either end of the scale may be of interest. The data were examined for occurrences of either situations and highlighted in the results. Individual scores were then divided into ranges and graphed to compare the anxiety level of students within the class. The aim here was to give a sense of what proportion of the class fell into each of the anxiety levels. Finally, the results from the two students who indicate the highest levels of anxiety are compared with the aim to indicate specific areas of anxiety.

The Mathematics-Related Beliefs Questionnaire

Research into the affective domain indicates that students with a more positive attitude towards mathematics are more likely to engage in class and persist in challenging situations. (Hannula 2002 and 2006; Malmivuori 2001 and 2006) I chose to modify and use a version of the Mathematics-Related Beliefs Questionnaire (see Appendix 2) developed by Op ’t Eynde and De Corte (2003). The authors make use of the work of Schoenfeld (1983) in indicating that the cognitive actions of students often result from subconsciously held beliefs about mathematics and the students’ perceptions of their own abilities. As such I felt this questionnaire could be the basis to gather information about the students’ beliefs. I wanted to include specific questions which would elicit responses about the initial stage of problem-solving however, and I incorporated these questions in the larger questionnaire so that they would not be an obvious focus. As in the original survey the same question is asked in different ways as a check on the responses. The revised questionnaire uses a 5 –point Likert scale:

Strongly agree	Some-what agree	Neither agree or disagree	Some-what disagree	Strongly disagree
1	2	3	4	5

Analyzing the MRBQ data

The responses to questions relating to the specific area of interest to this research, initializing thinking and problem-solving, were pulled from the full survey and analyzed by collating the number of students who responded to each level on the Likert scale. In addition, a class mean for each question was calculated to give an indication of trend in a particular area. At the end of the course the questionnaire was repeated and the results compared to examine any shift in attitudes by the student. The final questionnaire was supplemented with additional questions in order to examine changes in beliefs to problems solving ability in a more specific way. (Hannula, 2004)

VIDMA

I also administered a diagnostics test to the grade 10 and 11 class (**V**ancouver **I**sland **D**iagnostic **Ma**thematics **A**ssessment, 2008). The aim of this test was to give a general sense of how much the students had remembered at the start of the semester. Research indicates that a key element in successful problem-solver is the prior knowledge of the student (Chase and Simon, 1973, Mayer 1982, Schoenfeld, 1982) so the test was administered with the students having no preparation or warning prior to writing. I considered it useful to have a sense of how much is readily retained on a year-to year basis. The students wrote the test using the same pseudonym as the surveys and were told that the test was for purely diagnostic purpose. Following the test the students were asked to make a journal entry discussing the questions they did not know. They were asked to break down wrong answers in terms of 'mistakes', 'forgotten facts', and 'lack of understanding'. (This process was repeated for all unit tests during the semester and these were graded for an honest engagement with the task.)

Analyzing the VIDMA data

The results were examined both as class averages and also for individuals. The main objective was to look for specific areas of weakness as a class or individual so that attempts could be made to address these concerns. The results were also compared against the MARS-R and MRBQ for individuals and the class in general to see if there was a clear correlation between anxiety, beliefs, and performance at the start of the year.

Data Collection through assigned problems

The initial process was to explain to the class that a focus of the year was to develop their thinking skills and that their engagement in the thinking process would be a primary goal. Over the course of several lessons it was made clear to the students that the final answer in a problem was not the main target, but rather the methods of achieving an answer, and an understanding of the underlying mathematics, were the key point. Every attempt was made to reinforce the idea that starting out on a solution was an important step to finishing it. The group process was discussed and practiced so that students became comfortable with talking to others to share and build upon ideas. Students were encouraged to write down every idea they had, not just ones that led to a solution. Listening to other people's ideas and seeing alternative points of view was emphasized. Initially the class was given problems where the intention was to write down the first steps towards a solution and no answer was expected. Questions were given on the board, audibly, and on paper with the sole instruction being to write down the first thing they thought of when presented with the problem. The typical response was analysed and discussed with the class. At this stage the process was an attempt to get past the students' fear of giving a wrong answer. My objective was to make them feel comfortable in offering their thoughts and exploring from where these thoughts came.

Problems done in class and for homework were examined for trends, both by the classes as a whole and by individuals. A coding system was used to determine how students were writing their thoughts and ideas into their problem-solving journals. After examining several problem-solving attempts at the beginning of the year I decided to code the students' notes by what seemed clear demarcations in their approaches. The data were first divided into three sections corresponding to the entry phase, the attack phase, and the reflection phase as described by Mason et al. (1982). The entry phase was then subdivided to look for the expression of ideas, whether the student simply described a process, or neither of these – the ‘non-start’ behaviour. An early example of a student writing down their initial thinking is shown in Fig. 4-1. The student then went on to solve the problem using the method outlined.

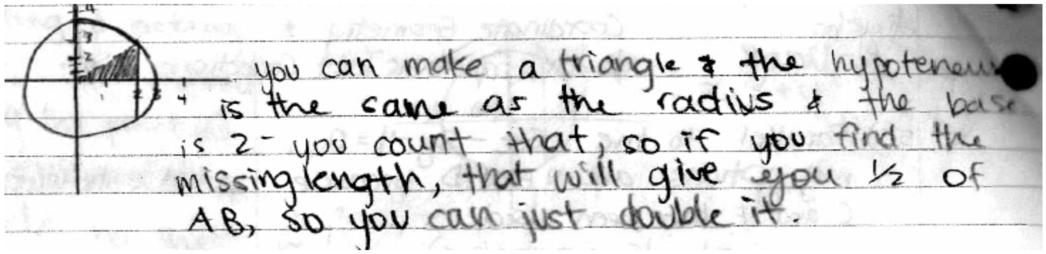

you can make a triangle & the hypotenuse is the same as the radius & the base is 2 - you count that, so if you find the missing length, that will give you ½ of AB, so you can just double it.

Figure 4-1: An early example of student thinking

The attack phase was subdivided into two further sections; whether the student became stuck, and if they subsequently became unstuck or gave up – potentially the ‘quick-stop’ behaviour. Given the nature of homework assignments it was impossible to determine how long the students spent being ‘stuck’ before either giving up or finding a solution, although evidence of the latter could be implied by the work shown. However, the reluctance of students to write down anything which is incorrect, or not to erase incorrect ideas, was very evident and difficult to overcome – especially at the start of the year. For this reason the written solutions were not a successful way to demonstrate quick-stop behaviour trends. The reflection stage of the problem was subdivided into three categories indicating a good, weak, or no reflection. A good reflection was determined to be one in which the student analyzed their solution – either correct or not – and commented on the possible alternate ways to find a solution or why they had found difficulties.

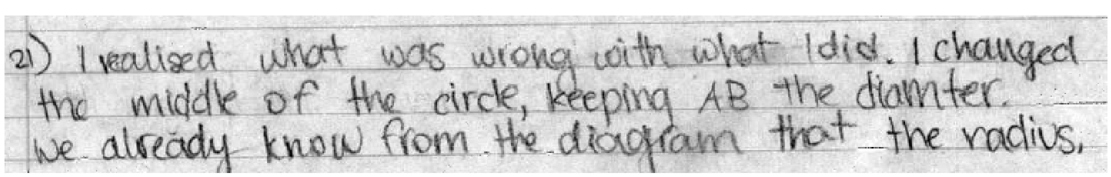

21) I realised what was wrong with what I did. I changed the middle of the circle, keeping AB the diamter. We already know from the diagram that the radius,

Figure 4-2: A reflection example

As an example, in the extract shown in Fig. 4-2 the student began their reflection by demonstrating an understanding of what they did wrong, and then went on to provide a solution using a different approach. A weak reflection was one which did not add much to the learning situation, such as merely commenting on the process. The aim of coding these problems in this way was to look for a trend in the nature of the students' initial response to problem-solving and being stuck in the problem. The problems used for particular reference in the final analysis were those given in tests and exams. In this way the progress of individual students could be determined without input from their peers. A significant change in this area would be a strong indication that progress had been made. The anecdotal comments recorded from field notes as indicated above were the primary source for analysing how the students' thinking was developing on a day to day basis. Due to the nature of the thinking process, and the limitations of the written word in this area, making notes on the verbal discussion conducted within group settings was a rich source of material in giving an insight to the students' thinking. In the same way, field notes made during and after individual tutorial sessions gave valuable information on individual cases and were used to highlight the observable trends. The students' reflection journals were also used to provide support for observable trends and snippets from these were either added directly if the journal was typed, or copied and added if hand-written. In all cases an attempt was made to select examples which highlighted a class trend, or a specific individual who was having difficulties in the areas of non-start or quick-stop behaviour (or was making significant progress in these areas).

Organization of the class: Establishing the classroom atmosphere

In this section I outline the methodology used to try and change the local affective goal s leading to a changing state of emotion around initializing problem solving (Goldin, 2000). The interventions used were aimed to set an appropriate classroom atmosphere that would encourage thinking and positively enhance the emotions students might have around problem-solving. (DeBellis and Goldin, 1997, 1999) The aim was to emphasize to students that thinking about the process of problem-solving was as important as getting the answer. (Burton, 194) It was important to establish that questioning and discussing options should be valued and thus would be evaluated. Reflecting on the process to examine why things were done and to examine alternative approaches was also emphasized. Initially the students, placed in groups, were given problems and only asked to discuss how they would go about solving them, without actually doing so. Discussion between groups was encouraged. They were also asked to consider how they would know if their answer was correct. As an instructor I moved around the groups and listened to the discussions, but did not offer advice at this time. I tried to emphasize that any ideas they had should be noted and discussed.

Stories of mathematicians, such as Wiles, Napier, and Boucherds, who spent years searching for answers to problems were used to show how and problem solving was often perseverance rather than inspired thought, or, that what appears as inspiration was often the opening of the mind to new ideas. The thought process was constantly being emphasized with questions such as "*what is the question asking us to do*?" (Burton, 1984, Fisher, 1990) Studies on the way mathematicians approach a problem (Schoenfeld, 1984) were shown to the students to help change their beliefs on the problem-solving process. '*Even if you cannot find the solution, outline your thinking*' was constantly repeated to the class in an attempt to move them away from the idea that only solutions had value.

Students were then introduced to the problem-solving rubric of Mason, Stacey, and Burton (1985). Students were asked to divide their pages into two columns covering 1/3 and 2/3 of the page, and use the smaller column show their *thinking* process behind the methods used in the larger column. The use of the terms STUCK! and AHA! were introduced and discussed. The importance of showing thinking, not describing method, was stressed and a reflection on the solution was included as part of the rubric. Students were given several problems and told that, while they should try as many as possible, they would only be formally required to write up three by the end of the unit. Each unit of the course was seen as a breakpoint to assess progress in using the rubric, while at the same time making sure that the process was not affecting the grades of the students in any adverse way. The 'What-if' strategy (Brown and Walter, 1990) was a useful guide to extending a problem for a student or group who found the initial problem quickly solvable. It also allowed me to give suggestions to students on how to think beyond a problem and so be 'like a mathematician'. I emphasized four steps to the students when talking about actual mathematicians and their problems (Hadamard, 1945), namely: *Preparation – Incubation –Illumination –Verification*.

Prior to the end of each unit the students were also given a Journal Entry to write. This allowed a simple questionnaire about the process to be given in order to access how it was working and students' response to it. The journal also contained a problem to be solved using the rubric. This gave a common problem to each student. The obvious question about students 'copying' solutions from each other was considered moot, as this will happen regardless of the precautions taken, other than in a test setting. Instead it was decided to try and incorporate this as part of the process by encouraging students who were stuck to ask their fellow students for clues. Any advice given should then be incorporated into the rubric. For example: '*STUCK! – could not think how to proceed; AHA! Albert gave me a hint* '. The students were told to keep all of their work, quizzes, tests, journals, problems, homework, and in-class assignments, and to reflect on their mistakes. At the end of the course they would be asked to complete a reflection portfolio. Incorporated into each unit test would be a question at the end which required them to use the problem-solving rubric. This question would not be for a significant mark, but would be enough to value the thinking process. The following guide was used to mark the test question, and was displayed on the test paper (adapted from the VIDMA diagnostic test.:

0. No thinking demonstrated
1. Some thinking is demonstrated, but it is not methodical or structured
2. The thought process is clear and may lead to a solution
3. There is clear evidence of thinking which would or does lead to the solution

Problem sets were primarily selected from Mason at al (1982) and Burton's (1984) books, or sourced from the Internet[2]. The style of the questions were selected to allow students easy access, but leaned towards an open ended or non-obvious result.

In addition to problem sets the methodology of teaching the class was also gradually shifted from the normalcy they expected to one which it was hoped would foster an independence of thinking. (Fraivillig, 1999)There was less of an attempt to feed students methods and more to encourage them to think of solutions for themselves, look them up, or solicit them from friends. At the same time I was very careful to watch that no student was becoming frustrated with the process. I tried to intervene to advance children's thinking, rather than merely elicit such thinking, and to be aware of the need to intercede if a student was moving off track. I did not want students to get stuck in the typical traps, such as the 'Oasis of false promise', or the 'Narrow canyon of exploration' (Perkins, 2000) whereby they would spend a long time getting nowhere, but I also did not want to be take away the nature of the problem or the experience of the struggle. At every turn there was an emphasis on creating a classroom atmosphere of acceptance of mistakes, their recognition, and of learning from them. That this was a 'thinking classroom' was as important a philosophy as it being a mathematics classroom.

Looking for emerging themes

The collected data were continuously examined to look for common themes which began to emerge. Observations were made in an attempt to demonstrate the themes further, and

[2] http://www.stfx.ca/special/mathproblems/ proved to be a very useful site.

problem selection and the methods used to ask students to record their thinking was adapted. In accordance with this I present the treatment for the data collected for each grade separately in chapter 5 (grade 11), chapter 6 (grade 12), and chapter 7 (grade 10), before looking at common themes across all three grades in subsequent chapters. This approach can be described as going with the grain and across the grain (Watson, 2000). Examples from students work, both anecdotal and extracted from reflection guides, are given to support the emergent themes in each case. The grades are presented in the order given because I felt the grade 11 class were a good representation of a mixed ability class who were not under the pressure of a standardized exam, and as such would indicate trends of non-start and quick-stop behaviour more immediately. As an elective class the grade 12's would make a good comparison following a discussion of the results from the grade 11 class. The grade 10 classes are discussed last because they were on a linear program as opposed to the semester timetable of the 11's and 12's. In this way it was possible to make an analysis of the emergent themes at the end of the first semester and then look at these themes with closer attention in the second semester.

Chapter 5. Grade 11

"To Thales the primary question was not what do we know, but how do we know it."

Aristotle*, Mathematical Intelligencer* v. 6, no. 3, 1984.

In this chapter I present a general overview of the data collected for the grade 11 class in order to identify the trends associated with this group. I present in detail an analysis of the data from the surveys administered, and a breakdown of the problem-solving questions done in test situations at the beginning, middle, and end of the course. I provide descriptive accounts of the data collected for general classroom problem-solving sessions, and tutorials. I will present a more detailed examination of the latter data in chapter 7, when comparing data collected across the grades to examine emergent themes.

The Mathematics Anxiety Rating Scale (MARS-R)

The survey consisted of 15 questions selected using the following 5 - point scale:

When I am … I feel ANXIOUS	Not at all	A little	A fair amount	Much	Very much
	1	2	3	4	5

The response for individual questions, as a mean for all students, is shown in table 5.1.

Table 5-1: G11 MARS-R: Average response for each question (n = 11)

Q#	1	2	3	*4*	5	6	7	*8*	9	10	*11*	*12*	*13*	14	15
Mean	2.2	2.0	2.1	*3.1*	1.8	1.9	2.0	*3.9*	1.8	2.0	*3.0*	*3.3*	*3.1*	2.0	2.0
	Looking at text	Coming into class	Read a formula	*Think about Test*	Watching teacher	Algebra explained	Start hw	*Writing Test*	Reading a graph	Start a problem	*Difficult Hw*	*Test returned*	*Stuck*	Start new chapter	Listen to student

The aim is to give a sense of the anxiety level for the whole class while highlighting particular areas of concern (See Appendix 1 for a copy of the full survey}. The highlighted boxes indicate higher levels of anxiety. While high and low scores can balance out, a mean skewed towards either end of the scale could be significant. In this case the class mean was 2.4, which is not a significant result to suggest overall class anxiety levels.

Fig. 5-1 shows the total scores for each student, grouped in ranges of 5, in order to give a sense of anxiety levels within the class. The maximum rating is 75 so a rating of 45 gives an average response of 3 - a *fair amount* of anxiety. A score of less than 30 would average 2 – *a little anxiety*. Two students can be said to show a fair amount of anxiety.

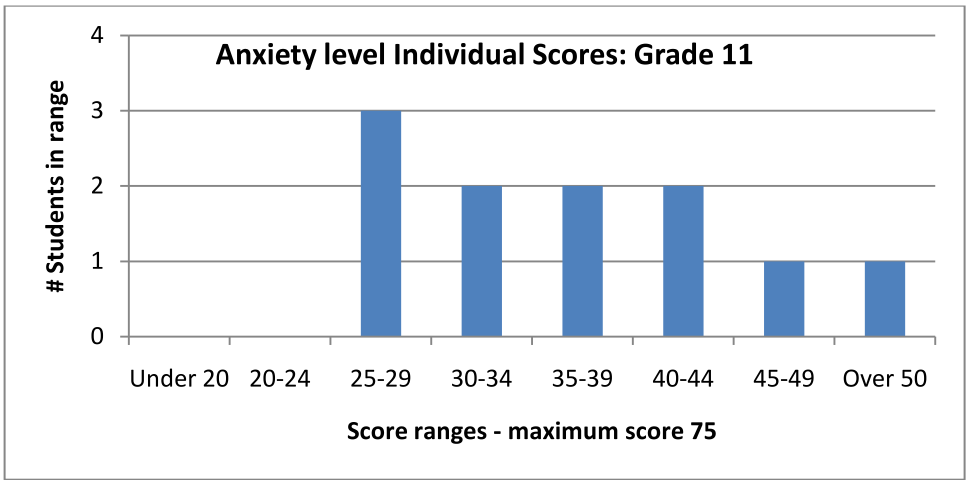


Figure 5-1: G11 MARS-R: Total scores for each student. (n = 11)

To give a sense of where these two students feel most anxious it is instructive to look at their individual breakdowns, which are shown in table 5.2:

Table 5-2: G11 MARS-R: Areas of anxiety for the two students scoring highest on the MARS-R

Question #	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	Sum
Moon Unit	4	5	3	4	3	5	3	5	2	3	4	2	5	3	3	53
Renesme	4	2	3	3	3	2	3	3	1	4	5	3	4	2	3	45
Note: A score of 5 represent s 'very much anxiety , 3 is a 'fair amount'	Looking at text	Coming into class	Read a formula	*Think about Test*	Watching teacher	Algebra explained	Start hw	*Writing Test*	Reading a graph	Start a problem	*Difficult Hw*	*Test returned*	*Stuck*	Start new chapter	Listen to student	

The difference between these two students may suggest their anxiety comes from different sources, with the second student being more concerned about difficult concepts as compared to a more general anxiety level from the first student. This table can also be compared to the summary of their beliefs around mathematics, shown in table 5.3 in the next section.

The areas where students reported feeling anxious concentrated around test taking, as might be expected, but of interest was that students noted a 'fair amount' of anxiety when they were stuck in a problem. They did not indicate the same level of anxiety when starting the problem. At this stage of the year the students' notion of a problem may not have been the same as mine. Over the past two years teachers in the school have been encouraged to do more problem-solving with students, but students may still interpret a problem as any question given in a mathematics class. Only one student reported feeling 'much anxiety' on starting a problem, but this is of interest as it was one of the top achieving students in the class (as correlated with the diagnostic test using the same pseudonym). Being stuck elicited one 5 and three 4's (a combined 36% of all students); waiting for a test elicited two 5's and four 4's (a combined 55%); taking a test elicited four 5's and two 4's (a combined 55%); and thinking about a test elicited no 5's and four 4's (a combined 36%).

Mathematics Related Beliefs Questionnaire

The Beliefs survey showed a mixed response to the questions related to starting problems. Over 70% felt they started problems confidently, but 70% also reported being unsure *how* to start a problem. This discrepancy may indicate that, while they start confidently and are able to tackle most questions, if the question is problematic they are then unsure of what to do. Given that the majority of questions they tackle are not what might be considered problematic this may explain their initial confidence.

Examining the responses from the student, 'Moon Unit', who indicated most anxiety in the *MARS-R*, shows a student entering the course with low expectations of success.

I believe I will receive a good grade this year	– strongly disagree
To me, mathematics is an important subject	– strongly disagree

I'm not the type to do well in math – somewhat agree
I have not worked very hard in mathematics – strongly agree
Mathematics is a mechanical \boring subject – strongly agree
I start problems confidently - strongly disagree
I am often unsure of how to start a problem - strongly agree
When I get stuck I tend to give up quickly - agree
When I get stuck I have good strategies - disagree

To give an indication of class beliefs around initializing thinking and problem-solving I extracted those questions to table 5.3, and included the number of student responses.

Table 5-3: Student responses within selected MARS-R questions.

Please respond to each statement	Strongly agree	Some-what agree	Neither agree or disagree	Some-what disagree	Strongly disagree	*Mean response*
Solving a mathematics problem is demanding and requires thinking, even from smart students.	6	3	1	0	1	*1.8*
I start problems and questions confidently	2	6	1	1	1	*2.4*
I am often unsure how to start problems assigned in class.	3	3	2	2	1	*2.5*
I like to know the answer to a problem before I start so that I can see where to go with the solution.	4	1	2	4	0	*2.5*
It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.	1	1	2	1	6	*3.9*
When I get stuck on a problem I tend to give up quickly	0	2	5	2	1	*2.9*
I tend to avoid starting a new problem until other students have done so to give me a clue.	1	2	1	5	2	*3.5*
When I get stuck on a problem I have good strategies to help me.	0	1	5	3	1	*3.5*

The mean value is found by assigning a value from 1 – 5 for each column, multiplying the value by the number of responses, summing the result, and dividing by n.

A mean value of 3 would correspond to a neutral response for the class as a whole and is used to indicate a trend for the class.

Diagnostics Testing

The class was given a diagnostic test in order to give a sense of how anxieties and beliefs correspond to achievement. The students used the same pseudonym for both surveys and the test. The test was administered without warning with a view to establishing how much knowledge had been retained. The average mark on this test was 57%, with 5 of the 11 students scoring under 50%. The indication here is that many of these students are starting the year with very limited recall of material from previous grades.

The response to the problems on the diagnostic test indicated poor initialization of the thinking process (as demonstrated on paper). Most students either left the problem question blank or did little more than summarize the data or draw a picture. In this, and for the initial problems given in class, the students' belief that they are unsure about how to start problems seemed evident. This seems to contradict their belief that they start problems confidently. Rewording or making a sketch was the common response, along with 'look for a formula'. One student wrote ''*I don't know what to do, and making a diagram makes it look like I'm moving forward in the question*.' 'Moon unit' wrote '*Paniced(sic) and didn't do anything. Mwaha!*' While the latter comment is an indication of anxiety, the former speaks more to frustration, a local affective step which DeBellis and Goldin (1997, 1999) suggests leads towards the more global affect of anxiety. The initial suggestion is that difficulty in developing ideas beyond re-writing the question may lead to entrenchment of non-start or quick-stop behaviour.

A comparison between the diagnostic and the anxiety survey shows no clear correlation between stated anxiety and achievement at this stage. 'Moon Unit', who scored highest on the anxiety test and indicated the most negative beliefs, scored lowest on the test with 30%. However, the student 'Renesme', who had the second highest anxiety score, achieved 60% on the diagnostic test, slightly above average for the class. In contrast, the student who achieved the second lowest mark on the diagnostic at 43% did not indicate a raised level of anxiety at the start of the year. Both of the two students who scored poorly on this test performed badly in the area of ‘computation’, indicating that they had a very poor grasp of the basic arithmetic processes. They were also weak in the area of algebraic manipulation. As I discuss later, this weakness in basic skills caused significant problems in the problem-solving ability of many students, who became bogged down in a solution because of computational errors rather than a flaw in their thinking.

Early problems assigned

As described in chapter 4, the main focus at the beginning of the year was to work on students’ emotional affective state around problem solving in order to determine what extrinsic factors could change this state. The student responses to the initial problems (See examples in Appendix 3.2) indicated that they had very little clear idea of how to start a problem, and found it hard to articulate their ideas. As a first step the responses were often limited, vague, or too general to be of help to them. Many of the students quickly became frustrated and gave up (demonstrating quick-stop behaviour). One student reflected: “*I cannot logistically think of this mathematically. It has been my number one cause of frustration when approaching a problem: no formula. So, I am without a plan of action*”.

While I observed I also noted that some gave up before even reading the question (demonstrating non-start behaviour). In the first sets of problems they were asked simply to write down their ideas on how to solve the problem, without actually doing so. We then discussed the ideas offered, but it was very apparent they were waiting to be told how to solve the problems by either myself or a student designated as ‘good at math’. The point at which students became frustrated was low, as illustrated by the student’s reflection in Fig.5-2:

It is most difficult when I do not know where to go with the problem. If I have no insight as to how to begin the problem, I am obviously stuck. When there is no method in sight that I can use, I just get frustrated and give up.

Figure 5-2: Example of student frustration

The students initially found it hard to apply a deeper level of thinking. While this does not mean they were not internally processing the problem, they had difficulty engaging the problem externally, as shown in Fig. 5-3, which was all the student produced.

thoughts

How does this
fit with a method
I've used?

Figure 5-3: Example of limited thinking

The students were introduced to Mason et al.'s 2-column problem solving rubric (using one column for thoughts and one for working) at an early point but initially found implementing the rubric challenging. On the first occurrence of its use, only two students had written down any ideas towards a solution, while three others had summarized the question. The remaining papers were blank. With practice and encouragement there was a general improvement in metacognition as the students became more familiar with the process and expectations. However, the responses beyond listing the information were generally more descriptive than a means to indicate thinking. Typical of responses was: "*I'd write down the information and then put it into a formula", or, "I formed an equation and used it to solve the problem". Many s*tudents still relied on others to stimulate their thinking and gave up very quickly on attempting a solution. Again, the problem of non-start and quick-stop thinking was very evident. At this stage the student reflection journals provided more insight than the working during problem solving. Students were making comments such as: "*I didn't think I'd be able to do it so I gave up and asked …. When he told me what to do I realized it was easy and I probably could have", or, "I* *had an idea about it but didn't think it would work. Then __ showed me how and it was the same thing."*

Similar examples illustrated that the students' confidence in their ability to work on problems was low. There was also recognition by the students that they often lacked structure in their work. One student wrote in his reflection journal: "I *don't think or put my thoughts down, I just do the work. Maybe I should put my thoughts down because when I get an idea to tend to forget what I am thinking about after a while*." One student's reflection in particular suggested at the anxiety she felt when reading a question: *"I feel like there are too many instructions and parts to this problem. I don't know where to start."*

The first unit test indicated that the student's marks were consistent with expectations, as indicated by their grades from the previous year. This showed that the greater time spent on problem-solving and thinking activities, at the expense of more typical practice-type questions, was not adversely affecting results. At the same time there was no obvious improvement in the way the majority of students were approaching their work. The problem on the test was well done for the most part, but there was a clear correlation between how the students who scored highly on the test showed a clear thought process in approaching this problem, while the students with lower scores showed thought processes that were not well structured or expressed. The more formal setting of a test may also have contributed to a greater willingness of some of the students to make an attempt to engage in the problem. This in itself raises an issue of the importance of striking a balance between raising anxiety in some students by making the problems count towards their grade, and using this as a stimulus to others who are unwilling to participate if there is no extrinsic reward. A reflection comment such as "*Writing the work Stuck just gives me a reason to give-up faster*" was a indicator that In some cases non-start behaviour was stemming from a lack of interest, or motivation, in pursuing the problems assigned.

On each test I added a problem so that the students would see that the process was valued. The problem used on the first test is shown in Appendix 3.4. As discussed in chapter 4, the rubric was designed to reward the student for indicating thinking about the problem and a possible solution method; even if they did not get time to finish the problem during the test. The marking rubric for this is reproduced in table 5.4 below:

Table 5-4: Test problem rubric

1	2	3	4
▪ A start beyond copying that shows some understanding.	▪ Correct answer but no work shown, or ▪ Appropriate strategy but not carried out far enough.	▪ Correct answer but unclear strategy or, ▪ Appropriate strategy but ignored a condition.	▪ Correct answer with clear strategy, or ▪ Incorrect solution with a copy error or minor computation error but not a misunderstanding.

A breakdown for the problem, using the coding described in chapter 5[3], is shown in table 5.5.

Table 5-5: Breakdown of the first test problem

Entry Stage	Description	6
	Ideas	2
	Neither	3
Attack Stage	Stuck	7
	Unstuck	2
Reflection	Good	1
	Weak	2
	None	8

The last column indicates the number of students in each category. Students were asked if they got stuck in the problem and whether they were able to get unstuck again before the problem was discussed. The students were then asked to reflect on the problem and its solution in their journals. (n = 11)

The data shows that the majority of the class had not progressed beyond simply describing what they were doing. However, all but three had made some attempt at the problem. An example of the eight students who had made an attempt at the problem, seven indicated they had become stuck and progressed, of which two managed to get unstuck. The others progressed no

[3] Entry stage coding was done based on whether the student simply described the process they were doing, such as 'I formed an equation', or if they indicated their thinking behind why they were adopting an approach or idea, such as 'I need to find a value for...'.

further. Of these only one showed any indication of trying an alternate method. The reflection process was done after the test and an in-class discussion of the problem. A follow-up analysis of their journals showed very little in the form of actual reflection about the problem. Eight of the students had simply written down the solution, misunderstanding the concept of reflection for 'test corrections'. I realized that the notion of genuine reflection was something that had to be stressed more, and modeled, if there was going to be a tangible improvement.

Working with lower-achieving students following the test

Following the test I had a tutorial with Rebecca, a girl who had performed weakly on the test. I went through the questions carefully with her to see how she was thinking about each one. Field notes were made during the course of this tutorial to aid recall of the information for both of us. It became very clear that the limitations to her thinking came from two sources:

1. Her basic arithmetic skills were very weak so that she made constant numerical errors in calculations - even when using a calculator (primarily due to a lack of understanding about brackets).

2. The structure and layout of her work was poor so that she often made copy errors when moving from one line to another. (See Fig. 5-4) In many questions she had a good understanding of the current work and knew what to do. She could explain the process verbally but not perform it mathematically. Inevitably she found her attempts at solutions got bogged down with awkward calculations due to earlier mistakes, and she would give up.

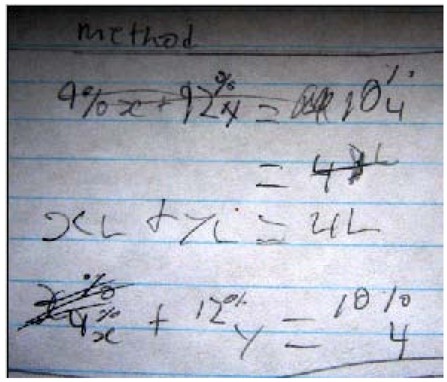


Figure 5-4: An example of poor layout leading to copy errors

I repeated the exercise with two other students, again recording notes, which were struggling to a lesser extent, and found a similar situation as mentioned above. Initializing their thinking was being helped by the metacognitive process but cut short by their weak arithmetic and organizational skills. One of these students had a particular problem in skipping steps which often led to the errors in manipulation, as illustrated in Fig. 5-5 below. In this case, when asked, the student explained that she had done the manipulation in her head. When asked to go through the procedure more methodically, however, she was able to do so flawlessly. It was not that she could not do the mathematics involved, but she was careless in doing so. This type of error was commonly noted when the students reflected on their tests and commented why they got a question wrong. In addition, the students often had only vague recollections of earlier algorithms learned and would remember them incorrectly, leading to error as in Fig. 5.5.

$$3x - 4y = 21 \qquad y = \frac{3}{4}x + 21$$

Figure 5-5: An example of skipping steps leading to a mistake.

I observed an increased willingness to engage with problems that did not at first have an obvious link to the course material. Problems based upon a logical approach rather than a computational approach, such as shown in Appendix 3.2.A, resulted in all students pursuing the problem longer than a curricula-based problem such as Appendix 3.2 .D. I subsequently made notes on one student in particular who was initially very reluctant to engage in any of the problems and noted that he was more willing to engage in a problem if the mathematics content was not immediately obvious. At this stage, it seemed he had conditioned himself to make a judgement on what type of problem he would attempt:

I know nothing about this stuff. I dont remember doing it last year.

Figure 5-6: Non-start behaviour based upon lack of knowledge

Once he had engaged in the problem he tended to persevere with it, even if the question developed a need to use a level of mathematics he would typically avoid. For this student, the initial phrasing of the problem was a key to his non-start behaviour. The '*Towers of Hanoi*' was an example of a problem this student became engaged in and persevered with. When he demonstrated non-start behaviour his typical comment was the oft expressed "*what's the use of this*?" Lack of obvious relevance, however, did not prevent him from attacking problems he became engaged in, while problems deliberately chosen to link to common experience, such as the 'Collision Problem' (Appendix 3.3), might fail to engage him. For this student, the problem needed to be one in which the first steps were very approachable, then, once hooked, he would persevere. Appendix 3.9 shows another example of a problem he spent a lot more time than usual with.

As the semester progressed the students continued to improve in starting a problem and writing down their ideas, but they were not improving significantly in moving past points where they became stuck (see Fig. 5-7). They continued to ask quickly other students how to do the problem rather than to think it through.

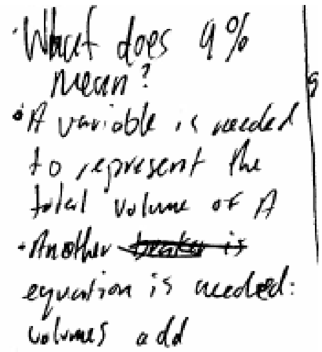

·What does 9% mean?
·A variable is needed to represent the total volume of A
·Another ~~beaker is~~ equation is needed: volumes add

Figure 5-7: An example of improving ideas

I did observe, however, that the quality of their discussion once within the group was getting better. The conversations were more on task and I had to spend less time bringing the students attention back to the exercise. I began to feel that students needed to focus more on the entry stage to the problem so I decided to adapt the 2-column rubric of Mason et al (1982) and introduce the *Organize-Entry-Shar*e (OES) method. As described more fully in Appendix 3, the students are given 5 minutes to write down their ideas about a problem; about 5 more minutes to start to work on the idea of their choice; then an option to continue working alone or discuss their ideas with a partner. If a student knows the answer then they can indicate that they are available to help. The objective was to try and encourage students to come up with their own ideas more, and have more than one idea about how to solve a problem, which they could use to help them or to provide discussion points in a group setting.

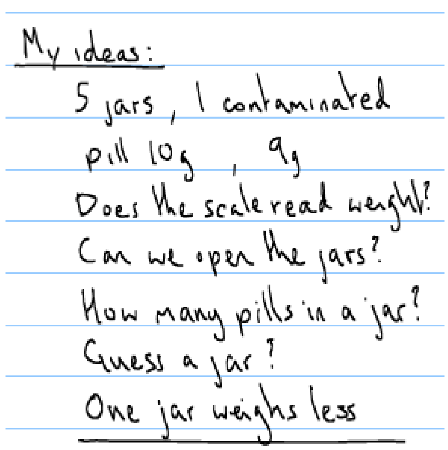
As an example, the class was given the following problem and asked, in small groups, to write out their thinking using the 2-column rubric:

> *You have eight pool balls. One of them is defective, meaning that it weighs more than the others. How do you tell, using a balance, which ball is defective in two weighings.*

Of the data collected from the 5 groups only one had written down initial ideas, while the others moved directly into attempts at a solution. In recording their conversations I noted that in 4 of the 5 groups one student took the lead and suggested a method, each time based on guess-and-check. At a later date the following similar problem was assigned and the students were told to use the OES method:

> *You have five jars of pills. All the pills in one jar only are 'contaminated'. The only way to tell which pills are contaminated is by weight. A regular pill weighs 10 grams, a contaminated pill is 9 grams. You are given a scale and allowed to make just one measurement with it. How do you tell which jar is contaminated?*

From the data collected, 8 of the 11 students had written down ideas\thoughts about the problem, 2 had just written down the information, and one student wrote he had no idea.



When the students moved together to discuss the problem their contributions were more balanced and I recorded only one of the groups having a dominant person. The groups also intermixed more readily in this problem.

Figure 5-8: Ideas expressed using the OES method.

The stronger students initially had difficulty using the allotted time to organize their thoughts, and to express alternative ideas, rather than just use the extra time to work on solving the problem. This was a good opportunity to stress how selecting from a number of ideas is beneficial when the problem is more complex and to discuss the chart from Schoenfeld's (1985) work, shown in chapter 3, which indicates how mathematicians approach problem-solving differently from students. The 'explore time' in the OES method seemed more immediately effective at focusing the students on the problem. When they were allowed to share ideas most students immediately sought out others to discuss the problem with, and I was encouraged by the discussions these students were having. In many cases the students who moved to problem discussion groups could complete the question and I recorded such comments as "*that's what I thought*", or "*I had that –look*". The students often seemed to need an external voice of agreement to be more confident of their result. This was also a good way to discuss the idea of reflection on the answer in the form '*how do you know you are right*?' or '*can you do it another way*?"

The problem question in the second test proved interesting as the majority of the students were able to correctly obtain the either correct solution, or indicate a correct method for doing so. A breakdown of this problem is given in table 5.6.

Initial Stage	Description	4
	Ideas	5
	Neither	2
Entry	Stuck	9
	Unstuck	6
Reflection	Good	3
	Weak	2
	None	6

The last column indicates the number of students in each category. Compare this with table 5.5 above to note the increase in the number of students starting with some ideas about the problem.

Table 5-6: Breakdown of the second test problem

Rebecca, the student mentioned earlier, was able to suggest the correct method for the solution and almost found the correct answer. She clearly demonstrated a willingness to persevere in her thinking and was able to describe her ideas. Unfortunately this progress was not continued with the other questions on her test, in which she demonstrated a similar pattern of errors as in the first test. The problem question is reproduced in Appendix 3.5. A copy of Rebecca's solution is shown in Fig. 5-9 below and demonstrated a much better effort than she was typically making in class. She did not shut her thinking down at the earliest opportunity in this case but rather explored some options. In discussion with her after the test I stressed to her the important fact that she was able to do the problem, even if she had not completed it. She was pleased with her efforts on this question and agreed that she understood the problem. She expressed doubt that this was likely to occur again however, and neither did she adopt the method to help her solve other questions on the test.

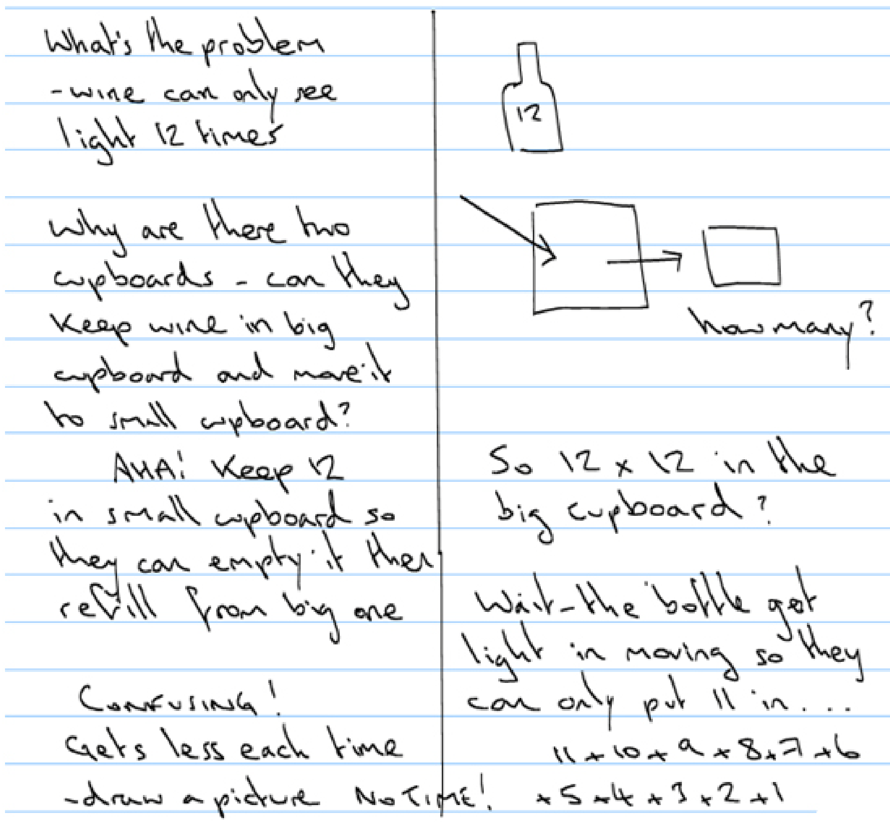


Figure 5-9: Rebecca's solution to the second test problem.

I tried to make the format of some problems as open as possible so students could develop their thinking about what was required for a solution. The method was successful in that it clearly set the students thinking about relationships between variables and to searching for what was needed. Most of the discussions around such problems were good but it also seemed that the question being too open prevented some students from getting a handle on the problem. This prevented them from initializing their thinking in a productive way, or caused them to shut down their thinking without exploration of ideas. It was apparent that at this stage several of the students still needed more scaffolding to the problem to get them going. I attempted to do this on a group-by-group basis, however, as some of the students were making good progress on their own. The Collision problem (Appendix 3.3) was a problem of this type.

Once the students got involved in the problem their interest level grew and the quality of work improved. Their reflection journal comments were favourable, such as:

> *"It was very neat because I knew most of the rules from studying for my L in March. I think more clarification could have been helpful even though overall I thought it went well."* and
> *"At first I hated the accident problem because I didn't fully understand what it was asking. But when I finished I liked it because it was a practical application of math."* and
> *"I found myself actually wanting to solve the formulas and create functions to prove our findings, so it was satisfying."*

A breakdown of the problem on the final exam is shown in tables 5.7 and 5.8. All but one of the students started the problem by writing down ideas. The problem given is shown in Appendix 3.6. The data shown in this table was in keeping with analysis of problems done in class towards the end of the semester.

Table 5-7 and Table 5-8: Breakdown of the problem on the final exam

Initial	Description	7
	Ideas	10
	Neither	1
Entry	Stuck	5
	Unstuck	4
Reflection	Good	6
	Weak	4
	None	1

ideas	#
2	4
3	5
4	0
5	1

Table 5.7 and 5.8 breakdown the problem used on the final exam. Table 5.8 indicates the number of ideas put forward by how many students.

Summary

The grade 11 group demonstrated difficulties in starting problems effectively. Although their initial surveys did not indicate an excessive anxiety, and their stated beliefs indicate they felt were capable of learning the material, many of the students demonstrated non-start or quick stop behaviour and relied on others to help them out at the beginning of the year. Their journal comments indicated that they had low opinions of their ability to solve problems and that they did not have a clear idea of how to structure a solution. Several of the students also reflected that they lacked real interest in the subject and found motivation hard, tying in with the ideas of Hannula (2004). Initially, many of the students made only a cursory attempt at a solution to a given problem. It was also evident from working with the students on an individual basis – and illustrated by their weak performance in the diagnostic test – that misconceptions and a general lack of background knowledge was also making mathematics difficult for them at this level. Confounding this was also a tendency to make avoidable mistakes in routine calculations which frustrated the students and caused them to give up.

Over the semester the group made progress in the area of initializing thinking, as indicated by their responses on the second beliefs questionnaire (p80), and borne out by their later problem-solving attempts. The students were generally better at putting forward ideas and structuring a solution to a problem. It should be noted that not all 'ideas' written down were of great merit, and there would certainly be a case of some students just writing anything down, given that was an expectation. However, I encouraged a free expression of ideas as a way to alter the emotional affective state, and tried to encourage those that had a meaningful connection to the problem. These ideas also provided a starting point for individual discussions with a student.

Chapter 6. Grade 12

"*I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors.*"

James Caballero, *Everybody a mathematician*?, CAIP Quarterly 2 (Fall, 1989).

In this chapter I present a general overview of the data collected for the grade 12 class in order to identify the trends associated with this group. I present in detail an analysis of the data from the surveys administered, and focus on data collected for classroom problem-solving sessions through journals, and tutorials.

The Mathematics Anxiety Rating Scale (MARS-R)

Table 6-1: G12 MARS-R: Average response for each question (n=26)

Q#	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
mean	1.3	1.4	2.0	2.8	1.7	2.2	1.5	2.9	1.3	2.1	3.0	3.6	3.3	1.6	1.8
	Looking at text	Coming into class	Read a formula	Think about Test	Watching teacher	Algebra explained	Start hw	Writing Test	Reading a graph	Start a problem	Difficult Hw	Test returned	Stuck	Start new chapter	Listen to student

The responses for individual questions are given in table 6.1 and were similar to the pattern seen with the grade 11 class, with test situations causing most anxiety and being stuck in a problem the only other significant showing.[4] Being stuck elicited three 5's and four 4's (a combined 35% of respondents); waiting for a test elicited five 5's and five 4's (a combined 38%); taking a test elicited two 5's and two 4's (a combined 16%); thinking about a test elicited one 5 and four 4's (a combined 19%). (n = 26)

[4] One student added 'It's more like frustration' after the question on being stuck.

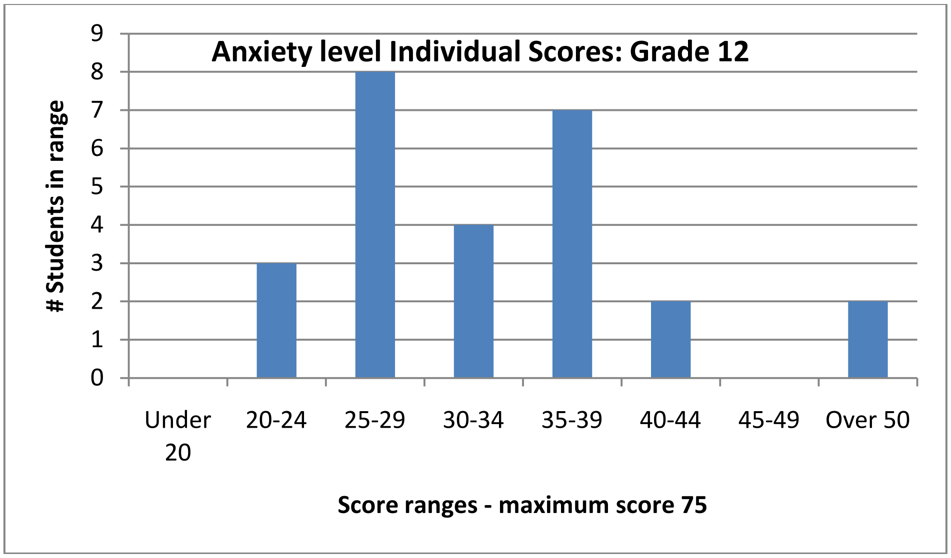


Figure 6-1: G12 MARS-R: Total scores for each student, maximum 15 x 5 = 75

The highest individual scores for the grade 12 students are shown in table 6.3 below. The class average was 2.2, suggesting that this was also not a class with a high level of anxiety in general. Fig. 6-1 indicates only two students suggesting anxiety could be a factor in their learning. The results for these two students are shown in table 6.2.

Table 6-2: G12 MARS-R: Areas of anxiety for the students scoring highest on the MARS-R

Q #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Sum
Ariel	2	3	3	5	4	3	2	5	1	4	5	4	5	3	4	53
Kitty	4	2	3	3	2	5	3	3	2	4	4	4	5	4	2	50
	Looking at text	Coming into class	Read a formula	Think about Test	Watching teacher	Algebra explained	Start hw	Writing Test	Reading a graph	Start a problem	Difficult Hw	Test returned	Stuck	Start new chapter	Listen to student	

The highest anxiety levels occur for: thinking about tests, writing tests, and waiting for tests to be returned. Only being 'stuck' in a problem gave a similar level of anxiety to test situations.

Mathematics Related Beliefs Questionnaire

The beliefs questionnaire indicated that most students felt confident in their ability to start solving problems, with 6 students (13%) disagreeing. However, 11 students (42%) felt that they were unsure of how to start a problem. When getting stuck in a problem 12 students (54%) believed they had good strategies to help them, but 8 (31%) also agreed that they gave up quickly.

Table 6-3: G12 MRBQ: Mathematical beliefs around problem-solving (n = 26)

Please respond to each statement	Strongly agree	Some-what agree	Neither agree or disagree	Some-what disagree	Strongly disagree	*mean*
Solving a mathematics problem is demanding and requires thinking, even from smart students.	6	20	3	0	0	*1.7*
I start problems and questions confidently	7	7	6	3	3	*2.5*
I am often unsure how to start problems assigned in class.	3	8	3	8	4	*3.1*
I like to know the answer to a problem before I start so that I can see where to go with the solution.	2	5	7	8	4	*3.3*
It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.	1	3	7	5	10	*3.8*
When I get stuck on a problem I tend to give up quickly	3	5	8	4	6	*3.2*
I tend to avoid starting a new problem until other students have done so to give me a clue.	2	3	9	9	3	*3.3*
When I get stuck on a problem I have good strategies to help me.	2	12	8	1	3	*2.7*

As might be expected in an elective class, the beliefs questionnaire in general indicated that the students were confident in their ability to do well in the course.

Problem-solving Examples: Grade 12

This class was very familiar with problem-solving activities as I had worked with many of them on some preliminary ideas the previous year. I had already introduced those to the Mason problem-solving rubric of splitting the page into two columns and writing down their thinking in the left column. I therefore began the year with a number of problems and took field notes on their progress. One of these problems was ‘Arbor drive’[5], which is reproduced in Appendix 3. The class was split into random groups and they worked on this problem with varying degrees of success. The discussions were lively and on-topic, with ideas flowing well. The majority of the students seemed to be making contributions with some being more vocal than others. As I walked around the class I observed that several students were writing down thoughts and ideas while others were still giving descriptions of what they were doing. Where appropriate I made comments to these students and tried to encourage them to write down as many ideas around the problem as they could before they went too far.

My interest was in those students who were having difficulty getting started with the problem so I focussed on one student who seemed to be in this category. Her pen hovered over the two neat columns on the page but the left column was blank except for a repeat of the question. The discussion around her was lively but she did not contribute. The other two members of her group were sketching diagrams on a sheet of rough paper which was also randomly covered with numbers. The sheet was a form of brainstorm but they had also written very little on their ‘divided paper’. I listened from a distance and recognized that their discussion was not progressing so I decided to step in. The students felt they needed more information. ‘*I notice you haven’t written down any thoughts or ideas yet*.’ I pointed out. ‘*Well, we haven’t got it clear in our minds yet*’ was the reply. I tried to reiterate the purpose of the rubric, to clarify thoughts and push ideas and not to be an ‘extra step’, but I could see they weren’t convinced. ‘*What’s happening now?*’ I asked them. ‘*We’re stuck*.’ They conceded. ‘*Well, why don’t you all try to write down your ideas, those you’ve had, separately and then compare notes again*’ I suggested. I then moved away and allowed them to try this but kept an eye on the group from afar while recording the conversation. It was evident that the student I had most interest in was

[5] Taken from http://www.stfx.ca/special/mathproblems/grade11.html

only writing down the ideas the others were adding to their page so I approached again. '*Jane, I notice you aren't giving me many of your thoughts here, what do you think is the best way to go?*' She was hesitant and mumbled and few random thoughts, one of which was '*we don't know how many missing houses there are*'. '*Yeah we do*,' one of the others added, '*we just don't know which ones'*. I noticed he had written down the number of the last house on the odd side as 55 (in fact it should be 53) as being the sum of 4 and 22, which made no sense (perhaps an example of distracted thinking), so I asked him to explain that. He was able to do that and recognize his mistake, at which point I suggested that if he'd explained his process to the others he would have similarly spotted that error. '*But I still don't know the missing ones*.' He stated. '*So try explaining why you don't know them*.' I suggested. He started to do so but I stopped him: '*Not to me*,' I said, '*to them*'. His explanation came down to the recognition that there were 11 even digits missing and that could be one single and five double-digit numbers, or any combination. I asked for further clarification of the 'others' and he gave me three single and four double digit numbers, or five single and three double digit, or seven single and two double digit, or.. '*It can't be those*' Jane interrupted – it was her first contribution, '*there's only four single numbers*'. The other two agreed before suddenly realizing that two and four were taken, giving them the solution. '*We cracked it!*' they declared with high fives.

I use the above example as demonstrating how the students often have difficulty expressing their thoughts clearly, even to themselves, but when they have to explain the process to another person it can help to clarify what they are doing. Often, however, in these groups I noticed that students interrupted each other before they could reach a point of clarity, especially if the argument faltered. My role as a teacher became to simply *start* to listen in many cases, and encourage students to listen to each other, as illustrated by the following journal extract (based in a different problem):

The next day, I asked Mr. Wells at school and told him that I thought the problem was impossible. He told me that there is a solution and gave me a subtle hint which I didn't get so I went on to tell him the reasons why the problem has no solution when I had my first "AHA!" moment. I actually said "AHA!" out loud and proceeded to do the problem. Eventually when we went over it in class, I found out that the method I discovered was just one among many so if I had thought of all the other methods, I could have listed them out as different methods. I was too busy writing down my answer that I forgot to check if there were other possibilities.

Returning to the first set of problems, I did notice a difference in approach between those who had experience with the two column method and those who did not. The next examples illustrate this. The following is taken from the reflections of a student who elected for this grade 12 class but was not in my previous year's grade 11 class:

These 3 problems were the first set of STUCK!AHA!CHECK! ones that I ever did. They are an example of how I was able to do the math, but not able to write down my thoughts and show my thinking. At the time, I thought I had written tons in my thinking column. I was surprised when I got my mark [6]back.

In this case, although the student had written quite a bit in the thinking column, it was not actually a representation of her thinking but rather a description of what she was doing, such as 'divide by 2'. The notes were clearly written after the work in the right column was completed and gave very little to thinking behind the mathematics. Compare this to a later journal entry from the same student:

It took me awhile to realize how much of my thinking actually needed to be written down, and I've found that it helps me, maybe not to solve the question in the end, but to gather my thoughts and get organized. I find it pretty soothing to write down what I'm thinking. I don't have to waste any space in my brain, thinking those things, when they're written on paper.

The following comment came from a student who was new to the school:

[6] The first few problems were only given a formative assessment and the 'mark' consisted of comments about the approach taken

‘My dad teaches students at university. When we talk about university life, I always hear that his students do not study at their room. They make a small group and work together and sometimes help each other. So when we work in the class, it is fun because we can discuss hard problem together and solve it.

This student’s comment was a little surprising as he seemed very reluctant to want to work with a group at first and seemed to work on his own within the group rather than share ideas. However, when the groups were rearranged he was with a student who spoke his mother tongue and could help him communicate with the rest of the group. I recognised the importance of having students comfortable in their group, but not having them necessarily working with their usual friends, if the aim is to stimulate dialogue quickly.

The following extract was taken from the reflection of a student who was in my grade 11 class the previous year:

I also experienced a very good AHA! moment while finding the solution to this problem. At the beginning I drew a visual diagram and a table since I think that helps me organize my information and also reminds me what information I am given and can use. At the beginning I had some difficulty deciding what to do next after drawing out my diagram and then I suddenly remembered that I needed to apply a formula for time ...I also think that finding a formula for the equation is usually the most difficult part of solving a problem for me because it takes me a long time to think of problems that are similar to the one given and how I solved those problems previously. I think it’s hard to apply some of the things we learn into problem as I am so used to doing the problems in the workbook where it tells us what we are practicing, and therefore should use.

This extract illustrates how many of the students had bought into the method when it was introduced the previous year, and were familiar with the terms and expectations of a problem and its reflection. I believe the level of insight – which was common among the students – of the reflection was high and this was very encouraging. The students still had a great difficulty in demonstrating their thinking at the early stage, however. If they thought they could do the problem they did not want to write down anything extra; if they couldn't see how to start the problem they didn't know what to write. One student, who asked to not use the columns because it 'cramped him', produced the solution in Fig. 6.2, which I used as an exemplar for discussion so that other students, who produced little in the way of thinking, could see an example:

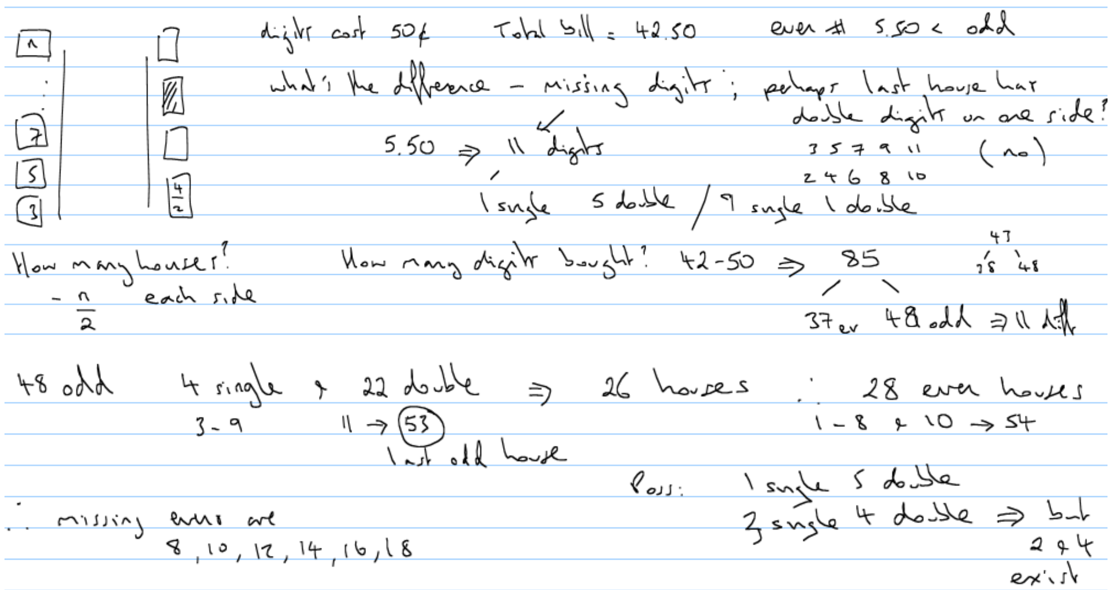


Figure 6-2: A good demonstration of the thinking process

I refer to examples from the students' journals for quotes to illustrate that they also began to see value in the process of writing down their ideas and thoughts

> *I felt that it (the rubric) was very unbeneficial and a waste of time to write out my whole thinking process. Now I have a more positive way of thinking about it because through these problems, I have also learnt a lot.'* (Student A)
>
> *This particular problem set (*See Appendix 3.8*) is very important to me because it is the first time I think that I have ever accomplished work in a group for math. We were split into groups for this question and I actually did this problem with the group. I normally don't enjoy group work so I am surprised that I almost finished the whole problem at school. Everyone in the group contributed and that is what I think allowed us to work together and accomplish the task.* (Student B)
>
> *For the actual question itself f (*referring to the problem shown in Appendix 3.9*), I must have spent about four hours trying to figure it out. Eventually it ended up as me and my*

> *mom coming up with all sorts of theories on how to make the shapes and values equal what we wanted it to equal. In the end, the first idea that I had but didn't even try to work with was the one that would have led me to the answer.* (Student C)

The comment of Student C is interesting as it gives an indication of how much time some of the students invested in the problems. In addition some were engaging family members in the problems, something that was also borne out by comments from parents themselves. However, the main point of using it is to illustrate how the student wrote down several ideas at the start of the problem and eventually came to the one which gave success. Another entry from the mid-point of the semester, which shows the growth of the student referred to as Jane above, reads:

> *This problem* (see Appendix 3.10*) gave me a lot of difficulty since I did not even think it was possible to find the odd ball after five minutes into the problem. I began by drawing visuals once again. I asked for help from my classmates at the beginning of the problem since I had no clue how to start. For this problem, I used both visuals and writing of the steps and the results to show my work.*

It is worth noting here the now casual reference to asking for help from classmates and acknowledging this. However, it is clear the she is still having problems initialising her thinking and I felt many students were still relying on others to do the thinking for them when in the group stage. It was at this time that I changed from the two column rubric to the OES method (Appendix 3) to encourage students to develop more ideas.

Charting progress through Journal Entries

With the grade 11 and 12 classes being on semester I was able to reflect on the process towards the end of the semester. One way to chart the progress of these classes was through their journal entries and final portfolios.

The first Journal entries illustrated how students' thought processes were limited in depth and often confused. Several responses to the question 'What is your first response?'

reflected instructions they had heard before, but were not being followed despite the authors claim. For example: "*I like to organize the information and draw diagrams so I can see the information more clearly*." Yet this particular student produced work which was anything but organized. Typical of students achieving lower grades in the class was: "*I figure out the easiest way to solve the problem*", or "*Read the question again then start to solve the problem*". Stronger students commented on preferring problems which made them think, while lower-achieving students preferred problems they could solve. The implication suggested is that the confidence of the student is a key factor at this stage (see the comment from *student A* above as an example). A typical response from a high achiever regarding a more challenging problem was: "*It was also very hard and required lots of thought, which is funner than just doing straight number solving*". This is a situation where extra motivation is provided for the student (Hannula, 2004), as they take up the challenge.

The second Journal responses showed that students were becoming more familiar with the process. Comments such as "The *questions helped me be more outgoing in class*," or "*I felt comfortable discussing my ideas"* started to appear, but most notably a very common theme was that students on the whole enjoyed working on the problems in small groups rather than at home, where they were often getting frustrated. "*I enjoy working in groups because you can see other thinking to solve a problem. It's fun and learning at the same time. I don't enjoy it at home because if I can't solve the problem I feel stupid,*" typified comments from lower to middle-achieving students. Those who preferred working at home recognized the chance to take breaks to think things through. At this stage higher- achieving students typically expressed a dislike for the two-column rubric, with comments such as "*I wouldn't say the process is helping me*." There was only limited recognition that the emphasis on problem-solving was beneficial: "*I do think* *they (problems) are helping me with other areas of the math class because the problems encourage thinking outside of the box. It teaches me to persevere*."

In the third Journal the grade 12 students were asked 'Have you found any advantage in using the Stuck-Aha-Check-Reflect rubric in your problem-solving yet?' A comment such as "*I have not found many advantages. I still feel it is sometimes annoying and tedious. However, I feel more organized…before, my information would be everywhere.'* Secondly*, "when I write 'aha' on my paper I feel relieved and really happy about it. It shows my thinking and problem-solving skills are improving. When I solve a problem, I feel proud of myself and to be able to write aha means a lot to me*" seem to illustrate the shifting opinions of many of the students as they wrestled with the extra time it seems to take to write out their thinking while at the same time acknowledge some benefits. Some students were becoming very positive: "*I have found the method very useful. I used to immediately give up on a method and get frustrated*." Perhaps most telling was a comment from a student who has traditionally struggled in mathematics: "*I have found an advantage. Originally I didn't.. until I came to the test problems solving questions. In the portfolio questions I would just add in the stuck and the aha at the end so it looked like I put it in but now I use it because it actually helps me.*" Overall the trend of the comments was moving from negative to positive when considering the emphasis on expressing their thinking. The only negative comment came from a student who considered himself to be very capable in mathematics. Interestingly he often avoided solving the problems in class initially and liked to produce the answers the following day using techniques that were often beyond the curriculum. As the course progressed, however, he did work better in class and was able to find solutions within the parameters of the course along with his peers. At no point did he embrace the concept of writing down his thoughts however, or acknowledge that such a process had been of help.

The next journal entries were clear on a couple of interesting points. Students expressed the opinion in class that they preferred to work in small groups, feeling that their voice got lost in a larger group. In addition they felt that it was a good thing to mix groups up so that they could learn from the others in their class. Several students commented on the advantage of seeing different learning styles and approaches at work in their journals. One particularly profound comment was: "*There has been many a time when I have learned something new or been enlightened by a different source. This has led me to see that everyone is capable of the same thing, it's just depends on whether they want to or not."* Higher-achieving students in the past have typically been less keen on working in groups, but with the increased emphasis on thinking and discussing their thoughts even some of these students were now appreciative of what the group setting could offer. "*We learned the quadratic formula, which was the most fun part, when we solved for x it still makes me surprise how that equation came out. We were doing some weird thing but last, it gave me the answer how the formula came out.***"**

The frankness of some of their responses was a very encouraging sign. Most of the students felt that they were much better at working with problems by the end of the course, and while most also put this down to doing more of them than ever before, several did comment that the thinking rubric had been of help to them, even those who were against it earlier. The comments indicated that the level of anxiety in doing problems was much lower. In this way the rubric and the de-emphasis on the final answer certainly seemed to be of help. It is apparent that the depth of thinking of some students is still too shallow however, and there is a danger that they are simply more comfortable with getting an answer which is wrong than struggling over getting a more acceptable answer. A particularly interesting comment was "*I share the feeling of doubts…sometimes I panic, especially during tests .. mostly I get frustrated, which makes matters worse. That way I won't be able to think clearly and I keep telling my brain* *that I do not know how to do this question! Knowing that I am stuck is good… it's one tiny progress towards success. Writing STUCK! helps me think what I can do or haven't done yet … but does not help me from distancing my state of being.. Or free me from incapacitating emotions."* What makes this especially interesting was that this was from a student who regularly scores close to 100% in tests! Another student writes: "*I think being stuck puts my thinking into a small box where I hit the walls and bounce back and forth*", but also "*If I am writing down the word STUCK it makes me think about why I stucked*." From a different student: "*I think I am developing more sophisticated strategies. Previously my strategy was to ask or give up. I had no idea that finding a solution contained so many important and helpful steps. I become much less frustrated*." Also: "*I feel much more confident about answering difficult questions now. I do not get as frustrated when I do not get the answer right away, and even if I know I don't have the correct answer in the end I'm not as worried. I know I will still get marks if I work hard experimenting various ways on how to get the end result*....*Although I didn't believe that writing the word STUCK would help me get an answer, it does*." This journal entry was very encouraging in showing that the process was on track, but not all students were onboard: "*In previous course we have never had problems to the extent we have in Math 12. I do believe I have progressed in my problem-solving skills but I do not think (the rubric) helps*" and "*I have not experienced the benefits of the thought rubric… yet. I'm very willing to keep trying because I would love anything that will aid me in solving strange math problems. My biggest problem is that I need to stop thinking about what I can write to get marks and what I can write to help myself*." The depth of the students' self-analysis progressed significantly, and I feel certain that this helps enable them more. "*I am less frustrated now when I get stuck because I have learned new ways to deal with being stuck. Before I used to just stare at the paper and hope the answer would come to me… I have learnt to ask myself different questions*"

The final portfolio indicated a shift in favour of the use of a thinking rubric by the students. Of the 26 journals, 18 made comments favourable to a rubric's effectiveness. There was also an indication that its implementation was successful in raising the confidence of many of the students, especially those in the lower- and midrange of success in the course. This is perhaps best illustrated with the comment:

> *I think that by using the thinking process that we have been using for the problems in this class, it has really opened my mind to the possibility of solving my own problems. At the start of the course I thought that once I became stuck, I either had to give up on a problem or ask other people. I think that using the system has really opened my mind to the possibilities of getting around being stuck."* and *"Before I went into this course, stuck was something bad. Something that makes me feel bad. Now it is just a feeling, a status I can change without getting angry.*

Another student writes:

> *The reflection portion allowed me to express my feelings towards the questions and answer them truthfully. It was like a self-analysis and it let me see what I was doing and what perhaps I wasn't doing too well with.*

A common acknowledgement of the grade 12 students was that *"to be stuck doesn't mean to fail. It is just a part of the way to pass."* The portfolios gave a good insight into the students' development over the semester. Having taught the students in grade 11 they were used to being given this type of problem, but without an emphasis on using the 2-collumn rubric or OES method. It was evident that the students were able to discuss their work freely and see areas of learning and development quite clearly. Their responses seemed open and honest, and appeared genuine. I felt that the portfolio was a successful and important step in the developmental process of fostering thinking skills.

Revisiting the Surveys

Towards the end of the semester the grade 11 and 12 classes were asked to do the *MARS-R* and the *Beliefs Questionnaire* again. The aim was to look for any significant changes in attitude which might have occurred over the semester.

The MARS-R indicated little change in the overall anxiety levels reported by most of the students, with the notable exception of one student who reported a noticeably higher elevation of anxiety levels (this may have had links to factors outside of the classroom).The levels of the two students in grade 11 and 12 who registered the highest levels of anxiety in September are compared with their original levels in table 7.4. Although the changes are not dramatic, they are noticeable. It is interesting to note the drop in anxiety for 'entering the class'. However, it is also notable that level of anxiety over starting a problem has *not* improved in three of the four cases. This was surprising as these were the students were given most help in this area.

Table 6-4: Anxiety related comparison values (5 is most anxious)

Moon Unit	4	5	3	4	3	5	3	5	2	2	4	2	5	3	3	53	Start
Moon Unit	4	3	1	4	2	5	2	4	1	3	5	2	4	1	2	43	End
Renesme	4	2	3	3	3	2	3	3	1	4	5	3	4	2	3	45	Start
Renesme	2	1	2	4	1	2	2	4	2	2	3	4	3	3	3	38	End
G11 ↑ Question G12 ↓	Look text	Into class	read formula	Think Test	Watch teacher	algebra exp	Start hw	Test	Read graph	Start prob	Diff Hw	Test return	Stuck	Strat chapter	Listn student	Sum	
Ariel	2	3	3	5	4	3	2	5	1	4	5	4	5	3	4	53	Start
Ariel	2	2	2	4	2	3	3	5	1	4	4	4	4	2	3	45	End
Kitty	4	2	3	3	2	5	3	3	2	4	4	4	5	4	2	50	Start
Kitty	2	1	2	3	3	3	2	3	1	4	4	3	3	2	2	38	End

If the class mean for each question is now examined a similar trend is observed, as shown in table 6.5 below. Although the difference in the mean is not substantial, 12 students (48% of students who did both surveys) from the grade 12 and 5 students (46%) from the grade 11 indicated an increase in anxiety, compared to 5 (20%) from the grade 11 and 2 (18%) from the grade 12 who indicated a decrease in anxiety. It would seem that, despite my focus in this area, the majority of the students were now *more* anxious over starting a problem than before.

To make sense of this it is instructive to examine the change in beliefs over the same period. I took the opportunity of adding in two more questions to the Beliefs survey which asked the students how they felt about writing out their ideas, and if they felt their problem-solving skills had improved. They were also asked to suggest a reason for any change in their problem-solving ability. The results are shown below with the results from the first survey as a comparison. The survey at the beginning of the year is shown in italics beneath the survey from the end of the year. The mean value is found by assigning a value from 1 – 5 for each column, multiplying the value by the number of responses, summing the result, and dividing by n. A change in the mean value indicates a general trend for the class where a mean of 3 would correspond to a neutral response. The individual numbers show how many students made the selection indicated.

The results of this survey were interesting as there was a indication that the students now believed they started questions *less* confidently and that *more* students felt they tended to give up quickly. More grade 11 students now felt they had better strategies to help them get past the stuck points in the problem, but fewer 12's. All of the students felt that their problem-solving had improved. In contrast, *more students* felt they had less idea of how to start a problem. These were surprising results. There was, however, a strong indication that they felt writing down their ideas was useful. Students' reflection journals suggest that the reason the students felt they were less confident and gave up to quickly was not because these categories had weakened over the semester but, as examined further in the next chapter, because they were now much more aware of what problem-solving meant.

Table 6-5: MRBQ comparisons (J = January; S = September

Please respond to each statement			Strongly agree	Some-what agree	Neither agree or disagree	Some-what disagree	Strongly disagree	*Mean*
Solving a mathematics problem is demanding and requires thinking, even from smart students.	G11	**J**	**7**	**4**	**0**	**0**	**0**	***1.4***
		S	*6*	*3*	*1*	*0*	*1*	*1.8*
	G12	**J**	**12**	**11**	**0**	**0**	**0**	***1.8***
		S	*6*	*20*	*3*	*0*	*0*	***1.7***
I start problems and questions confidently	G11	**J**	**0**	**5**	**2**	**4**	**0**	***2.9***
		S	*2*	*6*	*1*	*1*	*1*	*2.4*
	G12	**J**	**2**	**8**	**11**	**3**	**2**	***2.8***
		S	*7*	*7*	*6*	*3*	*3*	***2.5***
I am often unsure how to start problems assigned in class.	G11	**J**	**1**	**5**	**1**	**3**	**1**	***2.8***
		S	*3*	*3*	*2*	*2*	*1*	*2.5*
	G12	**J**	**0**	**7**	**6**	**13**	**0**	***3.2***
		S	*3*	*8*	*3*	*8*	*4*	***3.1***
I like to know the answer to a problem before I start so that I can see where to go with the solution.	G11	**J**	**3**	**1**	**3**	**3**	**1**	***2.8***
		S	*4*	*1*	*2*	*4*	*0*	*2.5*
	G12	**J**	**3**	**5**	**6**	**10**	**2**	***3.1***
		S	*2*	*5*	*7*	*8*	*4*	***3.3***
It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.	G11	**J**	**0**	**0**	**4**	**1**	**6**	***4.2***
		S	*1*	*1*	*2*	*1*	*6*	*3.9*
	G12	**J**	**3**	**2**	**3**	**8**	**10**	***3.8***
		S	*1*	*3*	*7*	*5*	*10*	***3.8***
When I get stuck on a problem I tend to give up quickly	G11	**J**	**0**	**4**	**3**	**4**	**0**	***3.0***
		S	*0*	*2*	*5*	*2*	*1*	*2.9*
	G12	**J**	**3**	**5**	**3**	**10**	**5**	***3.3***
		S	*3*	*5*	*8*	*4*	*6*	***3.2***
I tend to avoid starting a new problem until other students have done so to give me a clue.	G11	**J**	**0**	**1**	**1**	**5**	**4**	***4.1***
		S	*1*	*2*	*1*	*5*	*2*	*3.5*
	G12	**J**	**2**	**3**	**5**	**5**	**11**	***3.8***
		S	*2*	*3*	*9*	*9*	*3*	***3.3***
When I get stuck on a problem I have good strategies to help me.	G11	**J**	**1**	**4**	**6**	**0**	**0**	***2.5***
		S	*0*	*1*	*5*	*3*	*1*	*3.1*
	G12	**J**	**0**	**8**	**7**	**8**	**3**	***3.2***
		S	*2*	*12*	*8*	*1*	*3*	***2.7***
I have found writing out my initial ideas first helpful	G11	**J**	**4**	**4**	**2**	**1**	**0**	***2.0***
	G12	**J**	**2**	**5**	**5**	**11**	**3**	***3.3***
My problem-solving has improved since the beginning of the course	G11	**J**	**3**	**8**	**0**	**0**	**0**	***1.7***
	G12	**J**	**3**	**14**	**7**	**2**	**0**	***2.3***

Table 6-6: Free responses to ‘Explain your answer to the last question’

G11	
We have done more problem-solving	2
Learned to think better about the problem	3
Learned new methods\strategies	2
Increase in confidence	1
Use of reflective journals	2
Writing down initial ideas	3
G 12	
We have done more problem-solving	3
Learned to think better about the problem	8
Learned new methods or strategies	8
Better Understanding	1
Increase in confidence	2
No difference	4
No response	2

Summary

The students in grade 12, as might be expected from an elective class, were much more motivated to do well in mathematics than their counterparts in grade 11. There were, however, a significant proportion of the class who demonstrated non-start or quick-stop behaviour when faced with more challenging questions. Despite their generally high performance in mathematics over the years several of the students expressed a lack of confidence in their ability to solve problems. Several students mentioned getting frustrated with their work if they could not get the answer quickly, and there was a heightened sense of concern of the effect of this on their grade. Working with students individually it was clear that many of the students had a very rigid way of thinking which was focused on algorithms and found it difficult to ‘think outside the box’. While the technical ability of the majority of students was very high, those who performed less strongly tended to make numerical errors related to poor structure or bad habits. An important area lacking in these students was the willingness to reflect on their work and build upon their understanding. For the most part there was a sense of ‘getting it done’ and reflection was limited to checking an answer for correctness. For many of the students the learning was transient, but they possessed the ability to review topics and re-learn the processes so that performances on formal tests were always much higher than informal ones. I was, however, confused by the apparent increase in anxiety the students reported when starting a problem. This was an area that was explored further as the grade 10 students moved into the second semester.

Chapter 7. Grade 10 Semester One

"If I am given a formula, and I am ignorant of its meaning, it cannot teach me anything, but if I already know it what does the formula teach me?"

St. Augustine, De Magistro ch X, 23

In this chapter I present an overview of the data collected for the grade 10 classes in semester one. I begin by examining the data collected for this group and then examine their attempts at problem-solving. I then look across the grades for emergent themes.

Mathematics Anxiety Rating Scale (MARS-R)

The responses for individual questions showed a similar trend to the G11 and 12 classes in that issues relating to tests cause most anxiety. Getting stuck in a problem is also a point of stress. Again, there seemed to be little concern raised around starting a problem. Chart 7.1 below shows that overall level of anxiety for the group (n = 34).

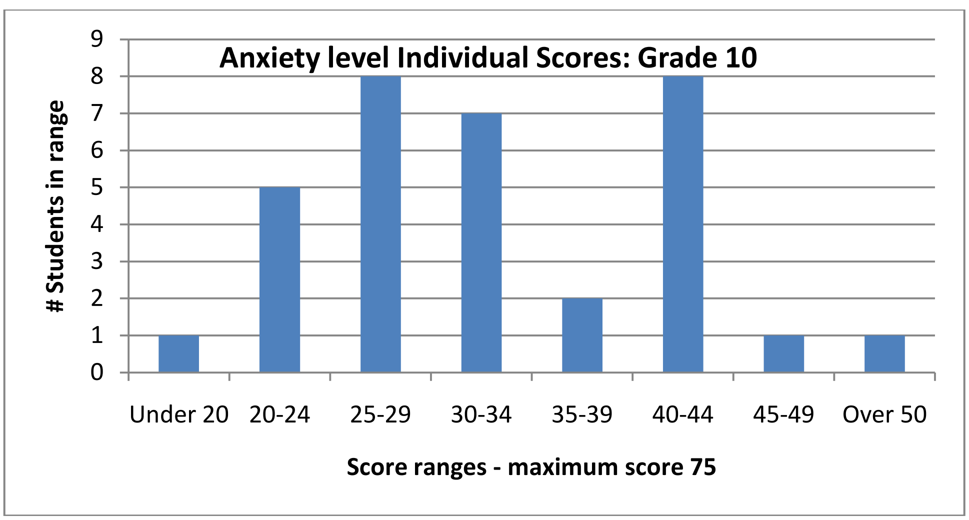


Figure 7-1: G10 MARS-R: Total scores for each student

Fig. 7-1 indicates that the overall level of anxiety of the group is not high, but there are two students who exhibit levels of concern, and a further eight who are close.

Mathematics Related Beliefs Questionnaire

The results from the grade 10 students initially indicate that, in general, they feel confident in starting problems, they like to think for themselves, and have reasonable problem-solving strategies. A closer analysis however indicates that 18% feel they are 'unsure' how to start problems and 32% are neutral on the point. Given that no particular problem was specified as an example this suggests that half of the students do not feel sure about their abilities to start a problem. There is a strong leaning to the belief that they do not give up quickly, but 26% responded in the neutral category and 8% felt they often did. Also, there is no indication in the survey about what constitutes 'quickly' when giving up on a problem. Most students seemed to think they had good strategies for solving problems and the validity of this claim is something I examined further in the course of this research. The question arises as to whether the students have a good sense of what is meant by a problem at this point in the year and on what basis they have for their beliefs.

Diagnostics Testing

The average mark on the diagnostic test was 61% with a standard deviation of 10%, between a low percentage of 38 and a high percentage of 77. This suggests the grade is a fairly homogeneous group. The first unit test on exponentials and radicals gave an average of 72% (low 36, high 98) with a standard deviation of 17%. Thus it would appear that knowing of a test in advance gives the higher-achieving students a chance to review and their mark goes up around 20%. In comparison the lowest scores were unchanged, suggesting a lack of preparation.

The lowest scoring student in the diagnostic test had favourable beliefs of mathematics as a subject but a low opinion of his or her own ability to do well. The next lowest student had similar views to the top two students in all areas except how much effort was being put in to achieve success. Both of the lower scoring students acknowledged that they did not prepare themselves carefully for past exams, supporting what was suggested above. For the lower-achieving students it is interesting to note that they believed that they did not start problems confidently and did not believe they had good strategies in solving problems. They did however believe that they did not give up quickly.

Problem-solving examples

The grade 10 classes had more success than the grade 11's in using the 2-column format at the start of the year, but also found it constricting to have to write down their thoughts while doing the problem. The thoughts column tended to consist more of short descriptive statements than actual thinking or ideas, often added after the thinking had been done.

An early problem given to a grade 10 class (See Appendix 3.12) is broken down in table 7.2 to give an idea of how the students were performing at this stage: (n = 17)

Table 7-1: **Breakdown of an early problem.**

Initial Stage	Description	9
	Ideas	4
	Neither	4
Entry	Stuck	14
	Unstuck	6
Reflection	Good	2
	Weak	12
	None	3

The last column indicates the number of students in each category. Students were asked if they got stuck in the problem and whether they were able to get unstuck again before the problem was discussed. Here, 6 of the 14 students could continue. (n=17)

In this case 'ideas' were two or more suggestions about the problem rather than a statement about the process, which I classified as a description. For example,' I *decided to try to find a way to solve it by calculating the profit...*' was coded as a description; as were statements such as '*I calculated the money spent on gas.'* In contrast, '*Need to find the costs. Need to find what he makes -x? Need to include the initial value (gas, insurance?)'* is an example coded as 'ideas'; as was '*Could plot a graph of take against cost, or make a table of values.'*

Writing down an idea first before acting on it was a difficult concept for many of the students, especially if the solution seemed immediately approachable, so I also discussed 'Fermat' style problems in class such as '*How much does the ice in a hockey-rink weigh?* 'The reflection was classified as weak if the student's response added little to the problem or its solution, such as: '*We discussed about our different attempts at a solution to the problem and looked for a correlation between the number and the shapes.'* A good reflection required the student to discuss why they think they couldn't do the question and to indicate how they could have completed the problem. Table 8.1 indicates the difficulty the students were having at expressing their ideas and reflecting on what they were doing at this stage in the first semester.

I observed one student getting into difficulties working with multiplying polynomials during a problem. It was clear that she had a good understanding of the process of distributing a number into a bracket and could perform the multiplication well. She was able to simplify the terms to the point of adding like terms, but then messed up on the last stage, constantly being unable to subtract two negative terms. It was clear she couldn't get past her notion that *adding* two negatives should give a positive, even though she was corrected on this several times and acknowledged it on each occasion. At some point in the past she had clearly mixed the concept with multiplying two negatives and the notion of 'two negatives make a plus' had become ingrained. The student was asked to show each step and talk through what she was doing. She was encouraged to put some symbol around each similar term and write out the calculation very clearly. When she did this she was successful and pleased with herself. She was then asked to try another question and she again began to jump steps. Also, as she was describing her thinking, it was evident that, because the terms were not yet placed in the order to group, her thoughts kept switching between x's and x^2 etc. so that she became confused. Reminding her to use the surrounding symbol to isolate the terms put her back on track. After two further examples she was able to go through the steps without prompting. Observed in subsequent lessons, however, she made the same mistakes when she opted not to be as systematic with her work. It appeared that the initial thinking process, where the material was new, was good – but older knowledge had become half-remembered and mixed up. Because the knowledge was a procedural in nature the process failed the student in completing the task. As a result she was hampered in her thinking, not as a reflection of her current knowledge, but due to a past misconception. Other students demonstrated similar issues, causing their problem-solving to falter at an early stage.

I also extended the idea of teaching through problems more, as noted in chapter 3, and at every opportunity stressed the notion of thinking carefully about how the mathematics was related to the problem. One student, who had already taken grade 10 mathematics at summer school, made a comment during a tutorial one day about how different the two courses were. '*I thought (the course) was going to be easy like the summer, but this is much more challenging*.' I asked her to say why she thought so, was it more difficult? '*Not really,'* she replied, '*but now I have to understand it. It's more stressful.*' This is an interesting point to which I return later in analyzing students emotions around initializing problem-solving.

In general it was noticeable that the students discussed their work more as the year progressed, and acknowledged the assistance of others (and in some cases were quite vocal in insisting others cited their help). I also observed that the quality of help given to each other was improving, moving away from telling to explaining and then to questioning. The range of responses demonstrated how some students were able to 'think it through' while other students struggled with the ideas. I recorded one boy say:

> *You know, I think I liked math better when the teacher just gave us a list of problems like 7 x 5 and we just wrote in the answers; that was easy.' 'Yeah,'* his partner replied, '*but we*

> *need to learn how to do this stuff, that other stuff is for the little kids.*' The first boy shook his head slowly. *'I know, but this is making my head explode*.

This comment again points to the idea that the students feel more pressured when asked to think deeply about a problem, while at the same time recognize the importance of doing so.

Reflecting on problem-solving was constantly stressed during the course. One student wrote: *"When I was first doing the questions I did not notice any patterns, but when I wrote the summary I realized I was writing about the same things over and over."*

Another student wrote in her journal:

> *One thing this assignment DEFINITELY made me do was thinking. Before I would just look at a problem, think of a formula and solve it. However the assignment made me think WHY more than HOW. For example, when I was looking at the area of various triangles, although I was only given two numbers to solve a shape that seemed impossible to solve with 2 numbers, by thinking and rearranging the area was able to be found. That taught me I always have to look beyond just what I see.*

The benefits of promoting the thinking process were again demonstrated in the problem used in the mid-year exam, (See Appendix 3.11). Students outlined and explored their thinking to the extent that the majority of the students either found a solution or came close. In addition, six different methods were found by different students to solve this problem, and this made an excellent discussion class after the exam.

A good way of comparing methods was to try different ways to give the same problem to each of the two classes. To gain a sense of how useful the OES method (Appendix 4) was I gave the following problem to both classes:

> **I am giving you a nut and bolt (See Fig. 7.2). Your task is to write down an equation for the position of the nut from the head of the bolt as a function of the number of turns of the nut. You must explain your final answer carefully and be prepared to justify it.**

Figure 7-2: One of one of the bolts given

In the first class I paired the students up but gave them no further indication of method. In the second class I asked them to follow the OES method. While in both cases the students were able to come up with a solution, the class using the OES method did so in half the time. I used this as an illustration of how sometimes ways you think might take longer may actually save you time. I tried the same process at a later time with another problem, this time reversing which class used the OES method. The result was consistent in that the students who were asked to think in the more structured way were able to come up with solutions faster. The students agreed that having more structure helped them, but when they were not directly asked to use the OES method they invariably returned to familiar methods, even though these caused problems. In contrast the students' responses after the tests, and their notes in their reflection journals, were positive about adopting a metacognitive approach. As one student wrote:

> *By always having to explain why we were doing something caused us to learn why we were actually doing the certain things. If we did not know why we were doing something in mathematics then we would have to figure it out causing us to have a greater understanding of math.*

Summary

At the beginning of the first semester there was clear evidence of both non-start and quick-stop behaviour in the work habits of these students. The issues related to this were common to both of these grade 10 classes, with a lack of self-confidence being a common theme. Poor core skills and organization was a particular problem which led to students having difficulties when working on problems and often they had a good strategy but were unable to execute it. At the beginning of the year the students engaged in no reflection of their work and were unable to critically examine solutions they achieved. A sense of learned helplessness was more apparent in the actions of a few members of this group than was readily apparent in the grade 11 or 12 class. This was particularly apparent at the beginning of the year when students were first given problems to work on.

I was encouraged by the success in identifying and helping to alleviate some of the problems students had around initializing their problem-solving process. It seemed that focusing on this aspect of problem-solving and working to reduce the impact of negative emotions at this stage was having a positive effect. A de-emphasis on a single answer and an emphasis on structure and reflection through the metacognitive procedures seemed to be enabling students to start and persevere with problem-solving. An examination of students' beliefs indicated an encouraging shift in an area described by researchers as difficult to make progress in.

Chapter 8. Initial Theory based on Early Observations

"Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost."

Anglin, W.S "Mathematics and History", *Mathematical Intelligencer*, v. 4, no. 4. (1978)

In this chapter I examine common data collected across all three grades with regards to their opinions on their own problem-solving ability. I then present my initial thinking on some of the issues surrounding initialization of thinking in the classroom. The objective of this chapter is to establish some ideas to explore further into the second semester.

How do students view their problem-solving?

Towards the end of the first semester I sought further feedback from all of the grades to get a sense of how they felt about problem-solving. The students were asked three free-response questions and results are summarised across all grades to give a more general picture of any changes which may be occurring.

'What do you think is the hardest part of problem-solving?'

Three key categories emerged from the data and so comments were coded into groupings corresponding to these. For example: '*I start moving ideas around in my head and it mixes me up' & 'Trying to work too fast'* were coded as 'Confusing myself or losing place'*;* '*How to organize my work so I am not jumping from step to step' & 'Knowing which math concept to use*' were examples coded as 'knowing where to start or what steps to take', while '*Understanding the wording and trying to make an equation'; &* '*Figuring out where the information fits into the formula'* were examples coded as 'making sense of the question'. Table 8.1 indicates the percentage response to each grouping (n=71). An individual may have suggested several reasons for their difficulty so the % inclusion is an indication of how often that choice is made overall.

Table 8-1: Students' opinions about the hardest part of problem-solving

	% inclusion in response	% for G10	% for G11	% for G12
Confusing myself or losing place	10	13	9	7
Knowing where to start or what steps to take	60	56	68	62
Making sense of the question	28	30	23	29
Other	2	1	0	2

The table shows the % for all of the students and for each grade separately for comparison. All of these responses are related to the initial stages of the problem-solving process. No student made reference to getting stuck in a problem once they had decided on a method of solution. That is not to say that being stuck doesn't happen of course, or that students do not see being stuck as an issue, but that their primary perception is that getting going is the hardest part of the problem. The elective grade 12 class, made up of higher-achieving students gave similar responses to the mandatory grade 11 or grade 10 classes, indicating that the level of achievement does not seem to be a factor here.

'What do you feel is your biggest weakness in problem-solving?'

The responses were coded into similar categories that emerged from the data (n = 71) and displayed in table 8.2. For example '*Figuring out which technique to use'* was coded as 'knowing where to start'; *'I don't have any faith in my ability to solve math problems'* was coded 'Giving up easily, self doubt, or panic'; *'Not using 3 or 4 techniques'* was coded as 'Can't think outside of the box or only one method'; and '*Knowing how to get an answer but not being able to explain it*' was coded as 'Not being able to explain myself'.

Again, a student may have chosen more than one reason so the percentage is inclusion in all responses. In response to this question there is a shift from the initial stages of the problem to the actual process of the problem. Although their recognition of 'giving up too easily' could relate to any part of the problem-solving process, observing the problem-solving process in action suggests that this occurs fairly quickly.

Table 8-2: Students' opinions about their biggest weakness when problem-solving.

	% inclusion in response	% for G10	% for G11	% for G12
Knowing where to start	28.6	20.6	45.5	30.8
Giving up too easily, self doubt or panic	25.0	32.4	27.3	15.4
Can't think outside of the box or only one method	28.6	32.4	18.2	26.9
Being unable to explain myself	17.9	20.6	18.2	15.4
Other	5.0	2.9	9.1	11.5

However, it is interesting to note that almost one third of the students fell into the category where they recognised a limitation to their thinking process similar to the ideas of the 'Narrow canyon of exploration and Oasis of false promise' discussed by Perkins (2000). One student summed up this category well when he wrote: *"Sometimes I really hate math, it's like a slap in the face when it's so obvious."* The students often commented that they understood the process clearly when they saw it written down or were told, and recognised that they ought to have been able to think of the solution but were 'stuck' in their own method.

What do you think is your first step in solving a problem?

Table 8-3: Students' notion of the first step in problem-solving.

	% inclusion in response	% for G10	% for G11	% for G12
Drawing\visualizing	19.7	14.7	18.2	26.9
Writing out the information	84.5	82.4	72.7	92.3
Looking for key facts	5.6	2.9	9.1	7.7
Look for a formula\equation	41.0	47.1	36.4	34.6
Nothing	9.9	11.8	18.2	3.8
Recognize the problem	2.8	0.0	0.0	7.7
Go with instinct	2.8	0.0	9.1	3.8

Responses to this question fell into seven categories as suggested in table 8.3. For example, 'writing out the information' consisted of several responses of a similar type:

Rewrite the question	Sort the information
Repeat the question to myself	Look for important information
Look at the numbers again	Write down what I know
Make sure I have read it correctly	Reread the question a few times
Write out information	Simplify the information
Put data in a chart/table	Write what the question wants

All these depicted a similar implied starting point and are significant in that nearly 85% of students list this as their starting point. This category is essentially that suggested by the problems solving heuristics of Pólya, and repeated in textbooks such as MathPower (Addison Wesley), as an initial step. However, how does this correlate with what the students actually *do* when they start the problem? The data examined earlier indicates that this process, while may occur mentally, does not initially occur on paper without stimulation.

Developing an Initial Theory

Over the course of the first semester I began to develop some ideas around the process of initializing thinking. At the end of the first semester my contact with the grade 11 and 12 students would come to an end, having taught them every day from September to January. The grade 10 classes, however, being on a linear system (one class every-other day from September to June) would continue. I collated the data gathered from all of the classes (n=71) over the first five months in order to guide my investigation into the second semester with the grade 10's.

My ideas were based on the concept of a 'governor' used in mechanical devices to prevent the device from being over-run. For example, a motorcycle might have a governor attached to limit the speed the rider can go. The student develops a self-governing mode of action over successive grades of education (cf. Malmivuori, 2001). The governor acts as a self-protection device to help prevent the student from feeling embarrassed in front of others and, in some cases, to his or herself. DeBellis and Goldin (1997) use the term *defensive mechanisms* to explain behaviours in this situation. The 'setting' of the governor varies between students and may be consciously applied or subconsciously developed. This setting will determine how long a student is willing to think about a problem before shifting to ways to acquire the solution through other means. I believe that a flexible concept of a governor allows control over the local affect of emotion (DeBellis and Goldin 1997, 1999) to be adjusted on a day-to-day basis. Malmivuori (2001) uses the term "socio-cultural mental regulators" to describe how students' mathematics-related beliefs, and their beliefs about self as mathematics knowers and learners, form a framework for the way they interpret and process mathematical learning situations. Panaoura, Demetriou, and Gagatsis (2009) also use the term 'self-regulation' to refer to the processes that coordinate cognition. In this case the term reflects the ability to use metacognitive knowledge strategically to achieve cognitive goals. I use the term 'governor' to distinguish between the concepts of students' self-regulating their thinking and having their thinking regulated subconsciously or by external factors. In other words, can we as teachers adopt processes that adjust the governor externally and thereby overcome the self-regulation aspect? Vygotsky (1978) saw self-regulation as something picked up informally in the social context, and in order to develop other-regulation students need to engage in other peoples' regulatory behaviours. As such, the teacher has a distinct role to play in demonstrating good regulatory behaviour. The use of the problem solving rubric of Mason et al. (1982) and my developed Organize-Entry-Share (OES) method are then instruments to help adjust self-regulatory behaviour through a sequence of interventions in adjusting the governor, what Schoenfeld (1994) describes as 'pushing the boundaries'.

The position of the governor can be advanced using a combination of methods designed to make the student more willing to start the problem. My research to this point had looked at altering the student's emotions in their approach to thinking. Using methods such as only starting a problem negates the fear that the student will be unable to finish it. Stressing that that process is *as important as* the answer reduces the fear of some students that they make silly mistakes. Working at first individually and then in groups allows a student to see – in a non-threatening way – that their ideas are shared by others, or that they can contribute to a solution. As discussed earlier, creating a non-judgemental atmosphere to students' ideas helps them to feel more comfortable in suggesting ideas.

Summary

Several themes were emerging from the data at the end of the first semester which I explore further going into the second semester. In particular, the apparent contradiction students were expressing between feeling that they were better at problem solving yet at the same time still finding it hard to start, was explored. For the grade 10 students especially there was a distinct feeling expressed (table 8.2) that they found it hard to 'think outside the box' when looking for a solution. This may indicate that the students have a growing awareness of what is required to be a good problem solver and that the traditional 'linear' thinking they had been familiar with was insufficient. The familiarity of the term 'outside the box' must also be considered, so that a deeper understanding of students' applicability of the term must be further explored. From the teacher's perspective, the difficulty lower-achieving students were having in recalling past knowledge accurately seemed to be an important stumbling block in the problem-solving process, and something to examine further. The methodology being employed in the first semester gave outward signs of being successful, but further investigation needed to be done to determine if students' beliefs around problem-solving had changed.

Chapter 9. Grade 10 Semester Two

"Teachers cannot make students learn – at best they can provide well-thought out situations which provide opportunities for pupils to engage with mathematical ideas and develop skills in using spoken and written language to that end"

David Pimm (1994, p. 145)

In this chapter I analyze data collected for the grade 10 classes in semester two. As indicated in chapter 4, the aim for this class was to build on the data collected from the first semester classes. In considering the results from semester one, I tried to focus more on the emotional state of students when given a problem and to develop a better sense of any local changes in anxiety. The data collected around problems and students reflections was analysed, followed by a case study of three individuals who are used as exemplars in problem-solving behaviour related to non-start and quick-stop activities. Finally I examine the data collected from the surveys.

Charting Progress through selected problems:

Table 9.1 gives an example of three problems (See Appendix 3.13) analysed at the end of the first semester for grade 10 students. They were asked to use the OES method (Appendix 3) to tackle the problem. The majority of students were able to write down ideas about the problem to a varying degree.

Table 9-1: Breakdown of problems in the middle of the course.

	Problem	1	2	3
Initial	Description	8	11	9
	Ideas	22	19	24
	Neither	4	6	3
Entry	Stuck	18	12	8
	Unstuck	15	10	8
Reflection	Good	8	10	10
	Weak	19	21	20
	None	7	5	5

In Chapter 7, Table 7.1 showed the breakdown for a typical problem at the beginning of the course. Table 9.1 shows a similar breakdown for typical problems in the middle of the course (***n*** varying depending on absence).

	n =	34	36	35

For the coding, expressions such as: 'I *decided to try to find a way to solve it by measuring the steps...*' without any further exploration before the solution was attempted, was coded as a description. '*Need to create a formula. Need to use f(x), need to find the change and initial values'* is an example coded as 'ideas'.

The table shows how students were becoming much better at expressing ideas at the start of the problem but were still poor in their reflection of the problem at the end. Table 9.2 shows a similar analysis done on a problem (see Appendix 3.15) at the mid-point of the second semester. The majority of the students were by now very good at writing down ideas at the start of the problem. On this problem only three students were still struggling to put down any ideas to paper. This was not to say they were sitting idly as two of these students had diagrams of possible situations sketched out. Only one student wrote down nothing of consequence. Table 9.3 breaks down the responses further to indicate how many ideas the students expressed at the start of the problem. This extra table was added to reflect the improvements the students were showing in the initialization of their thinking. In the earlier problem analyzed in table 7.1 the students listed only one or two ideas before pushing on with an attempted solution. Comparing table 7.1 to table 9.2 there is an indication that the students are continuing to improve in putting down their ideas in the initialization stage of the problem, and are also performing much better in the reflection stage.

Table 9-2 & Table 9-3: Breakdown of a problem in the middle of the second semester (n = 30)

Initial	Description	7
	Ideas	18
	Neither	3
Entry	Stuck	13
	Unstuck	4
Reflection	Good	19
	Weak	9
	None	2

ideas	# of Students
2	3
3	5
4	5
5	2
6	1
8	1
10	1

Table9.2 and 9.3 breakdown a problem used at the mid-point of the second semester. Table 9.3 indicates the number of ideas put forward by students alongside how many students wrote this number of ideas.

In this problem the students were coded for a good reflection if they discussed ‘how they knew their answer was correct’, ‘what they had learned’, and ‘how the problem related to the concepts developed in class.’ These were established prompts expected of the students so the results are also an indication of the students knowing the expectations better. Another indication of the development of the students was in the number of ideas the students were now writing down as part of the ‘*organization’* stage. At the beginning of the year the students would typically write down only one or two ideas. As table 9.3 shows the majority of the students were now writing down three or more ideas from which to work. Other problems done at this time demonstrated a similar trend and were encouraging in the way the students were initializing their thinking and reflecting about their ideas after the lesson was over.

Charting Progress through journal comments

As mentioned earlier, one aspect of the problem-solving process I wanted to pay particular attention to was the emotional local affect in starting. The results from semester one had suggested that, despite feeling they were better problem solvers, the students still found getting started a challenging process. A sense of the emotional state of students can be gathered by observation, but I refer to their journals for more concrete evidence. Early in the second semester, one student writes: “*This class hasn’t made me love math and consider it easy all of a sudden but it has made me think about things that I never would have on my own.”* This is an interesting comment which seems to speak to the conflict in emotion the student feels in the class. In a similar vein, another student noted: *“I feel a little less frustrated. I am slowly building confidence in my ability. I am aware of some better strategies for getting unstuck, I just have to persist. I need to get over feeling stupid and frustrated when I am stuck so I don’t quit. When I am successful it helps.”* A third student added a comment suggesting she was still struggling with her emotions: "*"I think I do have some better strategies but I feel the same frustration".*

Towards the end of the semester the students comments were becoming more positive: "'*If I'm stuck I like to think about the concepts*" and "*I'm no longer so hard on myself when it (getting stuck) happens."* Another student adds: "*Now I see that math is something everybody can think about. Thinking in some classes is just as valuable as doing."* However, the feeling was not universal, as one student noted when asked if he felt any change in his emotional state when presented with a new problem: "*"Not much change in my emotional state. I still do not like doing problems."* Perhaps more telling was the comment: "*I like it when I get it (the answer), but seeing the problem always makes me scared*." One student clearly still felt the pressure of expectation, even though the answer was de-emphasized: "*Math is a subject that does not come easy to me. It takes me longer to figure things out. I don't think I could do this on my own."* This comment is interesting as it comes as part of a solution where the student, in a group situation, achieved success in the problem. It seems clear that this success was still not altering her belief system about her ability to do mathematics. Another student reflected, when asked about his concerns over starting a problem: "*Math that I feel I don't know is scary to attempt. It makes me embarrassed."* He then added: "*I know that math is about thinking and not just doing, but I still want to get the answer."*

When asked, collectively, if they thought getting stuck was a normal part of problem solving, the students would all agree that it was. The earlier comments indicate that the students felt that they recognized the value of being stuck, but were still getting frustrated by being so if they could not quickly get 'unstuck'. As one student commented "*'I'm fine trying – it's once I get stuck that it's hard for me. I write what I think now, and I do try to think about the concept".*

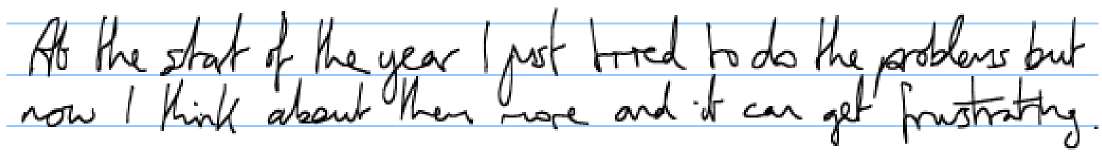


Figure 9-1: Getting frustrated thinking

At the end of the year one student reflected, see Fig. 9-1, that he was thinking about problems more and this was getting him frustrated.

Charting Progress through individual case studies

In this section I examine three case studies which I believe illustrate different aspects of thinking in the mathematics classroom. Stuart, an individual with latent ability who was finds difficulty through a lack of self-belief; Brittney, an individual who demonstrates a high level of anxiety and frustration in mathematics - but also an underlying desire to do well; and Jeanyus, who demonstrates little interest in the subject and could not be motivated to move much beyond his viewpoint.

The case of Stuart

Stuart stood out from the class at an early stage as a student who was quite sharp in his thinking and well focused in the classroom. Stuart also started problems well but reached an impasse where he seemed to lose confidence in his solution method. Stuart tended to shut out those around him when he felt frustrated by a problem. His frustration level grew if the partner(s) he was working with seemed to understand the problem he was struggling with.

In one particular class Stuart sat near the front and I could hear his mutterings quite clearly: '*How am I supposed to do this?*' he asked himself repeatedly. But he was not asking the question in any meaningful way, it had become a repeated mantra that was isolating him further from the thinking process that would help him. He had frozen himself into a state of inactivity because there were no – as he perceived – clearly defined steps for him to follow. He was, as Debellis and Goldin (1999) note, experiencing frustration which, without intervention, would lead to anxiety. This was the third and final lesson as part of an assignment on area and volume related to exponents done early in the first semester. Where they had been working in a group to investigate the properties, they were now asked to individually summarize their findings. I quietly walked up behind Stuart so that I could see what he had done and saw that he had all the information before him, all done correctly. All he lacked was the final summary. I asked him what was troubling him:

"*I don't know what to do,"* he replied, *"nothing makes sense."*

I repeated the instructions that he was looking for a single equation which could describe the area of any shape but I knew this would not suffice so I pointed to the work on his page.

"*What's this equation for?"* I asked.

"*Area of a triangle"* he stated, and I went through several of the shapes he had in a similar way before asking what he had noticed. *"It's the same equation every time"* was his response. "*But is that it? That's all you want?"* From this, and other observations, I recognized that Stuart lacked belief in his own ability. In a different lesson I recorded him saying to a partner: "*These problems make me feel dumb. Even when I get them I don't think I do.*"

Stuart found mathematics problems frustrating if he couldn't get an answer because he felt, in his own words "*dumb"*, and frustrating if he could get an answer because he didn't believe it could be a solution. I had noticed Stuart in previous, less formal investigations, as someone who became frustrated with a problem easily. He seemed a capable student but his frustrations soon rose to the surface when he was given an open-ended question. It seemed that Stuart did not expect to think about what he was doing. He wanted to be given questions that fitted into a well defined role and was uncomfortable if there was no answer he could check with. His sense of the didactic contract in a mathematics class seemed to be that I, as a teacher, was supposed to give him a worksheet of questions based upon a concept I had taught that was closely related to the relevant pages in his textbook. Diversion from this process led to bewilderment and frustration. '*If you just tell me the answer you want then I can get it.*' He demanded at a later time '*I like to work backwards*.'

Stuart is an interesting case because he was capable enough to feel that he should have been able to do the problems. He was clearly used to focusing on the answer and did not yet value the process he developed towards the answer. Stuart seemed open to a change in the way he approached problems. At a later date I noted him as singling out a student who declared himself as 'open to help' in a problem and asking the pointed questions 'What *are you doing? How do you know that?*'

Stuart was an example of someone whose initial thinking skills were strongly affected by the level of his self-esteem and past experiences. Stuart had a self-imposed governor to determine how far he would push his thinking, but it seemed that this governor could be adjusted by helping to remove the frustration and increase the self-belief in his work. Stuart had to recognize that there could be a better way to approach his thinking process but the key thing was that he seemed willing to do that. He had to realize that getting stuck is normal and his bewilderment could be a step in the solution if he managed it properly by asking himself good questions. He also needed to feel more comfortable using his peers as support when he was frustrated in a problem. This was a slow process but in comparing notes about his behaviour in open-ended problems towards the middle of the second semester and the early part of the first semester he was demonstrably less frustrated. He no longer muttered to himself in isolation but rather engaged in discussion with the members of his group. He was still focussed on getting the answer and was not satisfied unless he did, but he seemed more able to step back and re-focus his thinking than he did at an earlier stage. By the end of the year Stuart had become one of the better problem solvers in the class.

Choosing Stuart

What does the case of Stuart tell us? Stuart was not unique in the way he was responding to the problems given, but his frustrations were more obvious than other students. This made him an easier student to monitor and take notes on during class. I believe Stuart is a good example of how change can be affected in the initialization of thinking at the point where the student moves from mechanically approaching a solution to needing to think about that process. This cases illustrates how the shift can occur in students who demonstrate a capable nature in mathematics and who have achieved success to a certain point by focusing on a procedural method while at the same time recognizing the limitations of their thinking. I have used Stuart to exemplify this, but similar traits were noticeable in other students.

The following case was chosen to illustrate the type of student who suffers from a lack of self-confidence and had been performing poorly in the subject.

The Case of Brittney

Brittney was a student who stood out quickly as someone who was very insecure. In Brittney's case her insecurity often came in the form of verbal comments where she would profess to be "*not very good at math*." This was Brittney's second attempt at grade 10 and she was quickly overwhelmed by any complexity in a problem, giving up almost instantly at the start of the year. In an early journal entry she would write: "*The thing is, I keep blanking out. Even if I learn it I keep forgetting it.*" She did not get frustrated with her situation, or blame others, but quickly reached a limit to how far she developed her thinking - she exemplified the idea of quick-stop thinking. Often, it seemed she had the correct answer but lacked faith in her process. On a test reflection exercise early in the year she wrote:

In this question ~~an~~ I had the right answer circled and thought it was to easy to be right so I second guessed myself again !! Next time I need to NOT second guess myself!

Figure 9-2: An example of second guessing

In a journal entry a prompt was to write what she thought was her biggest weakness in mathematics, her response was "*The biggest problem I have and weakness is the fact that I doubt what I do. When solving a question I'm always doubting myself. And thinking what I am doing is not accurate."* On other occasions she would get flustered by the question (Fig. 9-2)

I think that for this question I didn't listen to the wording and write down the formulae and got confused. Next time i'll make sure i put the numbers into an equation, understand it and then figure it out.

Figure 9-3: An example of getting flustered by a question

Brittney would often stay behind to ask questions for clarification but often did not seem to be listening carefully to what was being said. On one such occasion I noted her as saying "*I got it, it's all so easy now. Wait, I forgot again."* Despite this, it was apparent that Brittney also wanted to achieve some level of success and was not content with her own perceived shortcomings. She responded very well to any success she did have and would always exclaim happily when she was able to understand something, often requiring a 'high five' from her usual working partner.

At the beginning of the year she wrote that her first step in solving a problem was to write down "*what I know from the question*", a similar response given by many of the students, but in observing her working at this stage this was certainly not the case. Her initial problems contained very little initialization at all and she frequently fidgeted around her materials before settling down to any kind of thought. When she worked with a partner she would not contribute to the initial stage of the solution at all. As I observed her progress however she began to be more comfortable in working with a group, sharing ideas and making suggestions, but still found it difficult to commit any ideas to paper – even a listing of the information. In the exercises at the start of the year where students were only asked to write down ideas about the problem rather than to solve it she found it very difficult to add anything to the blank page. I encouraged her to put down whatever thoughts she had about the problem. This seemed to help her and for the problem in the first test she was able to at least start with some description about her ideas. This process continued so that by the end of the first semester she was quite fluent in this process. For the problem in the mid-year exam she wrote:

> *Just reading the title 'Do Not Start[7]....." made me a bit scared about how difficult and lengthy this question would be. After reading the question I started thinking about ways I could solve the question. I realized I was going to have to take out the spool in the middle if I wanted to solve for the tape because it would have to be subtracted from the total of the tape.*

On observing Brittney during the first semester it was apparent that she was spending more time in the initial phase of the problem, and she was able to get further into a problem before her expressing confusion. She was also much more communicative when working in a group setting and was encouraged in this regard by receiving positive feedback from other students in the class. She still had a tendency to declare herself 'bad at math' in the class setting and at one point during a class I asked her a question, to which she replied "*You're asking me?*" She looked around the class as if to seek confirmation of the foolishness of my request. Fortunately, and significantly, nobody seemed amused.

As the class did more problems using the OES method she became more able to write down ideas in the organize stage (5 minutes working on her own), rather than describe her process. The following extract is from the 'Cornered Pole' problem done at this point in time and also serves to illustrate how Brittney had developed her thinking using the OES method.

[7] The open ended problem on a test always began with the instruction not to start until the students had completed the other questions on the test. This was done in response to an earlier occurrence when students would attempt this question first and spend too much time on it at the expense of the other questions.

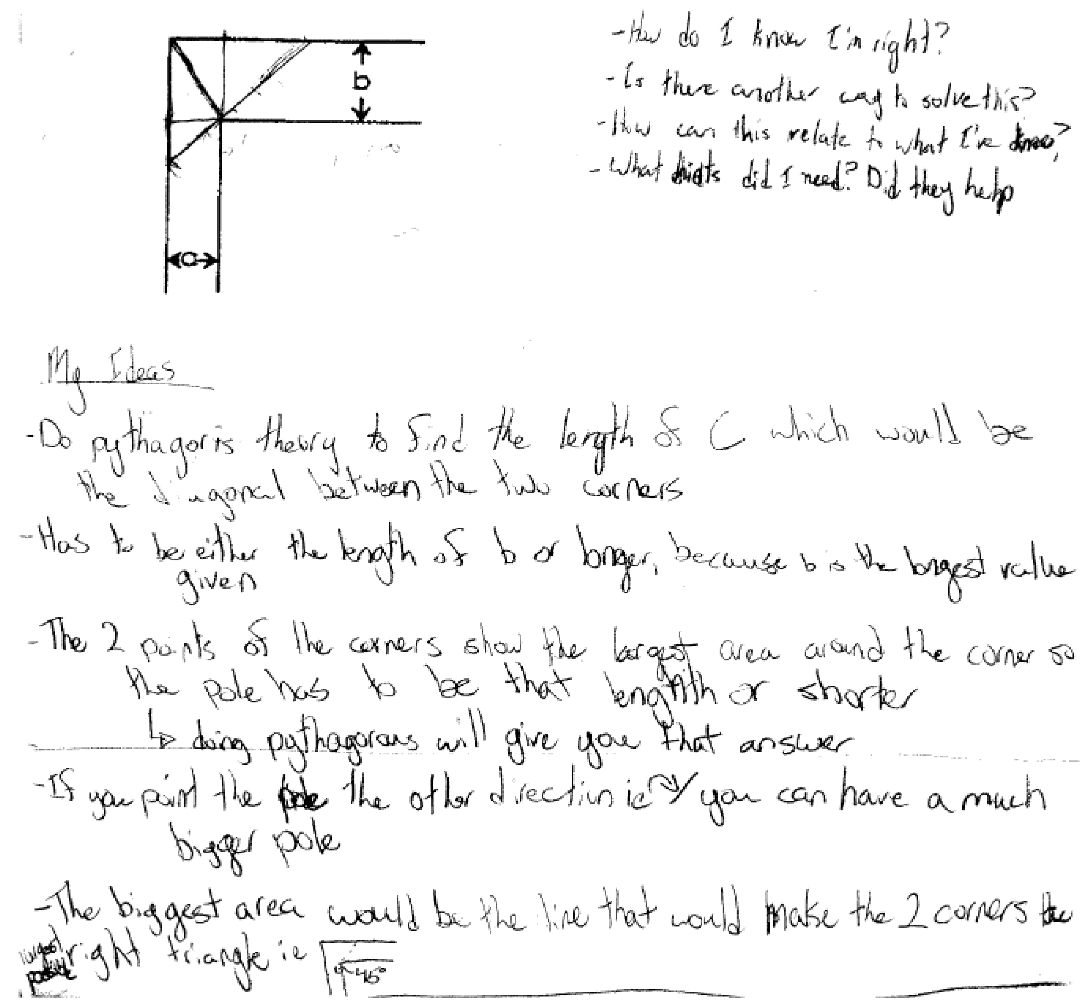


Figure 9-4: An example of the Cornered Pole question

By the middle of the second semester her work had developed well and it was noticeable that she set to work immediately and utilized the full time productively. She now contributed to the 'Share' stage of the problem-solving process with much more confidence.

Choosing Brittney

Brittney's case indicates that it is possible to work with students who lack confidence and structure and encourage them to start problems. In doing so they become more engaged in the process and there is a much greater chance that they will continue with the process. Although Brittney represents one particular student who demonstrated a very tangible behavioural shift there were elements of Brittney in several students to varying degrees. In this respect I believe Brittney is an example of how the teacher can positively intercede in the emotional local affect around initializing solutions.

The Case of Jeanyus

Here I present the case of a student who was noticeable because of the completely negative attitude he demonstrated towards the subject, illustrating the difficulty in helping to initialize thinking in such an individual. Jean stood out immediately as a student who had affected a similar air of disinterest, despite scoring well on the first test. Jean made it very clear from the onset that he did not wish to cooperate with anything which would cause him to do more than the bare minimum. In the early exercises where I asked students to listen to, or read, a problem and simply write down their ideas he typically responded with a single sentence such as "*Lay out information in a diagram. Because I don't know what to do, and making a diagram makes it look like I'm moving forward in the question."* At other times he would simply doodle or chat and make no attempt to address the problem. Jean was a student who liked to play up to a certain image and from experience I know it is important to work with that image rather than against it, so I had some early discussions with him where I thanked him for his honesty in these responses and told him that it was really helpful to me to see what him was thinking. I asked him to try to develop that more to give me a better insight to his thinking as he was progressing in a problem. His early contributions were still hit-and-miss, depending on his mood for the day, but I observed a shift from a non-start behaviour to a quick-stop behaviour as the class did more problems. His entries were often brief but showed some engagement, "*Write out the information in a more organized fashion. Because this way is too complicated to read, it needs to be in a chart format... Write some sort of formula.... to understand it?"* At this point he stopped, having given little time to develop his ideas further, and waited until I allowed the class to discuss the problem. In discussions he sometimes demonstrated a willingness to participate but often would see the time as an opportunity to chat. When asked to reflect on an early problem session his response was "*I didn't learn anything other than a monkey asking another monkey about percentages doesn't get either monkey anywhere."* He typically resorted to writing 'comedy' when he was stuck in a problem, such as suggesting that, in a problem about the logic of moving soldiers around a fort (see Appendix 3. 17) to keep a minimum number on each side, '*They should make some sort of paper mache dragon... so that the attackers will be intrigued by a giant mythological beast....'*.

With time, however, his quick-stop thinking delayed and, on some occasions, he worked fastidiously at a problem. In a journal entry he wrote:

> "*What works best for me is laying out the information in a format that allows me to see a formula. If I can see all the information properly I can better understand what formula and method to use in a problem."*

And later:

"It is most difficult when I do not know where to go with the problem. If I have no insight as to how to begin the problem, I am obviously stuck. When there is no method in sight that I can use, I just get frustrated and give up. "

What was clear from his journal entries was that he felt he needed to be able to see a formula quickly in order to solve the problem. I discussed this with him on several occasions and talked about the value of exploring options and developing ideas but he still saw mathematics as a study of equations. When he could plug numbers into an equation he felt he was having success. Given that the bulk of the Grade 11 course is centred on systems of equations his viewpoint was understandable and hard to shake.

I had selected some problems to teach through with this grade, such as introducing the unit of functions through a problem regarding a car's breaking distance. My hope was that in relating such problems to a 'real-life' situation, students such as Jean would respond more and be better engaged. While this was so to a degree, the problems an individual student seemed to engage with most were ones that intrigued them in some way, and most often ones for which they could see some path towards the solution. Jean became particularly engaged in a problem on shapes one day, despite the fact that he was not making a great deal of progress in the problem. He starts the initialization of the problem by recognizing its intent: "*This is somehow a function. It's trying to be a composite function, but how is this property shown?"* Later, after putting down other ideas which skirt around the core of the problem he writes: *"I got stuck. I'm still stuck. I can't get unstuck at the moment, really wish I could though."* He moved on to discuss the problem with another student after about 15minutes of thinking through it by himself, much longer than he would usually give a problem he could not solve. The discussion was on-task and lively and involved several scribbled drawings before the capitalized "*COUNT SIDES!! You have to count sides and put inside shapes to the power of outside shapes!!"* As part of his reflection he wrote: "*Well, that was really obvious but so hard. It's like a Where's Waldo book. He's there but the sneaky son-of-a-gun always seems to elude you."*

This situation, which occurred in similar ways to many of the students on one or more of the problems, illustrated that it is possible for to get drawn into a problem which can be quite abstract and not necessarily straightforward. Something in the problem seems to hook the student and it is not necessarily that the problem has some meaning or application to them. Where possible, giving the students a selection of problems to select from helped foster this approach, and Jean would be less likely to demonstrate quick-stop behaviour in these situations.

Choosing Jeanyus

Jeanyus illustrates the difficulty in changing beliefs a student has developed. I felt there were positive gains in his thinking, especially in the initialization stage, and he worked longer at problems - showed less of a tendency for superfluous comments by the end of the semester. He would still have days when he was totally disinterested and lacked focus, and his progress in solving problems was difficult to assess, as he seemed to solve about the same ratio as he started throughout the semester. However, I would suggest this case confirms is that progress can be made in changing beliefs using these methods, but that it does take time and is something that needs to be stressed from an early stage.

Examining the Surveys

To further investigate the changes in anxiety and beliefs suggested by the grade 11 and 12 students, the grade 10 students were asked to redo the Anxiety and Mathematics Beliefs Surveys towards the end of the course. These were not returned to the students so they had an eight month gap between the two sessions. In the case of the grade 11 and 12 students I had been surprised by the increase in anxiety expressed when starting a problem, and I was interested in seeing if this was an anomaly or a pattern of behaviour emerging.

Table 9.4 shows the mean score for each question in the survey (n=34).

Table 9-4: Mean sores per question for the MARS-R

Q#	1	2	3	*4*	5	6	7	*8*	9	10	11	*12*	*13*	14	15
Sept.	1.5	1.8	2.1	*3.0*	1.8	2.1	1.3	*3.0*	2.0	1.7	2.4	*3.6*	*2.7*	1.7	1.5
June	1.9	1.6	1.8	2.9	2.0	2.3	1.8	3.3	2.0	2.3	2.8	3.5	2.8	1.7	1.8
	Looking at text	Coming into class	Read a formula	*Think about Test*	Watching teacher	Algebra explained	Start hw	*Writing Test*	Reading a graph	Start a problem	Difficult Hw	*Test returned*	*Stuck*	Start new chapter	Listen to student

Once again there is an apparent *increase* in anxiety over starting a problem, despite all the work done to support this area. Examining the individual ratings for this question shows that 17 students indicated a rise in anxiety (9 by one point, 8 by 2 points) representing 50% of the students. Only 4 students (12%) indicated a single point reduction in anxiety. It must be noted that the majority of the increases were from a level 1 to 2 and as such represent little change and of course could not go down. However, for those students who had rated their anxiety at level 3 from the start, one had increased to a 5 and 2 had stayed level. Students who had rated their anxiety at level 4 maintained that level.

Table 9-5: Grade 10 Comparison of Mathematics Beliefs Surveys

Please respond to each statement		Strongly agree	Some-what agree	Neither agree or disagree	Some-what disagree	Strongly disagree	*Mean*
Solving a mathematics problem is demanding and requires thinking	**June**	**19**	**13**	**2**	**1**	**0**	***1.6***
	Sept	*13*	*15*	*5*	*0*	*1*	*1.9*
I start problems and questions confidently	**June**	**3**	**17**	**5**	**10**	**0**	***2.6***
	Sept	*9*	*14*	*10*	*1*	*0*	*2.1*
I am often unsure how to start problems assigned in class.	**June**	**3**	**12**	**5**	**14**	**1**	***2.9***
	Sept	*1*	*6*	*11*	*13*	*2*	*3.2*
I like to know the answer to a problem before I start	**June**	**2**	**11**	**10**	**9**	**3**	***3.0***
	Sept	*3*	*7*	*8*	*13*	*3*	*3.2*
It is a waste of time when the teacher makes us think on our own	**June**	**0**	**2**	**10**	**9**	**14**	***4.0***
	Sept	*1*	*3*	*6*	*12*	*12*	*2.3*
When I get stuck on a problem I tend to give up quickly	**June**	**0**	**8**	**8**	**14**	**5**	***3.5***
	Sept	*0*	*3*	*9*	*15*	*7*	*3.8*
I tend to avoid starting a new problem until other students have done so to give me a clue.	**June**	**0**	**3**	**12**	**11**	**9**	***3.7***
	Sept	*0*	*1*	*7*	*15*	*11*	*4.1*
When I get stuck on a problem I have good strategies to help me.	**June**	**1**	**16**	**12**	**6**	**0**	***2.7***
	Sept	*3*	*15*	*11*	*5*	*0*	*2.5*
I have found writing out my initial ideas first helpful	**Sept**	**6**	**10**	**12**	**7**	**0**	***2.6***
My problem-solving has improved since the beginning of the course	**Sept**	**9**	**19**	**6**	**1**	**0**	***2.0***

The Beliefs study (table 9.5) was given with a particular interest in matching responses to the surveys done with the other grades. Once again I present here those questions, picked from the full questionnaire, which are most relevant to this thesis. The top row for each question represent the number of responses from the final questionnaire (n = 35), while the bottom row represents the responses from the initial survey (n = 34). The last two questions were added along with the free response prompt: '*Can you suggest some reasons for your answer to the last question?'* This prompt referred to the last question at the bottom of this list.

These results are very interesting in that 28 of the students (74%) of the students felt that their problem-solving had improved while at the same time the individual categories suggest the students are more aware of their weaknesses in problem-solving. There is a *reduction* in the students' confidence in starting a problem and an increase in the students' awareness that they are unsure of how to start a problem and give up too quickly. There is also a stronger belief that solving mathematics problems is demanding. The fact that 74% feel that their problem-solving is better is an indication that, with this awareness, comes a belief that they have made progress in this area. For the free-response question asking them to what they attribute their selection for the change in their problem-solving ability, the results were coded into the following categories:

Table 9-6: Responses to the change in problem-solving ability

We have done more problem-solving	10
Learned to think better about the problem	7
Learned new methods or strategies	7
Increase in confidence	4
Learned new methods	4
No difference	2
No response	2
Was better before	1

Table 9.6 indicates an even stronger response with only five students not responding in a positive way, indicating 86% of the students suggest some improvement. Of these, 23% (20% of all students) attributed their improved success to learning to think more about the problem. In addition, the penultimate row in table 9.5 indicates that 46% of the students felt that writing out their initial ideas was helpful, while only 20% 'somewhat disagreed'. The OES method is seen as a helpful method by the students who feel an improvement in their problem-solving

skills, while at the same time a significant number actively recognize an improvement in the thinking skills in an area which is perhaps more challenging than they had originally believed.

Summary

The grade 10 classes observed over the full year indicated results similar to the grade 11 and 12 classes in semester one. They exhibited similar issues around initializing problem-solving but were able to develop in this area with a greater focus on, and practice in, doing so.
Table 9.6 indicates that the grade 10 classes had a similar response to the grade 11 and grade 12 classes in regards to their beliefs about starting and progressing in problems. The pattern is repeated that they generally report feeling less confident about starting while at the same time acknowledging that they are better at doing so. Their reflective comments suggest that there is a high level of anxiety – often expressed as frustration still being associated with problem-solving. Gaining success at this stage improved their belief that they were better at problem-solving, and had better strategies, but there was clearly still anxiety over the process.

Chapter 10. Discussion

"Everyone can start. "

Mason, Burton and Stacey, 1982. (p. 1)

The aim of this study is to examine students' initial thinking process when given a problem in a mathematics class. What factors affect the reaction of a student to the situation, and what can be done as a first step to help the student without leading them into the problem itself? How does the student begin to approach the problem, if indeed they do so at all? If they do not make any serious attempt to solve the problem then why do they develop this 'non-start' or 'quick-stop' behaviour? Several themes were emerging from studying this issue over the period and I analyze them further here. In addition, I compare the data collected from the surveys across all grades in the study in order to explore further the ideas discussed in chapters 5 and 6.

An examination of themes emerging from the study

Anxiety

The students indicated levels of anxiety centred on test taking, which is to be expected, but there were higher levels of anxiety reported around starting and being stuck in a problem as well. As a group the students did not indicate an excessive level of anxiety but some students did express a high level of anxiety which was not necessarily related to their performance in class. The effect of anxiety on test performance can be seen from the following post-test reflection:

> *For question 6, I just got really flustered and stopped thinking of what we could do and started thinking about all the things I didn't know.... after the test when I saw what we had to do I felt very stupid. For the next question I also got flustered with not knowing how*

to do the other question and I didn't once again think about that kind of factoring and just looked for a common factor. I just started thinking random things.

The above example also shows how the student felt that being 'flustered' on one question caused a repeat performance in the next question. Another student wrote:

I think that seeing this problem as something 'new' or 'scary' was what affected the way I looked at it, I think that instead of going into the problem thinking FACTOR I went in thinking scary/danger which affected the way I solved it.

In an analysis of comments from the students' reflection on their first test the grade 11 class indicated anxiety levels affected their performance in 3 cases (n = 12). Of these three one achieved a mark of 92%, while the other two achieved 56% and 51%. Only one student in the grade 12 class (n = 26) expressed feelings over anxiety and this student achieved a mark of 76%. In the grade 10 classes (n = 36) feelings of anxiety were expressed by 6 students with marks of 66%, 66%, 68%, 73%, 81%, & 83%. That any student should feel anxiety in a test is probably a natural occurrence, but is noteworthy in cases where the student acknowledges its affect on their performance. In test reflections towards the end of the semester comments in regard to anxiety affecting performance were not made by any students. This is not to say that students did not feel anxiety, but their lack of comment on it may be suggestive of a higher level of *management* of that anxiety.

In a tutorial with one student in September the following recorded conversation illustrates the nature of the anxiety for a particular student:

Student: "*If I can see things visually then I can maybe do them.'*

Teacher*: "What did you feel when you saw that question?"*

Student*: "Scared. I wanted to stop. "*

Teacher*: Did you stop and think anything about the math?"*

Student*: "No, I didn't know the minuses on the power."*

Teacher*: "Can you remember anything to do with them?"*

Student*: "Add to multiply."*

Teacher*: "That's right, and to divide?"*

Student*: "Subtract."*

Teacher*: "Good. So all that was stopping you was one part – a piece of knowledge?"*

Student*: "Yeah, I guess I could've done it. I didn't see it through. That's what happens."*

Figure 10-1: Illustrating anxiety

Fig. 10-1 illustrates how quick-stop behaviour can be a function of the student's perception of their own knowledge base. The student actually knows the answer but lacks the self-belief to acknowledge that. This in turn has led to a rise in anxiety level on seeing the question followed by a cessation of thinking. Occasionally students would make comments in their journals or in the reflection of problems which also allude to anxiety when they are stuck in a problem. For example, a student writes: "*Not being able to think of a way to solve the question I freak out."*

Lack of confidence, or self-belief.

> *I got stuck but I unstuck myself by thinking differently; however I am still unsure of my solution and (am) worried about my thought being the wrong thought.*

This quote shows how a student lacked the belief in her own knowledge to allow her to solve the problem, in this case based on one small aspect of the problem. This was seen earlier, where the students doubt the validity of their thoughts and are not used to asking themselves questions such as 'how do I know if I am right?' A familiar theme was evident in the work of the students, both in terms of their written work and the own comments, that the students often had good, and even correct, ideas about the problem but failed to pursue them.

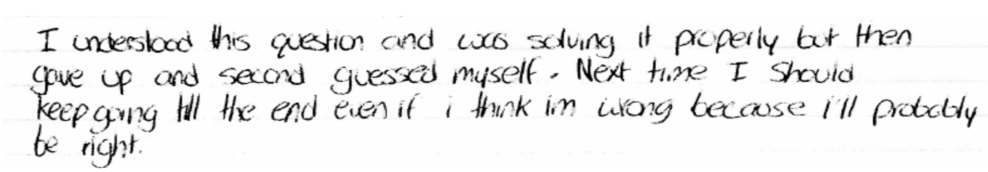

I understood this question and was solving it properly but then gave up and second guessed myself. Next time I should keep going till the end even if i think im wrong because i'll probably be right.

Figure 10-2: Failing to pursue an idea

In students' reflections about problems, comments such as shown in Fig. 10-2 were common, and supported by the partial work they had shown. This was sometimes evident when the students were asked to write down their initial thinking about a problem (comments such as '*I think I'm weak in the area of math itself', and 'My biggest problem is that I doubt what I do and thinking what I am doing is not accurate'* occurred several times in early problems) but was more often heard when students were discussing the problem. The following extract is typical of conversations I recorded with students, writing it down immediately after the particular student returned to his seat:

Student: "I don't understand this question."

Teacher: "Okay, what is it in particular you don't understand?"

Student: "Any of it."

Teacher: "Can you tell me what the question is asking?"

Student: "It wants the next number in the pattern."

Teacher: "What pattern?"

Student: "The tiles in the path. I know the number goes up by eight but I don't know how to write the function."

Teacher: "Can you remember what you need to make the function?"

Student: "No (pause). The change, (pause) and the start."

Teacher: "Good, anything else?"

Student: " (pause) the name."

Teacher: "That's right. So, what's the change?"

Student: "It goes up by eight each time."

Teacher: "Each time?"

Student: "Each time you add more steps."

Teacher: "So what's the function? What do you have written there?"

Student: "8n. I thought it's 8n plus what it started with, but I don't know."

Teacher: "Why not?" (Student shrugs) "How could you check?"

The student clearly had the correct function but doubted that it could be correct. This lack of confidence may be related to an internal sense of not fully understanding the concept, even though the stated facts could be recalled and linked procedurally to previous examples. Students who exhibit this kind of doubt may in fact be indicating more signs of thinking about the problem than others who simply accept their first answer. Consider the following response by one student as part of their reflection when asked to respond to the question '*how do I know this is right?*'

> *I know that this answer is correct, because simply by applying my knowledge of coordinate geometry I understood that .... I also used my knowledge of trigonometry and the Pythagoras theorem in order to help me solve the problem.*

Figure 10-3: Assumed correctness

The student is basing their assumption of correctness solely on the fact that they had used the mathematics and not whether the answer made sense. This type of thinking is particularly evident when the student uses a calculator to determine an answer. I noted a conversation with one student who was adamant that their answer must be correct *because* they had used a calculator to solve the calculation.

In problems posed in the early part of the first semester responses were common whereby the student believed their answer to be correct simply because they *had* an answer. One student writes:

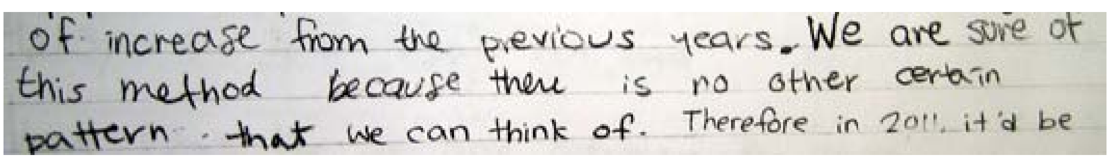
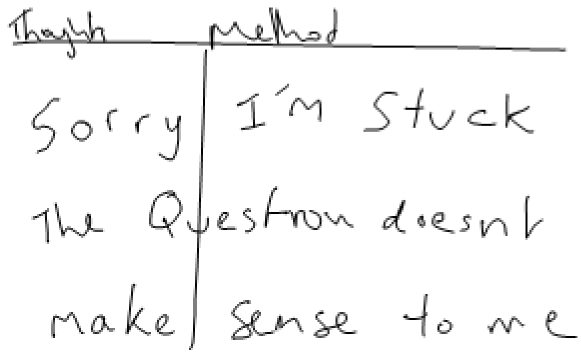

of increase from the previous years. We are sure of this method because there is no other certain pattern that we can think of. Therefore in 2011, it'd be

Figure 10-4: False confidence

The student indicates confidence in their answer by being unable to think of an alternative rather than having a sense of correctness.

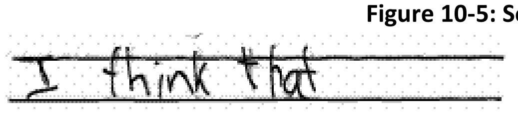
Lack of any systematic method of approaching a problem.

There was, in many cases, a reluctance to write down initial ideas which was only slightly showing signs of improvement at this point.



Many students also left the initial thinking area blank, or wrote nothing of consequence. Several simply repeated the question in the early part of the year.

Figure 10-5: Sorry I'm stuck



The rest of the area was left blank.

Figure 10-6: An example of leaving thinking blank

Fig.'s 10-5 and 10-6 indicate responses which were common at the start of the year. The students were either not comfortable with writing down the ideas they had, or were engaged in non-start thinking. Fig. 10-5 shows an example whereby the student makes a start to writing but can go no further. In other cases the 'ideas' were nothing more than a repeat of the question. Many of the students were unable to step back from the problem to think about the related ideas before starting, or expecting to start, a possible solution.

Lack of background knowledge.

Half-remembered rules and procedures from previous grades seem to be a constant issue for students. When the student tries to remember procedures for problem-solving, or their success is based upon finding identical problems with different values, then there is limited understanding and these methods are easily forgotten. Worse, half remembered rules seem to become concrete in the students minds as being correct and are hard to replace. Even when the students recognize the error they often seem to fall back to it in stressful situations such as tests. An example of this which was often seen was the inability to multiply binomials. Questioned individually the students could always recite the FOIL principle but frequently failed to use it in a question, often going with $(a + b)^2 = a^2 + b^2$ or the equivalent. As the grade level goes up the foundation of required knowledge obviously increases and it is often the case that a student demonstrates knowledge on the current topic but gets hung up with not knowing earlier material. This was often the case with students in the grade 12 class. Consider the following recorded exchange:

Teacher: Where does the $A = 2\pi r^2$ come from?

R: it's the area for the circle of the base. I know that.

Teacher: How can you be sure that's right? How can you check it? You're memorizing a formula and hoping it's right?

R: Yeah

Mixing up the equations for area and perimeter of a circle seemed a common problem which led to incorrect answers or calculations which became bogged down with awkward numbers. It was also apparent that for several students formulae were only linked to specific cases rather than a general application. For example:

A: So the idea is to create a formula? Don't we already know a formula? It's ... for a right triangle. But then for this I have to first find out this and subtract it from that.

Teacher: So why do you think this triangle is different from that?

A: Well, because..I don't know... it doesn't look proper.

Teacher: What's the formula for that (indicating a right triangle)?

A: Base times height.

Teacher: Why?

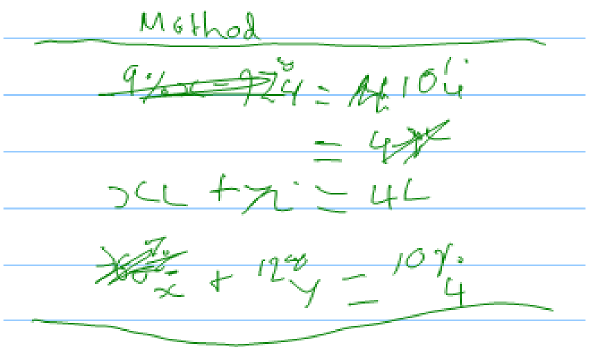
A: Don't know.

Teacher: And that's what you are trying to find out.

In many cases students felt they already knew a formula, even though their understanding of it was not there and their ability to apply it generally was lacking.

Poor Structure

Lower achieving students often make minor arithmetic errors which undermine their ability to complete a problem, even when they know how the problem should be solved. Typically these students have a poor understanding of basic arithmetic operations.



Weaker students often skip steps or litter their page with crossing out and random calculations. This approach often gives rise to arithmetic errors which interfere with the problem-solving approach.

Figure 10-7: An example of skipping steps

As a result the question becomes bogged down in an increasingly complex calculation which puts the student off their chosen method, often resulting in the student giving up on the problem.

In one example I sat with a student who persistently had difficulty rearranging equations. I walked her through the process, indicating the importance of not skipping steps. As I watched her working on her own she was able to get the answer correct every time. When I left her to work alone, however, she was again skipping steps and again getting the answer wrong. This continued to be a problem for her throughout the semester even though she recognised and agreed on the cause of her problem. She seemed unable to break the bad habits she had developed in the past. When later asked to reflect about her biggest problem in mathematics she acknowledged this as her problem (Fig. 10.8).

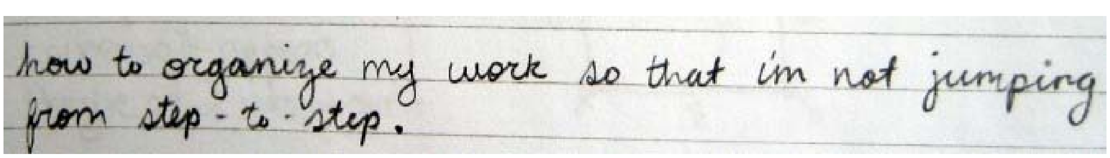


Figure 10-8: Acknowledgement of an issue

The comment may be a reflection of the number of times she had been told this rather than a truly insightful observation on her part, but she is showing awareness of the issue and she was aware that the situation could be improved. She just didn't seem to be able to do it.

A lack of reflection.

At the beginning of the year it was evident that the students were not used to reflecting on their answers other than to check them numerically, if an answer was supplied, or to compare them with another student if not. With this in mind I required all students to reflect on their answers to give an idea of why they thought they were getting questions wrong. As students worked on their problems I noticed that they were better able to both initialize their thinking and continue with their thinking if they were given the opportunity to reflect on their ideas at regular intervals. This can be done in dialogue with the teacher as described earlier, or in dialogue with a peer, as in an example of two students who were discussing a problem and started off with random guessing. This was quickly self-regulated as they combined their ideas to lead to more sensible suggestions. They then began to question each other and to correct their own and each other's mistakes.

Such dialogue is frequently heard when the students work together and reflect on what they are saying. I also observed however that many students are reluctant to engage in these rich conversations when a group is given the problem together from the outset. Certain students always remained quiet and allowed more vocal members of the group to initialize the thinking. It was with this in mind that I developed the idea of giving the students some time to work on problems themselves first before they joined a group. As I observed the class in these situations I paid particular attention to those students who had previously been quiet in the group initialization stage and noted that they contributed their ideas more frequently after they had been given some time to think first. I would hear comments such as 'that's what I had too', or ‘I was thinking that’. Even comments such as ‘I couldn’t think of a way either’ gave the students a chance to initialize with the group and be part of the discussion, drawing them in to make noticeably more significant contributions to the discussion.

Learned\Conditioned helplessness

The notion of ‘learned helplessness’ has been documented in other studies Seligman (1975), Pourdavood et.al (2005), McLeod and Ortega (1993), Wieschenberg (1994) in reference to students attributing failure to lack of ability and disengaging from the thinking process in the expectation they will be unable to find an answer. It was noticeable in this study that several students had developed techniques to avoid the initial thinking process and rely on others to support their thinking, in some cases through the complete problem. At the beginning of the first semester I observed certain students from each class who particularly demonstrated some degree of learned helplessness and monitored how this was manifest.

Kerry was a grade 12 student who combined a defence mechanism with conditioned helplessness. She was quiet and polite but never seemed to write anything down when a problem or challenging question was posed. If I came around to her, either working individually or in a group setting, she would always adopt the pose of thinking intently with pencil poised slightly above the paper and would softly start re-reading the question aloud. She would relax as I passed out of her ‘zone’ and resume watching or listening to others in the class. When I took her aside for tutorials and asked her direct questions it was clear that she lacked any strategy for thinking through her ideas. For example I asked her:

> What will you do when presented with this problem?
>
> Kerry*: “Write down all the information”*
>
> Ok, so what are you thinking now?
>
> Kerry: “*Umm........... (long pause) I get a formula you could use .... with the information*

that you are given .... "

So the first thing you are looking for is a formula is it? So what comes to mind in terms of that then?

Kerry: *(long pause) A formula that has the information in it. (long pause) You would....umm... this..umm .... (inaudible comment)......yeah...*

Her strategy was to give the appearance of thinking about the problem but she was basically waiting for input and further prompts to lead her along. She avoided writing anything down or saying anything to indicate actual progress until somebody else had done so first.

Alesh was a grade 11 student who avoided engaging in the initial stages of the problem by never being ready to participate. He never had paper, or a pencil, or it was deep in his bag, or he had forgotten to bring his bag. When I provided all the requirements for him he would immediately ask to use the washroom. When asked to wait until he had made a start it was clear that he was thinking of other ways to avoid engaging, from tying shoe laces to adjusting the position of the table etc. His actions seemed habitual and he seemed genuinely surprised when it was put to him that he seemed to be deliberately avoiding getting started. Whether he was fully aware of his actions is hard to judge but he now needed to address the issue and not hide behind it if he was to progress.

Suki was a grade 10 boy who had developed the technique of immediately questioning the question. He had an array of questions which were designed to draw a solution method from the teacher even as the problem was still being posed. 'What do you mean?', 'I don't understand the question', 'Why is it telling us that?' were more obvious starters, but he also had a more sophisticated line of questions designed to 'clarify' the question and remove the problematic aspect from the problem. In these questions he would seem to be suggesting an answer but be sufficiently vague and off-base as to challenge the teacher to clarifying the problem. Each question attempted to chip a little more off the mystery of the problem and render it sterile. He had also developed a line of sotto voice comments to his peers bemoaning the lack of teaching if he felt clarification wasn't coming his way.

In each of these cases, and others whose actions were less extreme, there was a noticeable improvement as the first semester progressed. Kerry progressed to first writing down information about the question, and then to making tentative suggestions. Alesh greatly curtailed his stalling actions and began to make progress in his initialization stage. At this point he was still looking to others but he was at least starting something for himself. It took Suki a long time to make progress but when he did change it was quite rapidly. While his homework assignments were still a challenge at this point his attitude in class altered significantly. The change fed on the success he found in one problem in particular (the Nutty Function, see Appendix 3.16) in which he quickly became absorbed and successfully brought to a conclusion. The tactile aspect to this problem may well have been a contributing factor to the success of this problem. After this he initiated his thinking and developed his problem-solving skills in a very encouraging manner.

Sitting down with the students and discussing their actions was a part of this process but placing an emphasis on the initialization process rather than the solution -and making it clear to the students that their initial thinking could not be 'wrong' but was part of the 'searching for a solution stage' was also helping. I regularly asked the class questions such as '*Does it matter if you make mistakes*?' to which the standard reply became '*Not if you learn from them and correct them*'. When asked '*What's the most important part of a problem*', the reply was '*thinking about the mathematics it's intended to show*.' In one lesson at the end of the first semester I asked the question '*What do we mean by a problem in mathematics*?' and was pleased with the first response of '*Something you have to think about*'. However, I was even more pleased when a student raised her hand and asked '*Don't we always have to think in this class*?' This led to a discussion where the class decided that the best way to think of a problem was as a question '*you couldn't do without thinking more deeply than you usually would*'.

Knowing the difference between thinking, ideas, and descriptions

Students can find it difficult to express their ideas and to show their thinking when asked to as part of the problem-solving rubric. As recorded, students' thinking is naturally broken, quick changing, and erratic. This makes it difficult to express at an early stage and, in doing so, can break the flow of thinking. The result is often that students will write their thinking and ideas after they have formulated their plan rather than as an integral part of doing so. As a consequence what is written is often a recall of their final plan and is an extra step rather than a helpful process in the plan. This can still have merit as a way of reflecting and reminding the student of what they did, or if the student becomes stuck in the process, but the real benefit to the students come when they can shift the process more towards a mind map of ideas in the initial stages and when stuck. Formal structure of these ideas is unnecessary but it must be stressed to the student they are using the initial stages of the problem-solving process to sift through possibilities before starting on a path. In addition the students initially had difficulty in distinguishing between describing the basic mathematical process and indicating their ideas, something which was discussed in earlier in chapters. Students would write down comments such as 'rearrange the equation' in the thinking column, which is an unnecessary addition. There was a distinct improvement in this area as the semester progressed.

Listening in to conversations led to some interesting observations that are not always apparent when students are asked a question directly. The external vocalization of the thinking process was very fragmented as the two students in a team were trying to sort through their ideas. Their conversations were broken as in the following example: (where the dots are pauses in the conversation and where they are pointing to the diagrams as they speak, but make few actual notes of their thinking)

P: That's the same shape as that... the base and altitude is the same... but how would it change, like....

C: So our pattern is, what we think it is.. so we move stuff right..to figure it out we like add on, ...so we change the shape..like for this one..we just do that.. yeah?

And again with a different pair of students:

A: It's kind of hard to see.. it looks longer.. So it's always going to be double, like here right?

P: So why do we need these things at all?

A: I don't know... just to prove some of these ones... they're different right. We have to show that. It should work, let's see if it works.

P: Don't we know that? It works cos all you have to do is go like this..

A: Yeah, but that's not the same as this.

P: Oh, right, sorry. Go like this and..

A: Oh yeah and then you cut this in half and put it there.

P: Yeah..... no....... well

A: Then it's like this then...

P: Where'd it go..?

A: Here, look here.......right

P: Yeah...

A: Yeah........ move it here and it becomes a square

P: All the areas are the same for these shapes.

The thinking about the process appears to almost be an intangible thing which is hard to vocalize and perhaps even harder to write down, which may be due to the apparent permanence of the written word or simply because the added task simply gets in the way. In other exercises it was noted that weaker students (and especially girls) were quite tentative in their actual writing when they were unsure, as if unwilling to leave a lasting impression on the page. For many students this resulted in the written output being a poor reflection of the richness of their discussion. In one example a student in a group I had listened to with interest produced the final result '*I used withheld knowledge and formulas from previous lessons to help solve this problem'.* Initially it was frustrating to see the disparity between what the students discussed and recorded, but the more the process was discussed and modelled then the better the students became at distilling their ideas into a more tangible form. More students were able to put down initial ideas rather than describe what they were doing or what they had done. However, I also began to realize that perhaps it wasn't so important that they write down extensive notes during the thinking process, but that the initialization stage before that, and the reflection afterwards, were the key areas to focus on.

Analysis of the surveys across the grades

Does the inclusion of an expanded problem set create increased anxiety amongst the students? It is interesting to note that waiting for the results of a test causes a higher level of anxiety than actually preparing for it or writing it. I would suggest that this may reflect on the importance the students place on a high grade. The grade, it would seem, is worth worrying about more than the requirement to be adequately prepared to achieve that grade[8]. Being stuck in a problem may also relate to this in that the student may be more concerned about how the failure to find an answer would be affecting their grade than concern that they do not fully grasp the mathematics behind the problem. If we then include problems which are 'problematic', so that students are required to think about the underlying mathematics more deeply, then we run the risk of raising the level of anxiety over how such activities affect a student's grade. Assessing problem-solving with an overly elaborate rubric may increase anxiety as looking for specific outcomes often results in a student receiving a lower mark. (For example, IB rubrics require the teacher to start at the bottom end of the scale and proceed up until the student misses a

[8] One student, Steve, approached me before a test towards the end of the first semester and informed me that he wished to raise his grade in mathematics. He wanted me to give him some bonus work which would allow this. I told him that he could raise his grade by improving his marks in quizzes (based on homework) and in the test. I advised that he prepare himself by thinking through areas where he was making mistakes and work on those. I indicated questions from his text which would support those areas and he asked if he could get bonus marks for doing them. I declined, pointing out that this was something he should be doing in order to be successful, and that his grade was an indication of how much he understood rather than how many examples he could copy out. Following the test, in which he did not perform strongly, I asked him about his commitment to do better, and if he had done any of the things we had discussed to prepare for the test. From his manner it was clear that he had not and, when pushed, he admitted as such. "I just want to get a higher mark. If you tell me what to do I will do it." I pointed out that I had told him what to do, to which he replied 'but they weren't for marks."

category.) With this concern in mind I used a much simpler assessment rubric f based upon the student being able to demonstrate that their thinking could lead to a solution. (I also felt that a detailed rubric provides the student with a road map of where to go in the problem so that the student matches their response to the desired outcome rather than freely thinking around the problem).

The points of anxiety highlighted by the surveys seem common across all three grades. In terms of this research the significance of initially not being concerned about starting a problem, but being quite anxious about being stuck in the problem is of interest. Across all grades only five students expressed an anxiety level of 4 when starting a problem (7%)[9]; nine marked their anxiety at 3 (13%); thirty responded with 2 (43%); and 26 with 1 (37%). This represents a significant majority of the students who reported not feeling stressed about starting a problem. This result made it even more interesting for me to examine *how* they went about starting a problem and to examine if their low level of anxiety was because they had already developed coping strategies for this initial stage. The way problem-solving is represented in mathematics textbooks used in the past has often been to imply that it is a secondary issue which did not reflect on the students' grades, and did not appear in tests or exams. Typically this type of thinking may have been relegated to 'mathematics contests' and so be dismissed by students as something for the high achievers only. If students are not used to thinking deeply about a concept then it may be that they do not get anxious about non-start behaviour in this area. In the *MARS-R* done at the end of the courses the trend showed an increase in the anxiety over starting a problem, as indicated by table 8.11 below, despite the fact that they considered themselves better at it, with 78% of the students in the second *MRBQ* indicating an improvement in problem-solving ability.

[9] The grade 12 class, whom I taught the previous year, and who were better informed about the intended nature of problem-solving, were proportionally represented in this number.

Table 10-1: Anxiety over starting a problem at the start and end of the course.

Anxiety level	1	2	3	4	5	
Start %	37	43	13	7	0	(n = 71)
End %	20	42	29	6	3	(n = 66)

These results suggest that the deeper students learn to think about problem-solving then the more anxious it can make them.

By contrast, the second *MARS-R* indicated that anxiety levels over being stuck in a problem did not significantly alter, as shown in table 10.2 below, although there were shifts in the categories.

Table 10-2: Anxiety over being stuck in a problem at the start and end of the course.

Anxiety level	1	2	3	4	5	
Start %	12	34	31	12	11	(n = 71)
End %	14	28	33	14	11	(n = 66)

Summary

In this chapter several themes have been mentioned which are manifest in non-start or quick-stop behaviour. Students who most clearly demonstrate this behaviour have a lack of belief in their own ability. This sets a governor on their process which limits the thinking they are willing to engage in. A student's perception, at an early stage, of the level of mathematics required to solve a problem is an important factor in this behaviour. For some, the governor is set so low that they do not start unless the problem has a 'friendly' entry level, such as a tactile or easy mental manipulation of the situation. The setting of the governor may be determined by several past experiences in problem-solving of which several factors play a role. A lack of background knowledge seems a key in allowing a student to subconsciously process the problem in order to determine its suitability to be started. Once started, however, poor structure and unsystematic methodology can quickly derail a student's progress, feeding back into their beliefs of inadequacy in this area. All students feel some level of anxiety when being presented with a problem; what is important is that students learn, with teacher support, to manage this anxiety.

Chapter 11. Conclusion

"The whole is more than the sum of its parts."

Aristotle, Metaphysica 10f-1045a

At the start of this study the following three questions were posed as a basis for research:

1. *What is the basis of 'non-start behaviour' in students?*
2. *What is the cause of 'quick-stop' behaviour in students?*
3. *Can' non-start' and 'quick-stop' behaviour be alleviated in students?*

In chapter 10 several emergent themes were outlined which point to a basis for non-start and quick-stop behaviour, while the idea of a 'governor' on students thinking was introduced in chapter 8. In chapter 9 further studies were completed in order to further explore these ideas. This study examines the problem from two aspects, a change in classroom methodology and an attempt to extrinsically alter the emotions around problem-solving. Burton (1984) looks to four components which make up a mathematics lesson, each of which is intertwined: Personal changes, Social changes, Process changes, and Mathematical changes. These provide useful categories into which the suggested causes for non-start and quick-stop behaviour can be divided. I would suggest that non-start behaviour is predominantly linked to the emotional state of the student so that *Personal* changes are required in order to change this state. I suggest that quick-stop behaviour is predominantly linked to the knowledge and technique of the student so that *Mathematical* and *Process* changes are required in order to change this state.

Non-Start behaviour and Personal changes

It was evident that a lack of confidence was certainly an issue with many students, and a lack of belief in their own ability to find a solution also caused both non-start and quick-stop behaviour. Surprisingly, however, as indicated in table 9.5 and 9.6 for the grade 10 students, and repeated for both the grade 11 and grade 12 students, the students' own *perceptions* around their ability to problem solve may not be the key factor in non-start behaviour. Some students perceived problems as more challenging now that they needed to think more carefully about them (for example, Fig. 9-1). While students reported considering themselves to be *better* problem solvers they also tended to feel they were *less* sure about how to get started on a problem. This effect may be attributable to the students moving deeper into a problem before getting stuck, but can still be seen by students as a condemnation of their ability. (cf. Fisher, 1990, Gordon, 1961, Mason et. al, 1982). For this reason it is important that teachers recognize that posing any problem can be a source of anxiety to students, regardless of their ability to solve them. What seems more important is creating the right environment for students to start or persist in their attempts despite their own feelings of anxiety. With low confidence and little support the students made little or no genuine attempt to start the problem, and gave up quickly. By working on the problem-solving atmosphere of the classroom, and encouraging the thinking process, students started better and progressed further into the problems posed.

The rise in anxiety may be tied to the concept of meta-affect (DeBellis and Goldin, 1999) whereby the feelings around anxiety are a key component to the process. An increase in anxiety around raised expectations, both intrinsic and extrinsic may be unavoidable and, if so, it is important that the teacher takes this into consideration. DeBellis and Goldin (1997, 1999) suggest that, in such a case, anxiety may be productive, if it engages the thinker cognitively. Using problem-solving as a tool without consideration of the meta-affects may do more harm than good. If the student is not coached to recognize and manage the rise in anxiety then it may negatively add to the student's belief system. Not providing problems which start from where the student is at (Burton, 1984) will quickly cause frustration, but even problems which the student is capable of doing may cause a rise in anxiety which the student needs to recognize as having the potential for improved performance. Individuals may or may not acknowledge the improvement in their confidence level, and problems posed can still be seen as a potential for failure, but most seemed willing to acknowledge an improvement in their overall ability to solve problems.

The Final Report of the U.S. National Mathematics Advisory Panel (2008) examined experimental studies demonstrating that '*changing children's beliefs from a focus on ability to a focus on effort increases their engagement in mathematics learning, which in turn improves mathematics outcomes: When children believe that their efforts to learn make them "smarter," they show greater persistence in mathematics learning* (p. 20).' Fisher (1990) also notes "*Research shows that boosting confidence increases the ability to solve problems and prevents the students from giving up too soon, before the right and bright ideas come up. "* (p. 113) Mason, Burton and Stacey (1982) also note that emotions can block recognition of a starting point.

A lack of interest in, or lack of relevance of, the subject matter is often cited as a reason for non-start behaviour in mathematics. While linking the material to common experiences helps to ground the learning for the students, the results of this study suggest that students are willing to engage in problems they sense they can be successful in even if they aren't grounded in everyday life. Students did not automatically show interest in topics which might be considered 'applicable' – such as the car collision problem involving the grade 11 students. Instead they became caught up in certain problems, or at certain times in a problem, when a sense of solution could be seen. If such a situation also links to an everyday experience then the learning is deeper as more connections are made to solidify the process. This is in keeping with Mason et. al (1982) who note the student must become engaged with the problem in order to move between stages of the solution, and the Final Report of the U.S. National Mathematics Advisory Panel (2008), which revealed that '*if mathematical ideas are taught using "real-world"* *contexts, then students' performance on assessments involving similar problems is improved. However, performance on assessments of other aspects of mathematics learning, such as computation, simple word problems, and equation solving, is not improved (p. 49)'*.

By developing thinking skills (cf. Fraivillig (1999); Rigelman (2007)) and strategies it is more likely that the student can become engaged in a problem and therefore become 'interested' in its solution. In many cases students are put off by the presentation of a problem, whether it is too wordy or too abstract, and need to be shown how to reframe the problem in a way that is more comfortable to them and encourages the initialization stage. Kahney (1984) notes that '*how a problem is presented has a powerful effect on ability to relate to previous knowledge'*; while Burton (1984) cautions that the teacher must be careful not to make the problem 'sterile'.

Non-Start behaviour, Process and Mathematical changes

Process changes

Consistent with the findings of Schoenfeld (1982, 1994), what does seem to be a key factor in quick-stop behaviour is the lack of a systematic process of approaching a problem. In particular, students do not take time to think through options before starting to solve the problem and expect to be able to see the solution immediately or not at all. By emphasizing this crucial stage in the classroom students were able to develop better strategies and a more systematic method of approaching a problem. By stressing a more structured and systematic approach to starting problems, asking students to think over options before sharing ideas, and giving credit for ideas as well as solutions, there was an increase in the time a student would spend working on a problem. By asking students to demonstrate and record their thinking they were able to add some structure to their ideas and were better able to follow the thread of their own thinking.

It was also evident that students generally did very little reflection on their work and did not take the time to learn from their mistakes. Encouraging the reflective process throughout the problem-solving process through discussion and journaling was an important part in the development of thoughtful students, and helped to solidify what was learned (cf. Burton (1984), Schoenfeld (1982, 1985)). By asking students to reflect on the learning process and to place an emphasis on *why* they thought they were unable to start or finish problems the students were able to recognize that mistakes were very important parts of the learning process - if they were reflected upon. By emphasizing the notion that all success and development comes through making, recognizing, and correcting mistakes the students were encouraged to see that getting stuck in a problem was a natural part of the process. By asking students to reflect on the various stages of the process allowed them to consider different options for success, or develop a richer line of thinking.

It was also evident that what is entailed by thinking about a problem was vague. At the beginning of each of the courses the idea of thinking about a question primarily entailed seeking out a relevant equation that could be used. If the equation was not quickly apparent then non-start or quick-stop behaviour was the norm. When the students were asked to write down their thinking the initial response was to describe the process they were performing - generally a procedure orientated approach. Students found writing their thinking difficult because they found articulating their ideas difficult. Students often made comments such as: *'Just tell me what to do and I can do it.'* Such a statement is often born from frustration on the part of the student caused by inadequate thinking skills, and what Novic (1995) calls inert knowledge, or knowledge without application. By developing the classroom atmosphere of thinking as an expectation these types of behaviour can be reduced by helping students to recognize information about the conditions under which a problem can be applied. I suggest that if students do not think about the process as they are carrying out the question, or attempting to solve the problem, then there will be little perceived relevance or connections built between concepts. By stressing the need to think about the underlying concepts and their relevance to the question there is a greater chance that longer lasting neural connections can be made, even if the process of solving a particular problem is short-term. Students also need to be encouraged to recognize the difference between having ideas and thinking them through. By encouraging students to reflect on their ideas and think about how to develop them then they are less likely to engage in quick-stop behaviour.

Mathematical Changes

A lack of background knowledge to give the students a platform on which to build was another key factor in quick-stop behaviour, but less so in non-start behaviour. Lower-achieving students often had as many ideas about the problem as other students but would be unable to recognize how to proceed with a solution. Burton (1984) notes that problems given should start from where the students are, but extending this to account for background weaknesses on assumed knowledge in higher grades can be the cause difficulties for students. In many cases important information from previous grades would be wrongly remembered so that the student would proceed with a solution using incorrect procedures. If the procedure did not quickly give success then the lower-achieving students would demonstrate quick-stop behaviour. Higher-achieving students however were likely to check their facts before proceeding and would use a wider base of knowledge to call upon. This result is in keeping with Chase and Simon (1973) who found that one key element in successful problem solvers was that they knew more. The diagnostic tests indicated that they may not possess this knowledge on day-to-day basis significantly more so than lower-achieving students, but they were more likely to seek out knowledge which would help. It may also be that students who are able to subordinate (Hewitt,

1996) their knowledge more are in a better position to use abductive reasoning (cf. Sinclair, Strickland, and Kozse , 2008) to see a solution path without formally recognizing it. In many instances a lower-achieving student was able to correctly solve problems that did not require connections to previous knowledge.

By promoting a sense of working together and group discussion there was less of a tendency for incorrect background knowledge to remain unnoticed, and more of a tendency for students to spark the recall of memories from previous classes. The Final Report of the U.S. National Mathematics Advisory Panel (2008) and Doise and Mugny (1984) also support the idea of social and intellectual support from peers and teachers . By asking students to work alone in the initial stage and consider various ideas there was a better chance of relevant background knowledge to surface. When students recognized that a problem did not require a deeper sense of knowledge than they possessed they were much more likely to work on it for a longer period of time, even if the problem was quite conceptually challenging. Students generally tend to use a smaller set of knowledge than they are capable of (Burton, 1984), preferring methods they are comfortable with even if they are less efficient. Discussing the merits of trying different options, and encouraging diversity with questions such as 'can you do it another way?' helped to encourage students to think about alternative methods.

Can 'non-start' and 'quick-stop' behaviour be alleviated in students?

The findings in this study suggest that non-start and quick-stop behaviour in students can be successfully addressed. In an examination of 71 students from grades 10 – 12 the majority demonstrated measureable and noticeable improvement in the important areas of initializing their thinking and being able to persist in problem-solving activities. By creating a classroom atmosphere focussed on promoting thinking and emphasizing the importance of seeing difficulties as learning opportunities, students can learn to develop strategies to aide their approach to problems. By focussing on the initialization (or entry) stage of a problem, and using methods to aid metacognition and reflection, students can be taught to concentrate on the process of problem-solving rather than a specific answer to a problem. Students can be led to broaden their search for ideas before pursuing a solution, although this is still a difficult step for them. Techniques such as *Organize –Entry- Share* and Mason et al.'s (1982) two-column reflection can help the students to be aware of their thinking and to recognise the difference between thinking about a solution and describing a process. Students also demonstrated a willingness to persevere longer with problems with which they were having difficulty, and to try alternative approaches both before a solution and afterwards. Reflecting on the solutions and failed solutions to problems gave students better connections to what they were doing and enabled the teacher and the students to see emergent patterns in the difficulties individuals were having.

A key component to success in this area is in establishing a classroom atmosphere in which the students feel comfortable to take risks and pursue their mistakes. Such an atmosphere does not necessarily mean that the anxiety of the students in engaging in problem-solving is reduced however. There is an indication that the more problem-solving a teacher offers then the greater the anxiety the student may feel around starting a problem. An important conclusion from this research is that teachers need to recognize this as an important emotion and help the student to manage it. As a local affective process it may always be a factor for the student regardless of the level of their achievement and so the student needs to develop their own regulation of the emotion. A key role of the teacher in problem-solving may be to foster that regulation.

Have the students gained?

Leone Burton (1984) suggests that there is '*no easy answer*' to this question and points to several reasons why this is so. In particular though the need for a concrete answer to the question leads to its own problems, as Burton points out, in that the things a teacher might look to in order to measure success may not be the same as those things the student feels contributes to their personal growth. In addition, the learning process is complex and interconnected and very individual. Burton notes that '*frequently, students perceive what is being taught quite differently from the teacher*'. The connections a teacher is intending for a class may or may not occur on that occasion but this does not necessarily make the lesson a failure. It may be that the lesson has laid an important foundation which allows the objective to be assimilated at a later time, or it may allow the connection of related pieces of information to fall into place which were not directly part of the lesson. Earlier, it was noted that a lack of background knowledge hampers the problem solving process. This may lead to the suggestion that a student needs to 'learn the basics' before any problem-solving can be attempted. In response I would suggest that the students at the start of this research had ample exposure to 'the basics', and yet the lower-achieving students in particular demonstrated a lack of knowledge. By utilizing problem-solving in a controlled and risk-free environment those same students were better able to work on problems using mathematics they struggled with in regular style questions. Their ability to transfer this knowledge to isolated questions is an unresolved issue and a topic for further research.

Appendices

Appendix 1: Mathematics Anxiety Rating Scale – revised[10]

When I am … I feel ANXIOUS	Not at all	A little	A fair amount	Much	Very much
1. Looking through the pages in a math text	1	2	3	4	5
2. Walking into a math class	1	2	3	4	5
3. Reading a formula in a science text	1	2	3	4	5
4. Thinking about an upcoming math test one day before	1	2	3	4	5
5. Watching a teacher explain a problem on the whiteboard	1	2	3	4	5
6. Being told how to interpret algebraic statements	1	2	3	4	5
7. Picking up a math textbook to begin working on a homework assignment	1	2	3	4	5
8. Taking an test in a math course	1	2	3	4	5
9. Reading and interpreting graphs or charts	1	2	3	4	5
10. Starting a new math problem	1	2	3	4	5
11. Being given a homework assignment of many difficult problems	1	2	3	4	5
12. Waiting to get a math test returned in which you expected to do well	1	2	3	4	5
13. Getting stuck in a math problem	1	2	3	4	5
14. Starting a new chapter in a math book	1	2	3	4	5
15. Listening to another student explain a math formula	1	2	3	4	5

[10] The revision of the revised scale is based upon the work of Derek R. Hopko **Confirmatory Factor Analysis Of The Math Anxiety Rating Scale–Revised,** *Educational and Psychological Measurement* 2003; 63; 336

Appendix 2: Mathematics-Related Beliefs Questionnaire

	Please respond to each statement	Strongly agree	Some-what agree	Neither agree or disagree	Some-what disagr ee	Strongly disagree
1.	Making mistakes is part of learning mathematics.	○	○	○	○	○
2.	Group work helps me learn mathematics.	○	○	○	○	○
3.	Mathematics learning is mainly memorizing.	○	○	○	○	○
4.	The importance of competence in mathematics has been emphasized at my home.	○	○	○	○	○
5.	Anyone can learn mathematics.	○	○	○	○	○
6.	There are several ways to find the correct solution of a mathematical problem.	○	○	○	○	○
7.	I am hard working by nature.	○	○	○	○	○
8.	Solving a mathematics problem is demanding and requires thinking, even from smart students.	○	○	○	○	○
9.	Mathematics is continuously evolving. New things are still discovered.	○	○	○	○	○
10.	There is only one way to find the correct solution of a mathematics problem.	○	○	○	○	○
11.	Mathematics is used by a lot of people in their daily life.	○	○	○	○	○
12.	My family has encouraged me to study mathematics.	○	○	○	○	○
13.	I'm only satisfied when I get a good grade in mathematics.	○	○	○	○	○
14.	The example of my parent(s) has had a positive influence on my motivation.	○	○	○	○	○
15.	I believe that I will receive this year an excellent grade for mathematics.	○	○	○	○	○
16.	I start problems and questions confidently	○	○	○	○	○
17.	I like doing mathematics.	○	○	○	○	○
18.	I am often unsure how to start problems assigned in class.	○	○	○	○	○

19.	I can understand the course material in mathematics.	○	○	○	○	○
20.	To me mathematics is an important subject.	○	○	○	○	○
21.	I prefer mathematics tasks for which I have to exert myself in order to find the solution.	○	○	○	○	○
22.	I know I can do well in math	○	○	○	○	○
23.	If I try hard enough, then I will understand the course material of the mathematics class.	○	○	○	○	○
24.	When I have the opportunity, I choose mathematical assignments that I can learn from even if I'm not at all sure of getting the final answer.	○	○	○	○	○
25.	I'm very interested in mathematics.	○	○	○	○	○
26.	Taking in to account the level of difficulty of our mathematics course, the teacher, and my knowledge and skills, I'm confident that I will get a good grade for mathematics.	○	○	○	○	○
27.	I think I will be able to use what I learn in mathematics also in other courses.	○	○	○	○	○
28.	I like to know the answer to a problem before I start so that I can see where to go with the solution.	○	○	○	○	○
29.	Those who are good in mathematics can solve problems in a few minutes.	○	○	○	○	○
30.	I am not the type to do well in math.	○	○	○	○	○
31.	My parents enjoy helping me with mathematics problems.	○	○	○	○	○
32.	It is a waste of time when the teacher makes us think on our own about how to solve a new mathematical problem.	○	○	○	○	○
33.	My parents expect that I will get a good grade in mathematics.	○	○	○	○	○
34.	I expect to get good grades on assignments and tests of mathematics.	○	○	○	○	○
35.	Math has been my worst subject.	○	○	○	○	○

36.	Mathematics enables people to better understand the world they live in.	○	○	○	○	○
37.	I have not worked very hard in math.	○	○	○	○	○
38.	Mathematics is a mechanical and boring subject.	○	○	○	○	○
39.	When I get stuck on a problem I tend to give up quickly	○	○	○	○	○
40.	I always prepare myself carefully for exams.	○	○	○	○	○
41.	Mathematics has been my favorite subject.	○	○	○	○	○
42.	I am sure that I can learn math.	○	○	○	○	○
43.	My major concern when learning mathematics is to get a good grade.	○	○	○	○	○
44.	I tend to avoid starting a new problem until other students have done so to give me a clue.	○	○	○	○	○
45.	When I get stuck on a problem I have good strategies to help me.	○	○	○	○	○
46	I get satisfaction from solving mathematical problems	○	○	○	○	○

Appendix 3: Selected Problems

Grade 11

1. Early in the year I gave the students a problem known as the Wason test[11] (Barkow 1995, Devlin 2000) in two parts with the aim of illustrating how the logic of a question can seem difficult if it is expressed in mathematical terms and yet easy if the same question is put in terms of a social situation. As in the original test given to university students the majority of the students could not get the answer in the first case but they all managed to correctly answer the question in the second case. There was a clear sense of 'ease' working with part two of the problem but I noted that they were also much more engaged with the first part of the problem than they had been with earlier problems given. There was more of a willingness to try the problem and to persevere with it, and the discussion was livelier. The two questions were:

1. ***Try this in pairs***: **Imagine I lay four cards on a table in front of you. I tell you that each card has a number on one side and a letter on the other. On the uppermost faces of the cards you see the four symbols: E K 4 7. I then tell you that the cards are printed according to the rule: if a card has a vowel on one side, it has an even number on the other side. Your task: which cards do you have to turn over to be sure that all four cards satisfy this rule?**

2. You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You, as the organizer, are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table are four young people who may or may not be over the legal drinking age. One person has a beer, another has a Coke, but their IDs happen to be face down so you can't see their ages. You can, however, see the IDs of the other two people. One is "under the drinking age" the other is above it. Unfortunately, you are not sure if they are drinking 7-Up or vodka and tonic. Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

2. A typical problem assigned at the beginning of the year, to start the students thinking was based on logic:

[11] I showed the students a video of this found at http://www.youtube.com/watch?v=sk71JqHpuX0

A] Two men and two boys want to cross a river. None of them can swim and they only have one canoe. They can all paddle but the canoe will only hold one man or two boys. How do they all get across? What is the *minimum* number of crossings required? [12]

Another example, in which the students were only asked to describe how they thought they would do the problem, was:

B] The average lifespan of American women has been tracked, and the model for the data indicates that every year the average lifespan increases by 0.2 years. The study was started in 1960 when the average lifespan was 73 years. What is the average lifespan today assuming a linear relationship?

Again: "I am going to give you another problem which I don't want you to solve. I want you to write down the first thing you think about after you ***hear*** the question: I'm going to say it once, so listen carefully.

C] The sum of the digits of a two-digit number is 7. When the digits are reversed, the number is increased by 27. Find the number

Now: How do you think you would ***start*** to answer this question?"

An example problem based on curricula content was:

D] The pool fun company has learned that, by pricing a newly released fun noodle at $3, sales will reach 5000 fun noodles per day during the summer. Raising the price to $4 will cause sales to fall to 4000 fun noodles per day. Assuming a linear relationship between noodle price and sales, how would you choose to price the noodles to make the most profit? Explain your thinking carefully.

3. In an attempt to teach through problem-solving I tried opening a unit with a problem. An example of this was:

[12] Taken From 'Thinking Things Through' by Leone Burton, 1984.

The Collision Problem. [13]

This problem was given as a way to introduce the unit on functions. It was adapted from an example given in the book 'Teaching Mathematics through Problem Solving: Grades 6-12' (NCTM). Since the concept of the function is an idea introduced in grade 10 the students had some background and the idea was to review this through the problem and extend the ideas to cover the concepts required for grade 11.

> ***The Problem:*** You work for the accident-investigation unit of your local police department. You have been called to the scene of an accident. A car coming around a blind corner has hit another car, which has broken down and was stopped in the street. Luckily, no one was injured. You will be asked to help determine whether the driver of the car was in some way at fault. Was he speeding? Since he was coming around the corner, did he have sufficiently early warning to stop?

The students were asked to write a report which the insurance company will use to determine fault. They were asked to consider the variables in this problem and to decide how to incorporate them. They were allowed to ask for information from the crime scene officer (which was me, but I did not *suggest* information, they had to think to ask for it) and use resources such as the internet or the library etc.

The students were asked to write a summary to answer the following questions:

I. On what assumptions did you base your model?
II. Compare your answer with that of other groups. Do you agree? If not, why not? Did the discrepancy arise because of the use of different information, a different method, or different assumptions?
III. If you are driving down the street and a ball rolls in front of you, what factors would determine how long, and thus how far you would travel, before you could come to a stop?

Reflect on what you think the mathematics is behind this problem, and comment on how useful you think it was in helping you understand the problem. Could you have found the solution in another way? Can you adapt these ideas to other areas?

4. On each test I added a problem so that the students would see that the process was valued. The problem from the first test, along with the mark allocation was:

3 people carry 5 pails (each with capacity 8L) to a place where there are 3 springs. One of the springs gives 2L/minute and the other two give 1L/minute. It is not possible to use one spring to fill two pails simultaneously. It takes less than 2 minutes and more than 1 minute to take a pail from one spring to another. What is the shortest time it takes for them to fill all five pails? How is it done?[14]

[13] Adapted from 'Teaching Through Problem Solving', NCTM
[14] Taken from http://www.stfx.ca/special/mathproblems/grade11.html

1	2	3	4
▪ A start beyond copying that shows some understanding.	▪ Correct answer but no work shown, or ▪ Appropriate strategy but not carried out far enough.	▪ Correct answer but unclear strategy or, ▪ Appropriate strategy but ignored a condition.	▪ Correct answer with clear strategy, or ▪ Incorrect solution with a copy error or minor computation error but not a misunderstanding.

Test Problem Marking Rubric[15]

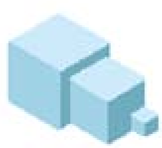
This was accompanied with the instructions: '*For this question you should show your thinking clearly in the way discussed in class for problem solving. You will be assessed as above*:'

5. The problem given on the second test was:

The Chateau family has two cupboards for its wine bottles, a small cupboard and one much larger. Being very fussy, they do not like their wine exposed to the light more than 12 times, including both the time they buy it and the time they drink it and each time they open the cupboard. If they drink one bottle each day, how often does the Chateau family need to buy wine?[16]

6. The problem on the final exam was:

Three cubes, whose edges are 2, 6, and 8 cm long, are to be glued together at their faces. Compute the minimum surface area possible for the resulting figure.



(It was clarified in the test that the diagram was just one possible solution and that the blocks could be put together in different ways.)

Grade 12 Example Problems

7. One of the first problems given to the grade 12 class was called *Arbor Drive*[17].

Arbor Drive has no numbers on houses, just names. Since the mailman was having some trouble with the mail, it was decided to number them. One side was numbered continuously with odd numbers starting with 3. The first building on the other side was a duplex, numbered 2 and 4, but somewhere on that side there was

[15] This rubric was adapted from the Vancouver Island Diagnostic Math Assessment package.
[16] http://www.stfx.ca/special/mathproblems/grade11.html
[17] http://www.stfx.ca/special/mathproblems/grade11.html

a gap where houses had still to be built, and allowance was made for eventual numbering.

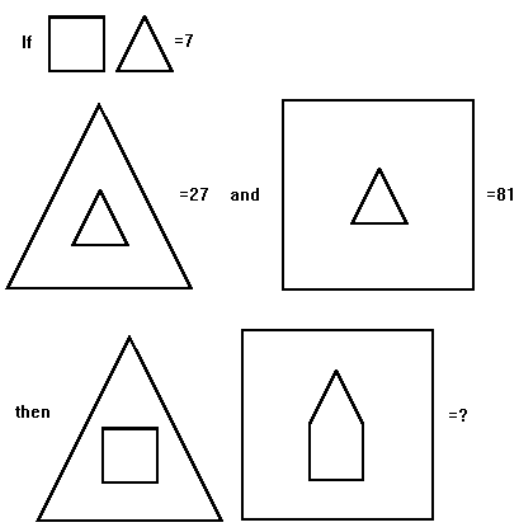
When contacting their local Canadian Tire they found that each digit cost 50 cents. The total bill for all the digits was $42.50 and the even-numbered side cost $5.50 less than the other side.

When the gap is filled there will be exactly the same number of houses on each side. What is the number of the last odd-numbered house, and what are the missing numbers on the other side?

8. The following two problems[18] were presented in a class in early October:

> **1. Draw four dots on a circle and join each dot to the nest dot with a straight line. Can you draw any more straight lines, joining up the dots *without* making any triangles*?* Try the same thing for 5 dots, 6 dots, and so on, getting in as many lines as you can. Can you fill in a table of the number of dots and number of lines? Predict how many lines can be drawn without making triangles when there are 13 dots. What sort of number never appears in the second column of the table?**
>
> **2. In a village there are three streets. All the streets are straight. One lamp post is put up at each crossroads. What is the greatest number of lamp posts that could be needed? Now try four streets and five streets. Predict the answer for six streets then check it. Can you see a pattern? Why does the pattern work?**

9. The following problem[19] was given as a homework Journal entry in early October, there were no instructions other than to 'Solve the problem.'



[18] Taken From 'Thinking Things Through' by Leone Burton, 1984.
[19] http://www.stfx.ca/special/mathproblems/grade11.html

10. An example of a problem given on mid-October:

You have eight pool balls. One of them is defective, meaning that it weighs more than the others. How do you tell, using a balance, which ball is defective in two weighings.

11. The following two problems[20] are examples given in November of the course:

1. **Desert Crossing : It takes nine days to cross a desert. A man must deliver a message to the other side, where no food is available, and then return. One man can carry enough food to last for 12 days. Food may be buried and collected on the way back. There are two men ready to set out. How quickly can the message be delivered with neither man going short of food?**

2. Desert Crossing revisited: A man must deliver a message to the other side of a desert, where no food is available, and then return. One man can carry enough food to last for 12 days. Food may be buried and collected on the way back. How wide can the desert be (in terms of days needed to cross) and still have the message delivered with no man going short of food?
Case 1: There are 3 men
Case 2: There are 4 men
Case 3: There are 5 men

Is there a pattern? Can a desert of any size be crossed if there are enough men?

Grade 10 Example Problems

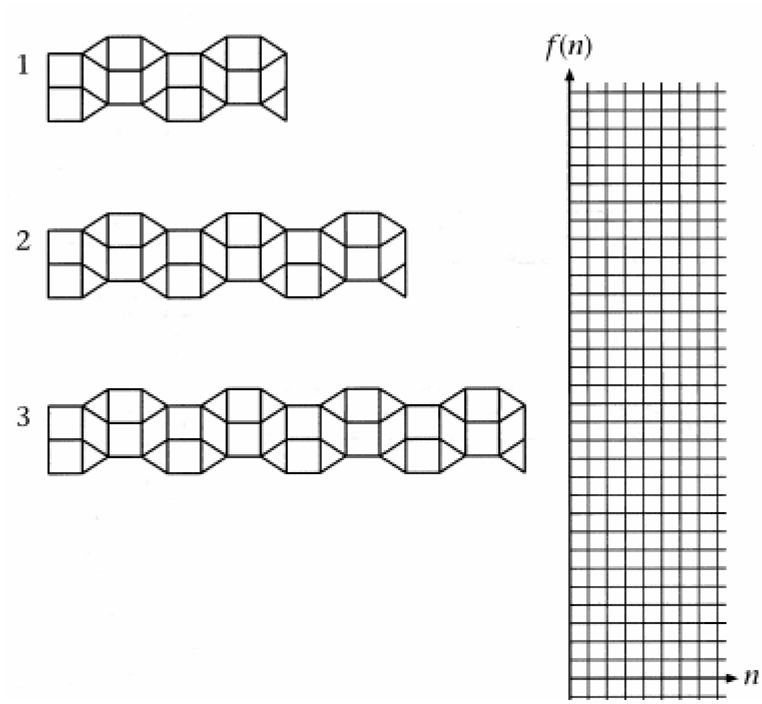
12. The following problem[21] was given early in the year

A student at St. F. X. decided to become his own employer by using his car as a taxi for the summer. It costs the student $693.00 to insure his car for the 4 months of summer. He spends $452.00 per month on gas. If he lives at home and has no other expenses for the 4 months of summer and charges an average of $7.00 per fare, how many fares will he have to get to be able to pay his tuition of $3280.00?

13

1] The following problem[22] was give as a one of a set of related problems towards the end of the first semester with the instructions to 'graph the pattern'.

[20] Ibid.
[21] http://www.stfx.ca/special/mathproblems/grade10.html
[22] Adapted from *The Pattern and Function Connection,* Key Curriculum Press (2009)



2] The Cornered Pole was a problem given early in the second semester

You are carrying a wide tray of food around the narrow corridor of a hotel. What is the widest tray you can carry given that you must carry it horizontally and the corridor narrows from width ‘b’ to width ‘a’. (You can think of the tray as a thin rigid pole)

3] The Pirate Problem[23] was another problem given in the second semester

You find a treasure map. It says on it:

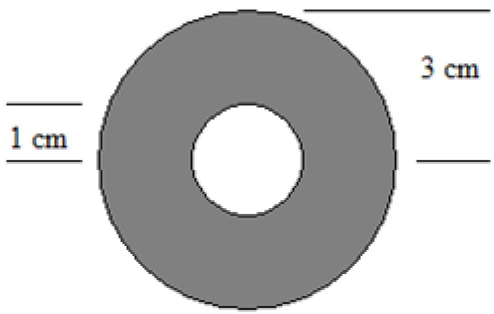
The treasure of Captain Bird is buried on the Island of the Parrot. Near the centre of this island three great trees form a triangle. The mightiest of the three is a great oak older than the treasure itself. Towards the west of the oak, some distance away there stands an elm tree, and towards the east of the oak there stands an ash. To find the treasure of Captain Bird count out the paces from the oak to the elm. When you get to the elm make a precise left turn and count out the same number of paces. Mark this spot with a flag. Return to the oak and count out the paces from the oak to the ash. When you get to the ash make a precise right turn and count out the same number of paces. Mark this spot with a flag. The treasure lays buried midway between the two flags.

[23] Provided by Nathalie Sinclair in a course at SFU.

So, you rent a boat a head off to the Island of the Parrot only to discover that there is no oak tree. You can locate the elm and ash, but the oak tree is gone. Where is the treasure?

14. The following problem[24] appeared on the Mid-year exam:

A manufacturer sells scotch tape on a spool with a radius of 1 cm. The tape is 0.02 cm thick and 1.5 cm wide. The combined radius of the spool and tape is 3 cm. What is the approximate length, in metres, of the tape on the spool?



15 The following problem was used in the second semester:

Al and Bob, who live in North Vancouver, are Seattle Mariner fans. They regularly drive the 264 km from their home to the ballpark in Seattle. On one particular day Bob drove to the game. On the return journey Al was able to increase the average speed by 10% and save 18 minutes on the travelling time.

a) Calculate the average speed at which Bob drove to the game.

b) Calculate the time it took Al to drive back from the game.

As an interesting side-note to this problem, one of the students brought in a similar problem he had found and wanted to share with the class.

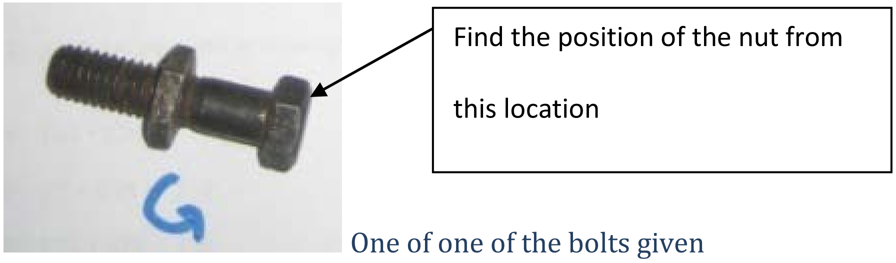
Adam and Bob are two old grandpas. They each leave Pottstown and Runsdown at exactly the same time. They pass each other at 12:00 but keep going. Adam reaches Runsdown at 14:15 and Bob reaches Pottstown at 16:00. At what time did they leave? Assume they both walk at a constant rate and do not stop or leave the straight road between the towns.

16. The Nutty Function[25]

I am giving you a nut and bolt. Your task is to write down an equation for the position of the nut from the head of the bolt as a function of the number of turns of the nut. You must explain your final answer carefully and be prepared to justify it.

[24] http://www.stfx.ca/special/mathproblems/grade11.html

[25] Adapted from *The Pattern and Function Connection,* Key Curriculum Press (2009)



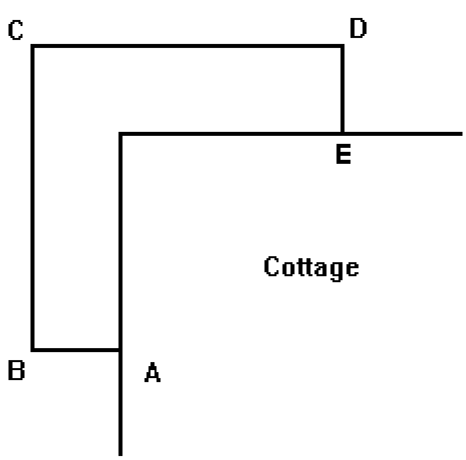
One of one of the bolts given

17. This problem[26] was used with both the grade 10 and 11 students:

> **Years ago, during a foreign war, a desert fort occupied by troops lay under seige. The fort was square in shape with 8 defensive positions - One at each corner and one in the middle of each side.**
>
> **The fort commander General Gregorie LeVangie knew that the enemy would not charge as long as they could see 15 active defenders on each side, so with 40 troops under his command, he stationed 5 in each defensive position. When one of his men was wounded he arranged the rest so that the enemy could still see 15 on each side. How did he do this?**
>
> **Further casualties occurred. Explain how, as each man falls, LeVangie could rearrange his troops around the fort to prevent a concerted attack. Reinforcement arrived just as the enemy was about to charge. How many active defenders did they find left?**

18. The following problem[27] was used on a unit test:

> **The Bunbury's want to build a deck on their cottage. The architect drew them a diagram which shows it built on the corner of the cottage. A railing is to be constructed around the four outer edges of the deck. If AB = DE, BC = CD and the length of the railing is 30 metres, then what dimensions will give maximum area?**



---

[26] http://www.stfx.ca/SPECIAL/MATHPROBLEMS/grade11.html
[27] Ibid.

Appendix 4: The OES Method

As the first semester progressed I recognised a need to adjust the 2-column problem-solving rubric suggested by Mason et. al (1982) into a format that would help students initialize their thinking better. With this in mind I developed the *Organize – Entry – Share* method.

The Organize-Entry-Share Method[28]

1. Organize

The students are given a problem, sometimes written out and sometimes orally. In this stage they are working independently. They are told they can only use a pen and any crossing out must be done to leave the underlying work legible. They are then given five minutes to process the information, and write down any initial thoughts and ideas they have about how they think they might solve the problem, but they are asked not to start their solution. They are advised to keep their work easy to follow and think about any of the problem-solving strategies that have been discussed or are on the display boards. After the set time they are told to draw a line under their work and not to go back and edit anything. All new work is to be done under the line.

2. Entry

The students are now told that they have 5-10 minutes to work on their selected approach to the problem. (This was flexible and I tried to judge the room for engagement) They are still working independently at this point. They can of course change their idea at any time and work on a new approach. They are reminded again not to erase any work and to indicate their thinking at key points in the solution using the problem-solving rubric used in previous classes. After 10 minutes they are told to once again draw a line across their page. It is of course stressed to them that how far they have gotten into the problem at this stage does not affect their mark on the problem. The purpose has been to help initialize their thinking.

3. Share

The students were supplied with a piece of paper divided into three rows. In each row was written the large words DISCUSS, or DO NOT DISTURB, or I CAN HELP. The paper was then

[28] Attending a conference towards the end of the year I was made aware of the 'Think - pair - share method used with younger children and recognized some similarities in the approach.

folded and taped to form a triangle. The students were asked to decide if they wanted more time to work alone, or if they wanted to form a discussion group, or if they wanted to give help on the problem.

If the student felt they had the solution then they would display the words 'I can help' and they would then be able to work with other students to give them a hint in beginning the solution. They were asked to indicate, after their solution, what help they had given during this time and if it proved useful. Any students who received the help were asked to cite their source and also reflect on the help given. Initially, I did not introduce the caveat that students could not take notes from someone giving help, and it was clear that that this simply became a copying session which was not useful to any of the students. In subsequent problems using this method the process was made more effective by not allowing the students to write down notes when being given help. They were at least forced to listen and the student giving help had to do a better job of explaining his thinking so that the others could make their own sense of it. I also spoke to the one student in particular, who was often in this situation, and he became much more adept at asking the other students about their thinking rather than simply telling them what he did.

If students displayed the 'discuss' sign then they were able to work together to form a solution. These students were also asked to note who had what ideas so as to acknowledge their sources. They were also asked to reflect on the usefulness of their collaboration. The students were allowed to keep notes during their discussion sessions. I tried to monitor these groups to make sure that no particular student was doing all of the thinking.

If a student preferred to work alone for a longer time then they would display the 'do not disturb' sign. They could then proceed to work through their ideas until they reached a point where they felt they were stuck beyond being able to see a solution route, or if they found the solution. They were allowed to change their sign at any time. I found it useful to check on these students when they did switch however and ask them why they thought they couldn't get unstuck (and to reflect on that), or to ask them check their answer and see if they were sure their answer was correct or ask them if they thought they could do it another, perhaps easier, way.

Reference List

Ball, D. L. (2000). Bridging practices: Intertwining Content and Pedagogy in Teaching and Learning to Teach. *Journal of Teacher Education*, 51, 241-247.

Barkow, J., Cosmides, L., & Tooby, J. (1995).*The Adapted Mind: Evolutionary Psychology and the Generation of Culture*. NY: Oxford University Press.

Bay-Williams, J., & Meyer. M., (2005, March) Why Not Just Tell Students how to Solve a Problem? *Mathematical Teaching in the Middle School,* 340-341.

Bloom, B.S. (1976) *Human Characteristics and School Learning,* NY: McGraw Hill

Bransford, J.D., Brown A.L., &Cocking, R.R. (1999); *How People Learn -Brain, Mind, Experience, and School* (pp. 39-66) Washington, D.C.: National Academic Press

Brown, S. & Walter, M. (1990). *The Art of Problem Solving*. NJ: Lawrence Erlbaum associates

Burton, L. (1984). *Thinking things through: Problem Solving in Mathematics*. Oxford: Basil Blackwell

Buschman, L. (2003, May). Children who Enjoy Problem Solving. *Teaching Children Mathematics*. NCTM.

Chase, W.G., & Simon, H.A. (1973) Perception in Chess*, Cognitive Psychology* 4, 55-81;

DeBellis, V. A., & Goldin, G. A. (1999). Aspects of affect: Mathematical intimacy, mathematical integrity. In O. Zaslovsky (Ed.), *Proceedings of the 23rd International Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249-256). Haifa, Israel.

Denton, L.F. & McKinney. D. (2004). *Affective Factors and Student Achievement: A Quantitative and Qualitative Study*, Proceedings of the 34th ASEE/IEEE Conference on Frontiers in Education, pp. TIG 6-11.

Devlin, K. (2000). *The Math Gene: How Mathematical Thinking Evolved And Why Numbers Are Like Gossip*. Weidenfeld & Nic)

Doise, W., & Mugny, G. (1984). *The Social Development of the Intellect.* Oxford, Penguin Press

Duen H. Y. (1998). Learned Helplessness, Retrieved December 21st 2008. Web site: http://www.noogenesis.com/malama/discouragement/helplessness.html

Dweck, C.S. (2000). *Self-Theories: Their role in Motivation, Personality, and Development*. Philadelphia: Psychology Press.

Farmaki, V. & Paschos, T. (2007). The Interaction between Intuitive and Formal Mathematical Thinking: a case study, *International Journal of Mathematical Education in Science and Technology*, Vol. 38, No. 3, 353–365.

Fisher, R. (1990). *Teaching Children to Think*, Oxford, Basil Blackwell.

Fraivillig, J. (1999). Advancing Children's Mathematical Thinking in Everyday Mathematics Classrooms. *Journal for Research in Mathematics Education*, Vol. 30, No. 2. pp. 148-170.

Goldin, G.A. (2002). Affect, meta-affect, and mathematical belief structures. In G.C. Leder, E. Phekonen, & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* Netherlands: Kluwer Academic.

Gordon, W. (1961) *Synectics: The Development of Creative Capacity,* NY: Harper & Row

Hadamard, J. (1945) *The Mathematician's Mind: The Psychology of Invention in the Mathematical field*, Princeton University Press; (paperback edition, 1996).

Hannula, M.S. (2006). Motivation in Mathematics: Goals Reflected in Emotions. *Educational Studies in Mathematics* 49(1), 25–46

Hannula, M, Evans, J., Philippou, G., & Zan, R. (2004) Affect in Mathematics Education -Exploring theoretical frameworks. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* Vol I pp 107–136 Retrieved September 5th 2008. Web site: www.emis.de/proceedings/PME28/RF/RF001.pdf

Harel, G., & Sowder,L. (2005) Advanced Mathematical-Thinking at Any Age: It's Nature and its Development, *Mathematical thinking and learning,* 7(1), 27–50, Lawrence Erlbaum Associates.

Harpaz, Y. (2005) Teaching and learning in a community of thinking. *Journal of Curriculum and Supervision, 20(2), 136-157.*

Hewitt, D. (1996) Mathematical Fluency: The Nature of Practice and the Role of Subordination*. For the Learning of Mathematics* 16, 2

Hopko, D. R. (2003) Confirmatory Factor Analysis of the Math Anxiety Rating Scale--Revised : Educational and Psychological Measurement . Sage Publications Retrieved September 15th 2008. Web site: http://epm.sagepub.com/cgi/content/abstract/63/2/336 Accessed 15th August 2008.

Houssart, J., Roaf, C. , & Watson. A. (2005) *Supporting Mathematical Thinking.* London: David Fulton .

Jackson, C., & Leffingwell, J. (1999, October). The Role of Instructors in Creating Math Anxiety in Students from Kindergarten through College. *The Mathematics Teacher.* Vol. 92, No. 7.

Kahney, H. (1984). *Problem Solving- a cognitive approach*. Buckingham: Open University Press.

Kaminski, J. , Sloutsky. A. , Vladimir M., Heckler , A. F. (2008, April). The Advantage of Abstract Examples. *Learning Math Science* 25. Vol. 320. no. 5875, pp. 454 – 455.

Lawrence, D. (1988) *Self-esteem in the Classroom.* London: Paul Chapman.

Lester, F.K. , Garofalo, J., & Kroll, D.L. (1989) Self-Confidence, Interest, beliefs, and metacognition: Key influences on problem-solving behaviour. In D.B. McLeod & V.M Adams (Eds) *Affect and Mathematical Problem Solving: A New Perspective (pp75-88).* NY: Springer-Verlag.

Mack, A. (2006, December). A Deweyan Perspective on Aesthetic in Mathematics Education. *Philosophy of Mathematics Education Journal, No. 19*.

Malmivuori, M-L. Academic Dissertation, May 2001. Retrieved January 6th, 2009 from University of Helsinki, Faculty of Education web site: http://ethesis.helsinki.fi/julkaisut/kas/kasva/vk/malmivuori/

Martin, L., Towers. J., & Pirie. S. (2006). Collective Mathematical Understanding as Improvisation. *Mathematical thinking and learning, 8(2), 149–183,* Lawrence Erlbaum Associates, Inc.

Mason, J. & Watson, A. (1999). *Questions and Prompts for Mathematical Thinking*. Derby: Association of Teachers of Mathematics.

Mason, J., Leone Burton, L. , & Stacey. K. (1982). *Thinking Mathematically Thinking Mathematically*, London: Addison Wesley .

McLeod, D. B. (1998). Affective Issues in Mathematical Problem-solving: Some Theoretical Considerations. *Journal for Research in Mathematics Education* v. 19, 134-41

McLeod, D.B. & Ortega, M. (1993). Affective issues in mathematics education. *In P.S. Wilson (Ed.), Research Ideas for the Classroom: High School Mathematics*. (pp. 21-36). NY: McMillan.

Nardi, E. and Steward.S. (2008). Is Mathematics T.I.R.E.D? A Profile of Quiet Disaffection in the Secondary Mathematics Classroom. *British Educational Research Journal*, 29, 3, 2003, pp. 4-9. Quoted from the BC Ministry of Education IRP's for Mathematics 8 and 9 (2008)

Nisbet,J. & Shucksmith, J. (1986) *Learning Strategies.* London: Routledge & Kegan Paul

Novick, L. (2006). Research on Procedural Transfer in the Solution of Mathematical Word Problems Retrieved December 17th 2008 Website: http://www.vanderbilt.edu/peabody/novick/proc_trans.html

Novick, L. R. (1995). Some Determinants of Successful Analogical Transfer in the Solution of Algebra Word Problems. *Thinking and Reasoning*, 1, 5-30.

Novick, L. R. (2006). *Research on Procedural Transfer in the Solution of Mathematical Word* Problems.Retrieved July 16th, 2008, Web site: http://www.vanderbilt.edu/peabody/novick/proc_trans.html

Novick, L. R., & Holyoak, K. J. (1991). Mathematical problem solving by analogy. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 17,* 398- 415

Op't Eynde, P., & De Corte, E. (2003). Students' Mathematics-Related Belief Systems: Design and Analysis of a Questionnaire. Paper presented at the annual meeting of the American Educational Research Association (Chicago, April 21-25, 2003). Retrieved September 9th 2008 from the Educational Resources Information Center.

Perkins, D. (2000). *Archimedes' Bathtub: The Art and Logic of Breakthrough Thinking.*NY: W. W. Norton & Company*;* 1st edition

Pimm, D. (1994) Spoken mathematical classroom culture: Artifice and artificiality. In *Cultural Perspectives on the Mathematics Classroom*, S. Lerman, ed. pp. 133 – 147. Kluwer.

Poincare , H. (1914). *Mathematical Discovery.* Collection of essays Science and Method, Maitland, F (trans). London: Thomas Nelson & Sons.

Polya, G. (1957) *How to Solve It*. (2nd ed.). Princeton University Press.

Pourdavood, R., Svec, L.V., Cowen, L.M., & Genovese, J. (2005). *Culture, Communication, and Mathematics Learning: an Introduction.* Retrieved December 22nd 2008. Web site: Center for Teaching - Learning of Mathematics http://www.thefreelibrary.com/Culture%2c+communication%2c+and+mathematics+learning%3a+an+introduction-a0136154745

Purkey , W.W. (1978). *Inviting School Success.* Belmont, CA:Wadsworth.

Purkey, W.W. (1970). *Self-Concept and School achievement*. NY: Prentice Hall

Rigelman, N. (2007, February). Fostering Mathematical Thinking and Problem Solving – The Teacher's role, *Teaching Mathematics*, pp. 308-314. NCTM

Rogers C. (1961). *On Becoming a Person*. Boston: Houghton Mifflin

Romberg, T. (1994). Classroom Instruction that Fosters Mathematical Thinking and Problem Solving. *Mathematical Thinking and Problem Solving*. Hillsdale, NJ: Lawrence Erlbaum Associates Inc.

Rota, G. (1985, Spring/Summer). A Dialogue with Gian-Carlo Rota and David Sharp: *Mathematics, Philosophy, Artificial and Intelligence*. LOS ALAMOS SCIENCE

Rota, J-C. (1997) *Indiscrete Thought*s. Boston: Birkhauser*,*

Schoenfeld, A. H. (1985). *Mathematical Problem Solving.* San Diego: Academic Press Inc.

Schoenfeld, A.H. (1994). Mathematical Thinking and Problem Solving. NJ: Lawrence Erlbaum Associates Inc.

Schoenfeld, A.H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics. In Douglas A. Grouws (ed.) *Handbook of Research on Mathematics Teaching and Learning*. A project of the National Council of Teachers of Mathematics. NY: Macmillan Publishing Company. pp. 355-358.

Seligman, M.E.P. (1975). *Helplessness: On Depression, Development, and Death*. San Francisco: W.H. Freeman.

Sinclair, N, & Watson, A (2001). Wonder, the rainbow and the aesthetic of rare experiences. *For the Learning of Mathematics, 21(3), 39-42*

Sinclair, N., Strickland, S. K., & Kozse, L. (under revision). Abductive inferences in mathematics thinking: Necessary risks. *Educational Studies in Mathematics*. (Copy obtained from Sinclair, N., 2009.)

Smith, W.J., L. Butler-Kisber, L., LaRoque, L. Portelli, J., Shields, C., Sturge Sparkes, C. & Vilbert, A. (2008) Student *Engagement in Learning and School Life: National Project Report.* Montreal, QC: Ed-Lex., 1998. Quoted from the BC Ministry of Education IRP's for Mathematics 8 and 9.

Stein, M. & Burchartz, M. (2006)The Invisible Wall Project: Reasoning and Problem Solving Processes of Primary and Lower Secondary Students, *Mathematical thinking and learning,* 8(1), 65–90. NJ: Lawrence Erlbaum Associates.

Stroup, W., Kaput, J., Hegedus, S., Ares, N., Wilensky,U., Roschelle, J., & Mack, A. (2002). The nature and future of classroom connectivity; the dialectics of mathematics in the social space. In D. Mewborn et al (Eds.); *Proceedings of the 24th annual meeting of the north American Chapter of the International Group for the Psychology of Mathematics Education* (Vol 1. pp. 195-203). Columbus, OH: ERIC Clearinghouse.

Suinn, R. M., & Winston, E. H. (2003). *The Mathematics Anxiety Rating Scale, A Brief Version:* Psychometric Data. Psychological Reports 2003, 92, 167-173. Retrieved July 5th 2008, from web site: www.msme.us/2008-1-6.pdf

Tobias, S. (1993). *Overcoming Math Anxiety*. NY: W.W. Norton & Company

Uswami, G. (2002). *Analogical reasoning in Children.* Retrieved December 17th 2008, Web site: http://www.nbu.bg/cogs/personal/kokinov/COG501/ANALOGICAL%20REASONING%20IN%20CHILDREN.pdf.

Van de Walle, J.A & Folk, S.(2007). *Elementary and Middle School Mathematics*. Second Canadian Edition Pearson Education Canada.

Wang, H. (2001). Aesthetic Experience, the Unexpected, and Curriculum. *Journal of Curriculum and Supervision, 17(1), 90-94.*

Weiner, B. (1973). *Theories of motivation: From mechanism to cognition*. Chicago: Markham.

Wieschenberg, A. A. (1994). *Overcoming Conditioned Helplessness .* Mathematics College Teaching, Vol. 42, (Spring 1994) p. 51-4

Williams, K. (2003). Writing about the Problem Solving Process, *Mathematics Teacher,* Vol. 96 No.3