

Students' Mathematical Modelling Behaviors: Strategies and Competencies

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Abstract

Mathematical modelling has recently taken the spotlight in mathematics education as a means to prepare students for the challenges they face in the modern world, and there have been numerous proposals on the modelling cycles describing students' approaches to solve modelling tasks. Within these proposed modelling cycles, researchers emphasize the importance of building a real model to describe the real situation and the application of extra-mathematical knowledge to highlight the relationship between reality and mathematics. However, the concept of extra-mathematical knowledge and the process to establish a real model have only been described in broad strokes and these descriptions lack details. This thesis aims to add to the descriptions of extra-mathematical knowledge and the process to develop a real model based on empirical data by closely examining students' *mathematical Modelling* behaviors. To achieve these goals, I administered two rudimentary mathematics complex tasks, a special type of tasks that present a complex situation but allow the audience to apply their well-worn tools in mathematics to establish a solution, to two groups of junior secondary school students. These tasks allow me to tip the balance of between reality and mathematics in mathematical modelling in order to focus on students' modelling behaviors. With regard to the process leading to a real model, my analysis indicates that students hold different intentions in building a real model and these intentions affect the strategies they use and therefore their modelling process and the quality of their solutions deeply. In the analysis of these strategies, I also apply flow theory to understand these intentions. As for extra-mathematical knowledge, my analysis demonstrates that extra-mathematical knowledge is a multi-faceted, complex construct composed of various competencies, that contains different characteristics and can deeply affect students' engagement with the tasks.

Keywords: mathematical modelling; reducing reality; reducing complexity; extra-mathematical knowledge; flow

For my family

Acknowledgements

In an episode in *Doctor Who, Heaven Sent*, the *Doctor* was trapped, alone, was confronted with his fears, and had to figure a way out. After being chased by a demon, the *Doctor* figured his only way out was through the Azbantium wall that was thousands of times harder than diamond. Determined to escape, the *Doctor* threw a few punches at the Azbantium wall just before he got caught by the demon. He died, regenerated, and repeated the process. It took him thousands of years to punch a hole through. He did, eventually, escape.

This thesis writing experience reminds me of this episode. Unlike the *Doctor*, I was not trapped, nor was I ever alone. However, I was, most definitely, confronted with my fears. I doubted myself numerous times and I was at the edge of giving up countless times. And I am glad that I did not give up, for it has been an amazing journey. And thank goodness it took me less time to finish this thesis than the *Doctor* to punch through a wall.

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Preface: Please mind the gap

A trip to Asia a while back reminded me of something I have not heard in a long time. At train stations, the PA systems announce every time a train approaches the platform, “Please mind the gap between the train and the platform door”. This simple reminder prompted me to think about my role and my practice as a classroom teacher, and drew my attention to the mathematics I teach students inside the mathematics classroom and the mathematics I also wish students to know or to notice. It seems to me that sometimes the mathematics I teach my students, no matter how beautiful it may be, exists only inside the mathematics classroom. There seems to exist a gap between the mathematics classroom and the rest of the world: mathematics is confined inside the mathematics classroom, and what is applicable outside of the mathematics classroom does not necessarily relate to what students learn inside the mathematics classroom. This gap also seems to contribute to many of the negative interpretations about mathematics I’ve observed over the years.

These thoughts about the gap saddens and scares me. If what I teach students in mathematics class remains in mathematics class, what am I teaching? What are my students learning?

These thoughts also inspires me to reflect on my practice as a mathematics teacher: How does my teaching contribute to students’ mathematics learning experiences? Is my practice strengthening this gap, to confine mathematics in the classroom, or closing it? As a mathematics education researcher, I also wonder what are some of the possible ways to close the gap?

Chapter 1.

Introduction

1.1. The need of mathematical modelling

... there is no point in educating human automata; they are losing their jobs all over the world. Society now needs thinkers, who can use their mathematics for their own and for society's purposes. Mathematics education needs to focus on developing these capabilities. (Burkhardt, 2006, p. 183)

Mathematics teaching used to focus on students' mastery of procedural skills such as basic computations, factoring, graphing, trigonometry, etc., to promote students' efficiency and accuracy in performing these calculations and computations, and to use these skills to solve mathematical problems. Under this mathematics education system, graduating students can solve complex mathematical equations and problems, spot patterns and make generalizations, and have the fluency to apply their mathematical knowledge to solve mathematical problems.

Due to developments in technology, procedural calculations are available at the touch of our fingertips in the modern-day society – calculators, cell phones, smart devices, etc. These technology devices allow students to determine the answers to procedural calculations efficiently and accurately. This is not to say procedural fluency has lost its importance in mathematics education. Rather, the advancements in technology require students to be capable of much more than being proficient at the procedural skills mentioned above. These procedural skills, in Bauersfeld's terms, serve as "the bricks for the building, but the design for the house of mathematizing is processed on another level" (as cited in Yackel and Cobb, 1996, p. 4).

In the modern world students are often bombarded with an overwhelmingly large amount of information that is constantly changing and complex systems of information that require them to sort and tease out the information they need to make suitable decisions, such as choosing a cell phone plan, purchasing a vehicle, paying off a student loan, etc. As such, students need to be more capable than being proficient at procedural skills. Students are required to "use mathematics to recognize the issues, understand the

problems, synthesize the available information, analyze the results, and communicate solutions in order to make decisions” (van de Walle et al., 2015, p. 2), and to “activate their mathematical competencies to solve problems they encounter in life” (OECD, 2004, p. 35). In other words, with the current technology advancements, it is not sufficient to be fluent only in procedural skills. What is required, along with fluency, includes a deep understanding and flexibility – knowing multiple approaches to solve mathematical problems, and being able to recognize and use the appropriate and effective mathematical tool to solve these mathematical problems (Yakes & Star, 2011). When dealing with unfamiliar mathematical problems:

someone with only superficial knowledge of procedure likely has no recourse but to use a standard technique, which may lead to less efficient solutions or even an inability to solve unfamiliar problems. But a more flexible solver – one with a deep knowledge of procedures – can navigate his or her way through this procedural domain using techniques other than that are over practiced, to produce solutions that best match problem conditions or solving goals. (Star, 2005, p. 409)

As a result of these changes in demands, mathematics educators call for mathematics curricula that widen the focus from fluency in procedural skills to include flexibility in students’ use of mathematical knowledge.

Many current mathematics curricula push for students to have a curious mind, to have perseverance when dealing with difficult problems, to ask *what if* and *why* questions, to verify and make sense of their answers, etc. Unfortunately, despite the demands in these mathematics curricula, many of our graduates seem to lack these attributes. Many students are completing their compulsory education without the mathematical skills to cope with the demands that life and work will require of them. In school, mathematics is often taught as a subject that is isolated from reality. School mathematics often value accuracy and speed over struggle and understanding; focus on procedure and techniques rather than multiplicity of ideas. Mathematical problems often come after students master specific mathematical knowledge or techniques. These problems are often designed for students to apply these particular mathematical techniques. While these mathematical problems may require the use of a number of different mathematical approaches or strategies, they are usually clean, and students are most often given all that they need to solve the problem (Niss, 1987). These problems are used for students to practice particular mathematical skills and knowledge,

and do not resemble the world we live in, where problems which require mathematics are usually messy, require students to seek additional information, and to activate the necessary mathematical or real world knowledge to complete the problem (OECD, 2004, p. 36).

As such, a simple curricular change is insufficient to prepare our students for the challenges they face in life. Mathematics education also needs to provide students with opportunities to practice dealing with messy real-life problems where they need to analyze the situation given to them, to organize and to identify the information that would help them solve the problem, to identify and to apply the mathematical skills they possess to the situation on hand, to solve the problem mathematically, and to verify, evaluate, explain, and to communicate the solution in the context of the original problem.

A possible way to address these skills is through mathematical modelling.

1.2. Mathematical modelling and our current mathematics education system

A mathematical model is a representation of a situation via mathematics. They are “some kind of mathematical statement about a problem originally posed in non-mathematical terms” (Howison, 2004, p. 4). Models are created, often, to make quantitative descriptions and predictions of a situation and to explain the situation (Howison, 2004; Fowler, 1997). They draw attention to mathematics in what seems to be non-mathematical situations, they provide a way to represent a situation in reality in a simplified manner, and they highlight and bring forth specific aspects of a phenomenon (NCTM, 1989). In terms of mathematics education, mathematical modelling can be used as “a learning milieu where students are invited to take a problem and investigate a situation with reference to reality via mathematics” (Barbosa, 2003, p. 230). From this perspective, mathematical modelling brings forth the relationship between reality and mathematics, and highlights the use of mathematics in the process of solving problems situated in reality. As such, mathematical modelling has an important role in mathematics education: to promote mathematical understanding using realistic situations or scenarios inside the mathematics classrooms along with the application of mathematics in realistic situations outside of the mathematics classroom.

Mathematical modelling has become increasingly prominent in mathematics education in various countries around the globe in the recent past (for example, see Blomhøj, 2011; NCTM, 1989; Singapore Ministry of Education, 2006; California State Board of Education, 2015; English & Watters, 2004; Houston, 2003). Some of these mathematics curricula “integrate models and modelling in the teaching of mathematics both as means for learning mathematics and as an important competency in its own right” (Blomhøj, 2009, p. 5).

For example, the California State Board of Education highlights the importance of mathematical modelling as a tool “for solving everyday problems, making informed decisions, improving life skills (e.g. logical thinking, reasoning, and problem solving), planning, designing, predicting, and developing financial literacy” (California State Board of Education, 2015). NCTM (1989) points out that it is necessary for school curricula to provide students with opportunities to experience mathematical modelling and develop their mathematical modelling skills in order to become successful in various disciplines. In Germany, the *Steigerung der Effizienz des mathematisch-naturwissenschaftlichen Unterrichts* project (*Increasing the efficiency of math and science teaching*, abbreviation: *SINUS*) was launched soon after the TIMSS results in 1997 were released (Blum & Ließ, 2005). Some of the goals of the project include: to provide students with modelling opportunities, in which students are encouraged to make vertical connections (within mathematics) and horizontal connections (between mathematics and reality); to foster students’ communication skills and skills to work cooperatively with others; and to foster a *learner-centered and teacher-directed* classroom (Blum & Leiß, 2005). Eventually, modelling competencies become a focus of the German mathematics education curriculum.

In Canada, the recent changes in the British Columbia mathematics curricula (K-12) also highlight the flexible use of mathematics as a part of students’ daily problem solving processes. The curricula value students’ communication, thinking, and personal and social competencies, and aim to develop both students’ content knowledge in mathematics and their processed-based or curricular competencies to solve complex problems that arise in students’ daily lives through flexible teaching and learning. Some of the competencies are repeatedly seen throughout the curricula include students’ abilities to interpret, analyze, apply, solve problems, and to communicate ideas with

others, which line up well with the idea of mathematical modelling (British Columbia Ministry of Education, 2017).

1.3. An overview of this thesis

This introduction illustrates the challenges technological advancements bring for our students and argues that mathematical modelling provides students with the opportunities to flexibly solve problems beyond the mathematics textbooks. As such, tasks that encourage mathematical modelling could be a means for students to apply mathematics to investigate and solve messy problems situated in reality and to build connections between reality and mathematics.

Given my focus and interest to highlight these reality-mathematics connections through mathematical modelling, I naturally aim to investigate the challenges students face and the competencies students develop in solving such tasks. Prior to discussing my aims of this thesis and presenting my investigation in detail, I provide readers with an overview of mathematical modelling literature from Chapter 2 to Chapter 5.

In Chapter 2 and Chapter 3, I present literature pertaining to the many purposes of mathematical modelling and the use of modelling tasks to achieve these purposes. In general, mathematical modelling highlights the relationship between reality and mathematics, develops students' ability to use mathematics to deal with everyday challenges, and to enhance students' mathematics learning experiences. These goals can be achieved through the use of various types of modelling tasks.

In Chapter 4, I focus on literature that discuss students' mathematical modelling behaviors and the modelling cycles which researchers offer to describe these behaviors. In this chapter I also highlight the similarities and differences between these modelling cycles. In Chapter 5, I dwell into literature that focus on modelling competencies, or skillsets which students apply to solve modelling problems. I examine the ways which mathematical modelling researchers propose to target the development of specific modelling competencies. It is also in Chapter 5 that I discuss the necessity to develop a new type of tasks as a means to investigate students' mathematical modelling behaviors along with the challenges students face and the competencies they develop during the process of mathematical modelling.

After an overview of mathematical modelling literature, I provide a description of this study and the data collected in this study from Chapter 6 to Chapter 7.

In Chapter 6, I discuss the framework to be used in this thesis and the intention of this thesis. I also present two research questions which I aim to answer in this thesis.

Chapter 7 provides details to the setting of the study. I present the two tasks used to study the research questions, the participants involved, the data collection process, and my role in this study. Chapter 8 and Chapter 9 focus on the first task used in this study, the *Door Project*. In Chapter 8, I present four groups of students' modelling processes as they worked on the *Door Project* and my analysis of these four groups of students' modelling processes. In Chapter 9, I discuss the modelling competencies that lead students to successfully solve the task. Chapter 10 and Chapter 11 are set up similarly to Chapter 8 and Chapter 9, but focus on the second task used in this study, the *Design a New School* task. Chapter 10 presents the modelling process of three groups of students and my analysis of these processes. In Chapter 11, I discuss students' approaches and the results of these approaches to solve the task along with the modelling competencies that allowed students to successfully solve the task.

Chapter 12 to Chapter 15 provide an overall analysis of the data, focusing on the strategies students employed in this study, the difficulties they experienced, and their mathematical modelling competencies.

In Chapter 12, I discuss two specific strategies students employed in the process of simplifying the tasks. These strategies have roots in students' different intentions and lead to very different results in their modelling process. Chapter 13 to Chapter 15 are written to address some of the specific modelling competences discussed in modelling literature. In Chapter 13, I identify the facets of extra-mathematical knowledge in solving the tasks. I first bring the discussion to students' activation, acquisition, and application of these extra-mathematical knowledge during their modelling processes, and then discuss the characteristics of these extra-mathematical knowledge observed in the data. In Chapter 14, I discuss students' validation process and highlight two types of validation processes found in this study. Chapter 15 is a response to literature's description of extra-mathematical knowledge related competencies, including the competencies closely tied to the modelling process and the those as a complex construct. Finally, Chapter 16 provides a conclusion of this study. I provide a summary of the findings,

discuss the limitations of this study, and reflect on my journey to use mathematical modelling to bring forth the relationship between mathematics and reality.

Chapter 2.

The Goals of Mathematical Modelling

A precise clarification of concepts is necessary in order to sharpen the discussion and to contribute for a better mutual understanding [of mathematical modelling from a mathematics education perspective]. (Kaiser & Sriraman, 2006, p. 308)

Mathematics education researchers approach mathematical modelling from a wide range of perspectives and identify a number of different goals of mathematical modelling. In the 1980s, Kaiser-Messmer identifies two major perspectives of mathematical modelling – a *pragmatic perspective* and a *scientific-humanistic perspective*. Kaiser-Messmer's *pragmatic perspective* is closely related to Pollak's pragmatic perspective and emphasizes the development of students' ability to solve practical problems through the use of mathematics; and the *scientific-humanistic perspective* draws attentions to the "relations between mathematics and reality" (Kaiser & Sriraman, 2006, p. 302).

In the early discussions of mathematical modelling in mathematics education, researchers also identify *epistemological goals*, an *emancipatory perspective*, and an *integrative perspective* as purposes of mathematical modelling. *Epistemological goals* are closely related the scientific-humanistic perspective and focus on the "development of mathematical theory as an integrated part of the processes of mathematizing" (Kaiser & Sriraman, 2006, p. 302). An *emancipatory perspective* draws on the critical aspects of mathematical models that are used to describe issues in society. This perspective is similar to Barbosa's sociocultural view of mathematical modelling. Barbosa (2003) proposes that mathematical modelling provides students with opportunities to "deal with situations that are part of their day-to-day lives and part of the working world" (p. 230), and to "empower citizens in debates and in decision-making that involves mathematical applications" (p. 229). Finally, an *integrative perspective* suggests mathematical modelling could be used to achieve a variety of purposes, including scientific, mathematical, and pragmatic ones (Kaiser & Sriraman, 2006).

While Kaiser-Messmer (Kaiser & Sriraman, 2006) focuses on describing specific mathematical modelling goals, Julie (2002, as cited in Barbosa, 2003) and Galbraith and

Stillman (2006) suggest two general goals of mathematical modelling: *modelling as vehicle* and *modelling as content*. *Modelling as vehicle* uses models and modelling activities to introduce curricular material and to help students develop their understanding of the material. As a result, modelling as vehicle focuses on the curriculum rather than the modelling process, and uses modelling as a method to promote content and enhance students' mathematical learning experiences. This is similar to *vertical mathematization* in realistic mathematics education, RME. A discussion on RME can be found in Chapter 4.

Conversely, *modelling as content* aims to connect students' real life experiences with mathematics (Blomhøj, 2008), help students see the connections between mathematics and reality, and uses mathematics to describe and understand situations in life. This is similar to *horizontal mathematization* in RME, and is related to Pollak's pragmatic perspective of modelling. These two ideas, modelling as vehicle and modelling as content, represent two vital goals of modelling from a mathematics education perspective – to build and to promote students' skills in mathematics and to connect these skills with students' real life experiences.

In the short discussion thus far, I highlight a few researchers' views and interpretations of modelling goals from a mathematics education perspective. Other than modelling as vehicle which is closely related to vertical mathematization in RME and draws attention to the learning of mathematics, the rest of these interpretations draw attention to the relationship between mathematics and reality, and bring awareness to the use of mathematics to solve problems and make informed decisions on a daily basis. Reflecting on these different views and interpretations of modelling goals, Kaiser and Sriraman (2006) identify four major goals in the discussion of mathematical modelling:

- *Pedagogical goals* – to improve learners' understanding of reality;
- *Psychological goals* – to promote learners' positive attitude and motivation towards the learning of mathematics;
- *Subject-related goals* – to improve the structure of mathematics learning by introducing mathematical concepts using realistic scenarios;
- *Science-related goals* – to provide insight into the relationship between mathematics and reality in the history of mathematics.

Reflecting on these four major goals and the various perspectives of mathematical modelling, Kaiser and Sriraman (2006) create a classification system to distinguish between these goals and perspectives, to provide continuity for discussions in mathematical modelling, and to organize coherently these perspectives according to existing literature. Kaiser and Sriraman's classification system consists of six modelling perspectives: *realistic or applied modelling*; *contextual modelling*; *educational modelling*; *socio-critical modelling*; *epistemological or theoretical modelling*; and *cognitive modelling*.

Realistic or applied modelling aims to use mathematics to solve realistic problems and to promote mathematical modelling competencies. It is related to Pollak's pragmatic perspective, which aims to promote students' ability to "apply mathematics to solve practical problems" (p. 302). *Contextual modelling* carries with it psychological and subject-related goals, and tends to focus on the learning of school mathematics, including solving word problems.

The third perspective, *educational modelling*, is closely related to the integrative perspective, which serves to promote the relationship between mathematics and reality and the learning of mathematics. Educational modelling can be differentiated into *didactical modelling* and *conceptual modelling*. Didactical modelling focuses on students' learning processes, and conceptual modelling focuses on the introduction and developments of various mathematical concepts.

Socio-critical modelling carries with it pedagogical goals and deals with developing students' understanding of the world surrounding them. It is closely related to the emancipatory perspective of modelling and Barbosa's (2003) view of sociocultural view of mathematical modelling. *Epistemological or theoretical modelling* "[emphasizes] the development of mathematical theory as an integrated part of the processes of mathematizing" (p. 302). Finally, *cognitive modelling* can be classified as a type of meta-perspective that focuses on the "analysis of cognitive processes taking place during modelling processes and understanding of these cognitive processes" and the "promotion of mathematical thinking processes by using models as mental images or even physical pictures or by emphasising modelling as mental process such as abstraction or generalization" (p. 304).

These perspectives highlight a wide range of foci and goals of mathematical modelling. Mathematical modelling brings awareness to the relationship between reality and mathematics, develops students' abilities to use mathematics to deal with everyday challenges from an individual and a societal perspective, to analyze and to critique mathematical models, and to make educated decisions based on their understanding of these models (Blomhøj, 2009). It emphasizes the development of students' abilities and competencies to solve realistic problems, including "a mental 'modelling infrastructure' so that they can become users of their mathematical knowledge in the sense of being able to independently address problems in their world" (Galbraith, 2012, p. 4). Mathematical modelling also serves to provide students with a positive mathematics learning experience: to provide structure in the learning of mathematics and to illustrate mathematical concepts. As mathematical modelling carries with it all these goals and benefits to students learning of mathematics, it is important to investigate the development of these skills and to understand the process of mathematical modelling through tasks. In Chapter 3, I look into the various categories of tasks and the classification of these tasks that could be used to promote mathematical modelling. In Chapter 4, I bring forth the idea of cognitive modelling and discuss the process of mathematical modelling, with a focus on modellers' expected behaviors during the modelling process.

Chapter 3.

Develop Modelling skills through tasks

Tasks are a fundamental part of mathematical lessons and thus have a central position within mathematics education. (Maaß, 2010, p. 286)

In Chapter 1, I highlighted the importance of mathematical modelling and the incorporation of mathematical modelling by various mathematics curricula around the globe. These mathematics curricula suggest the use of tasks to foster students' modelling skills. For example, the California State Board of Education (2015) suggests to promote mathematical modelling skills by providing "opportunities for students to tackle real-world problems of different complexity" (p. 808) and developing students' "understanding of, and experience with, all stages of the mathematical modelling process" (ibid.). The Consortium for Mathematics and Its Applications (COMAP), a project under the NCTM, promotes the idea of mathematical modelling as a process through the use of "messy, open-ended problems that require students to make genuine choices about how to approach problems mathematically, what assumptions to make, and how to determine the effectiveness of the approach used" (Hernández et al., 2016, p. 337).

In order to promote students' mathematical modelling skills or modelling competencies, it is important for teachers to set clear goals of the mathematics lessons, receive adequate and continuous professional development, and above all use effective tasks that aim at the development of these skills (Maaß, 2010). There are many types of tasks. One of the types of tasks that is commonly found in a math classroom is word problems.

3.1. Word Problems

Word problems can be loosely defined as problems "in which a more or less familiar situation is described and a quantitative question is posed that can be solved with the help of mathematics" (Maaß, 2010, p. 290). They serve as a mean for students to practice specific skills in mathematics, usually the skills students are currently learning or have recently learnt. They are well-defined, describe an "idealized real-world situation in mathematical terms" (Pollak, 2012, p. viii), require students to choose a basic arithmetic

operation based on the key words in the problem, extract numerical values from the problem text, apply the operation to these numbers, and determine the mathematical solution to the operation. While students should, under ideal conditions, verify their solutions by comparing their answer to the problem to see whether the solution is reasonable, students under most circumstances do not do so (Greer, 1997).

Word problems are a good starting point to encourage students to apply their mathematical knowledge to solve problems. However, word problems are only a starting point, because they do not necessarily represent situations which students may experience in their daily lives. They are also often simple and artificial, in which students' lived-experiences can be ignored while solving these tasks, and an unambiguous and exact answer is found through the combination of the numbers given in the text using one or more mathematical operations (Maaß, 2010). These problems are sometimes disconnected from reality and do not represent the challenges students face on a daily basis, where situations are often messy and ambiguous, and solutions to the problem situations may not be as exact as those in word problems. Furthermore, "if students are only faced with easy mathematical and situational problems, they will not have the opportunity to learn how to apply reason to the relations between the quantities present in a problem or to the situational limits that the context imposes" (Sánchez and Vicente, 2015, p. 698). The following is an example of a word problem:

If two watermelons cost \$13.98, how much does it cost to purchase 55 watermelons?

This question involves students' understanding of unit rate and allows students to practice their multiplication skills. However, it is distant from reality and focuses on practicing students' specific mathematical skills rather than to make sense of the situation. In particular, why does one need 55 watermelons? Can one purchase these watermelons individually, or can one only purchase them in pairs?

To encourage students to think beyond what the question has given and relate the question to their everyday experiences, tasks that have a stronger connection to reality than word problems are required. These tasks can be referred to as reality-related tasks (Maaß, 2010). A type of reality-related tasks that can be used for these purposes are modelling tasks.

3.2. Modelling tasks

Literature suggest that modelling problems are “authentic, complex and open problems which relate to reality” (Maaß, 2006, p. 115). They provide an avenue for modellers to successfully reach one or more modelling competencies which will be discussed in Chapter 5. They allow for the investigation of a problem situated in reality from a mathematical perspective; the deepening of understanding of mathematical concepts and how these concepts could be applied to solve realistic situations; and highlight the mathematics in what seems to be non-mathematical situations (Barbosa, 2003; Borromeo Ferri, 2006; Howison, 2004; Fowler, 1997; Swetz and Hartzler, 1991).

Compared to word problems, modelling tasks are messy problem solving questions that are situated in reality. Barbosa (2006) describes modelling tasks as problematic activities (as compared to exercises) that are not purely mathematics. These problems require modellers to investigate the situation “with reference to reality via mathematics” (p. 294), and relate what happens in reality to mathematics by applying their mathematical knowledge and skills to determine a possible solution to the problem.

Modelling tasks can be classified based on their focus of modelling activity, data, nature of relationship to reality, situation, type of model used, type of representation, openness, cognitive demand, and mathematical content (Maaß, 2010). The last three classification schemes – openness, cognitive demand, and mathematical content, are not limited to modelling tasks. In what follows I provide a brief description of these classifications.

3.2.1. Modelling Activity

Modelling tasks can be classified based on their focus of modelling activity: to carry out either the entire modelling process, or a specific step/steps within the modelling process (Maaß, 2010). These tasks are used to promote modellers’ understanding of the steps taken to solve a modelling task.

3.2.2. Data

Data tasks require modellers to investigate and sort out the data, to critically decide the information relevant to solve the task, and to acquire the missing information required to

solve the task. Data tasks can be classified into five categories: superfluous data tasks contain both relevant and irrelevant data and require modellers to sort out the data; missing data tasks do not provide modellers with all the information they need and require modellers to acquire the missing information or make estimations about the missing information; missing and superfluous tasks contain relevant and irrelevant data but also require modellers to acquire missing information about the task; inconsistent data tasks aim for modellers to reflect on the context of the problem and may contain missing and superfluous data; matching data tasks carry all information required to solve the problem and are similar to embedded word problems (Maaß, 2010).

3.2.3. Nature of Relationship to Reality

Five categories of modelling tasks are found under the nature of relationship to reality classification scheme: authentic tasks, realistic tasks, embedded word problems, intentionally artificial tasks, and fantasy tasks. Authentic tasks are context-based tasks which are relevant to those who work on the tasks. Also, the task itself, the data presented, and the way the data is presented must be genuine in order for the task to be considered authentic. Realistic tasks are closely related to reality, but the data or the problem presented is not necessarily genuine. These tasks may ignore certain aspects of reality. Embedded word problems are tasks in which mathematics is embedded in a chosen situation. The situation presented in embedded word problems are simplified situations and may be exchanged with other situations. They do not represent genuine contexts. Intentionally artificial tasks are created to allow for students to reflect on the task situation. Finally, fantasy tasks are based on a fantasy world which students may find appealing.

3.2.4. Situation

The situation of the tasks refers to the circumstances under which a task takes place. It ranges from personal (closest to a student's life), to educational/occupational, public, and scientific (furthest away from a students' life).

3.2.5. Types of Models

Modelling tasks can also be classified based on the types of models produced.

Descriptive models describe the situation as close to reality as possible; while normative models only describe certain aspects of reality while ignoring other aspects.

3.2.6. Type of Representation

Another way to categorize modelling tasks is based on the way they are presented: text; pictures; text and pictures; materials; and situations. Tasks that are presented via text contain written work, those presented via pictures contain pictures and/or photographs; text and pictures tasks use pictures and/or photographs to illustrate specific aspects of the task; tasks presented via materials include printed materials such as newspaper articles, invoice or bills, radio broadcast, etc. Tasks presented via situations include those which require modelers to physically explore the situation.

3.2.7. Openness of a Task

There are seven types of tasks under this classification scheme: solved examples, ascertaining tasks, reversal tasks, ascertaining problems, reversal problems, finding a situation tasks, and open problems. Solved example tasks provide modellers with the solution of a task and may require modellers to reflect on the solution provided. Ascertaining tasks provide modellers with the situation and with sufficient information about the situation to allow modellers to easily identify the steps need to be taken to solve the problem. These are similar to embedded word problems. Reversal tasks provide modellers with a problem situation and the end situation of the problem, and require them to reflect on the possible ways to connect the original situation to the end situation. For example, given a price list of pastries, what can one purchase with \$10? Ascertaining problems provide modellers with a problem situation and require them to reflect on the situation to determine the steps taken to reach the solution for the problem situation. Reversal problems provide modellers with the solution and require them to reflect on the possibilities of the original situation. Finding a situation tasks ask modellers to determine a problem situation which requires them to use a particular tool to solve the problem. Finally, open problems are geared towards advanced modellers, where a general problem is recognized but no details are provided.

3.2.8. Cognitive Demand

How cognitively demanding a task is depends on its target audience. For example, what young modellers or students find cognitively demanding may be trivial for older or more mature modellers or students. There are six categories under this classification: extra-mathematical modelling, inner-mathematical working, *grundvorstellungen*, dealing with texts containing mathematics, mathematical reasoning, and dealing with mathematical representations. Extra-mathematical modelling focuses on all the work done in reality and the work done in transitioning between reality and mathematics. The level of cognitive demands depends on the complexity of the situation and the mathematization steps. Inner-mathematical working focuses on the complexity of the mathematics involved. *Grundvorstellungen* refer to the “mental objects [required for the] transitions between reality and mathematics” (Maaß, 2010, p. 300). They “act as mediators between the two worlds” (ibid.). The more *grundvorstellungen* are involved, the more cognitively demanding the tasks are. Dealing with texts containing mathematics influence how cognitively demanding the task is by the structure the task is presented. This includes the written structure of the task and the order of the information presented. Mathematical reasoning refers to the complexity and amount of the reasoning involved to solve the task. Finally, dealing with mathematical representations refers to the complexity of ways to convey mathematical information, including graphs, sketches, etc.

3.2.9. Mathematical Content

From a mathematical content perspective, how cognitively demanding a task is depends on the mathematical area (arithmetic, algebra, etc.) and the school level (primary, secondary, tertiary, etc.) the task involves.

Maaß's (2010) classification scheme emphasizes that these categories of tasks could be used to help with the development of modelling skills for various groups of audiences, as different groups of audiences bring with them different skillsets and different needs, and these tasks could be used to target the audiences' specific needs. Other than these categories of modelling tasks that originate from mathematical modelling research and serve various modelling goals, there exists another genre of modelling tasks that originates from the need to promote students' ability to solve realistic quantitative

situations, and require students to consciously make assumptions and decisions about the situation. These are called numeracy tasks.

3.3. Numeracy Tasks

Numeracy can be loosely defined as “a concrete skill embedded in the context of real-world figuring” (Cohen, 2001, p. 25). It refers to the understanding, appreciation, and the communication of mathematical information, and the effective use of the tools in one’s mathematical toolkit to deal with life’s “diverse contexts and situations” (Orrill, 2001, p. xviii). In addition to a toolkit full of well-worn and familiar tools, numeracy also points to a disposition – a willingness to engage with the day’s problems through the use of mathematical tools (Steen et al., 2001), because just as having all the tools does not make one a handyman, having an extensive mathematical toolkit does not make one numerate. As the term ‘handyman’ implies doing hands on work, a numerate person is willing to use his/her mathematical tools to solve problems.

The importance of numeracy has been recognized in recent years and numeracy has become a focus in the British Columbia K-12 mathematics curricula. *Numeracy Tasks* have been designed specifically to meet the numeracy goals, which are included in the British Columbia mathematics curricula’s core and curricular competencies. *Numeracy Tasks* are crafted around “an aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication abilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work” (Steen et al., 2001, p. 7).

Numeracy tasks can be classified as a type of modelling tasks in the sense that they are meant to invite students to “investigate a situation with reference to reality via mathematics” (Barbosa, 2003, p. 230). These tasks aim to develop students’ ability to solve realistic problems via mathematics; to draw attention to the possible connections between reality and mathematics; to use mathematics to describe and understand realistic situations; to critically analyze a realistic situation via mathematics; etc. These goals align well with various modelling goals as discussed in Chapter 2, such as realistic or applied modelling, educational modelling, and socio-critical modelling (Kaiser & Sriraman, 2006).

Numeracy tasks provide students with a low floor, a high ceiling, a huge degree of freedom, some fixed constraints, and inherent ambiguity (Liljedahl, 2010). A low floor refers to the minimal amount of prerequisite knowledge required to approach the task. It allows all student participants to become engaged with the problem situation. A high ceiling refers to the possible (mathematical) extensions made to the task. Other than a low floor and a high ceiling, numeracy tasks also provide students with a huge degree of freedom along with fixed constraints. While these may seem to contradict each other, they are both necessary in numeracy tasks. A huge degree of freedom allows for multiple approaches and strategies to the problem. However, when “faced with huge degrees of freedom students can become paralyzed by the overwhelming number of possibilities available to them. Constraints are therefore used to give them somethings to organize their thinking around – thus liberating them to begin” (Liljedahl, 2010, para. 15). Finally, inherent ambiguity allows students to become aware of the assumptions and decisions they make regarding the situation. Overall, numeracy tasks are open-ended tasks which allow students to interpret and make assumptions about the situation. They allow for a variety of answers and may not lead to a single best answer.

There are at least four kinds of numeracy tasks (Liljedahl, 2010):

- 1) Fair Share Tasks – Fair share tasks involve a scenario in which students need to develop a strategy to fairly share something. However, the notion of fairness is not clearly defined in the situation. Students are therefore required to generate a solution which account for fairness in sharing;
- 2) Planning Tasks – Planning tasks require students to sort out a situation and to generate a plan as a solution. These plans may involve money, time, distance, spatial orientation, etc.;
- 3) Estimation Across a Large Number of Variable Tasks – These tasks require students to make estimations and assumptions about the situation to generate a solution. These estimations and assumptions may be restricted to a single unit of measurement but involve a range of contexts; and
- 4) Data Tasks – In these tasks, students are given a set of data or information and a scenario. Students then study the data set and develop a strategy based on the data set for the scenario. Students may be required to follow up on their strategy and make the necessarily adjustments and improvements.

As these tasks outline the possible types of numeracy tasks, numeracy tasks may involve a combination of the aforementioned types.

3.4. Summary

Modelling tasks could be classified based on their nature and their purposes. No matter which classification method is being employed, modelling tasks always involve a situation and require modellers to apply their knowledge outside of mathematics along with their mathematical skills to solve the tasks. In the next chapter, I shift readers' attention from modelling tasks to modellers' modelling behaviors as they solve modelling tasks.

Chapter 4.

Modelling cycles and modelling routes

Looking at the literature on modelling and applications one can find different modelling cycles. These cycles are different, because they are dependent on various directions and approaches of how modelling is understood ... In general these modelling cycles and the description of the phases are normative and are seen as an ideal way of modelling. (Borromeo Ferri, 2006, p. 86)

Modelling processes paint an overall picture of the stages or steps modellers take in the midst of solving modelling tasks. Modelling processes proposed in literature are theoretical and empirical outcomes of researchers' work (for example, see Klamkin, 1971; Pollak, 1979; Griffiths, 1976, Borromeo Ferri, 2006; Schukajlow et al., 2018), where researchers observe and analyze the overall general process which modellers take to generate a satisfactory solution to a modelling problem while taking into account their own understanding and perspective of modelling. These processes are "ideal-typical" (Kaiser, 2005, p. 101) procedures modellers take to solve modelling problems. For example, Pollak (1979) describes the modelling process as follows:

there are typically a number of distinguishable steps in the process. These consist of a recognition that a situation needs understanding, an attempt to formulate the situation in precise mathematical terms, mathematical work on the derived model, (frequently) numerical work to gain further insight into the results, and an evaluation of what has been learned in terms of the original external situation. (pp. 236-237)

As modellers go through these steps to generate a solution for a modelling problem, they may find their generated solution unacceptable and repeat certain or all of the stages in the process. As such, what Pollak describes here resembles a cycle, which modellers repeat until a satisfactory solution is generated. Modelling literature refers to this as a modelling cycle.

Pollak paints a picture of mathematical modelling using broad strokes. His descriptions highlight the use of mathematics to solve a situation or problem in reality and the need to relate the mathematical results to the original context, but downplays the process modellers go through to determine the mathematical tools s/he would use to solve the problem. Since Pollak's work on mathematical modelling, researchers have worked

tirelessly to develop various representations to illustrate the modelling process. These representations could be illustrated through the use of various diagrammatic representations (for example, see Blum & Leiß, 2005; Borromeo Ferri, 2006; Kaiser, 2005; Mason & Davis, 1991; Pollak, 1979). These diagrammatic representations “illustrate key stages in an iterative process that commences with a realistic problem and ends with the report of a successful solution, or a decision to revisit the model to achieve a better outcome” (Galbraith, 2012, p. 8). They highlight various aspects of the modelling process, describe general modelling behaviors, and provide educators with a way to look into students’ learning during the modelling process. From an education perspective, these modelling processes serve to help educators “better [understand] what students do when solving (or failing to solve) modelling problems” (Galbraith & Stillman, 2006, p. 143), and allow educators to better diagnose students’ difficulties and provide students with valuable interventions. Prior to moving on to an in-depth discussion of the modelling process, I first bring readers’ attention to researchers’ various perspectives on the modelling process, including the worlds of modelling cycles and some of the critical stages of the modelling process.

4.1. The Worlds of Modelling Cycles

Modelling cycles describe the ideal iterative process to solve a modelling problem. To accurately describe the modelling process, we must begin with the world where the modelling problem is grounded, and the world where the modelling process leads to.

Literature suggests that two or three worlds are involved in modelling cycles: mathematics and rest of the world (Pollak, 1979); reality and mathematics (for example, see Blum & Leiß, 2006; Borromeo Ferri, 2006; Kaiser, 2005); material world, the world of mental imagery, and the algebraic-mathematical world (Mason & Davis, 1991). Reality, material world, and rest of the world refer to the origin of the modelling task or the world where modelling processes are initiated; and mathematics and the algebraic-mathematical world focus on using equations and calculations to represent the modelling task. Mason and Davis’ third world, the world of mental imagery, exists in between the material world and the algebraic-mathematical world, and allows for the interaction between these two worlds.

Other general characteristics of modelling cycles include: a problem situated in reality (the modelling problem), mathematization and the creation of a mathematical model (moves the modelling process from reality into mathematics), and interpretation and validation of mathematical results/solution (or de-mathematization) based on the mathematical model (moves the modelling process from mathematics into reality). Some researchers also include a situational model (Blum & Leiß, 2005), a mental representation of the situation (Borromeo Ferri, 2006), or a real-world model (Kaiser, 2005) as a step between the initial problem and mathematization, and some include a written report (Maison & Davis, 1991; Galbraith & Stillman, 2006) as a final step of the modelling cycle.

4.2. Mathematization, Interpretation, and Validation

Galbraith and Stillman (2006) outline the general difficulties students experience during their modelling process and suggest that the most challenging and critical steps in the modelling cycle include the process of mathematization which leads the modelling process from reality to mathematics, the interpretation of a mathematical result as a real result which leads the modelling process from mathematics back to reality, and the evaluation of the real result.

4.2.1. Mathematization – From Reality to Mathematics

The term “mathematization” is first used by David Wheeler to describe “the mental processes which produce mathematics” (Wheeler, 2001, p. 50). Mathematization occurs “in situations where something not obviously mathematical is being converted into something which obviously is” (p. 51). This way of thinking allows modellers to draw awareness to the mathematics in situations where the mathematics is hidden and make the mathematics explicit. It also allows modellers to organize a situation in such a way to highlight the mathematics that is hidden within.

Wheeler (2001) argues that mathematization is a form of mathematical thinking, and includes the process which one takes to generate a mathematical model. Mason and Johnston-Wilder (2004) further discuss Wheeler’s idea of mathematization as a way of thinking in their book, *Fundamental constructs in mathematics education*:

To think mathematically is to mathematize situations and apply (mathematical) powers in order to model situations inside and outside of mathematics itself. It means to pose and resolve problems by following chains of necessary deductions, even if you sometimes work empirically in order to locate appropriate conjectures. (p. 193)

Mathematization occurs “in situations where something not obviously mathematical is being converted into something which obviously is” (Wheeler, 2001, p. 51). It concerns the construction of a mathematical model based on a real model. It is the process where modellers develop a representation of a situation in reality using mathematical languages. It requires modellers to recognize the relationship between the various elements within the situation and the relationship between the situation and the mathematics required to represent the situation. During the process, modellers determine the appropriate mathematical vocabularies, symbols, and notations to communicate the situation, and make conjectures and generalizations about the situation (Rosa & Orey, 2015; Sekerak, 2010). It aims “to enable a logical, traceable and rational treatment of the given artefacts and situations with the help of mathematical knowledge and tools” (Grigoras, 2010, pp. 2206-2207).

Mathematization is one of the most important and difficult phases in the modelling cycle, as it requires modellers to choose a mathematical tool to represent the real model. It also requires modellers to move away from the original context (reality/rest of the world) and use only mathematical languages to represent the situation (mathematics) (Borromeo Ferri, 2006). To successfully mathematize a real model into a mathematical one, modellers need to have a clear understanding of the original situation and the problem they are trying to solve, to recognize the various mathematical tools that are available to solve the problem, and to have a good understanding of the reasons for their selections, including what these mathematical tools are capable of and what they are good for, prior to making a conscious decision about what tool(s) to use.

On top of choosing an appropriate mathematical tool(s) to represent the situation, mathematization also involves the use of extra-mathematical knowledge (EMK), a term coined by Borromeo Ferri (2006), along with modellers’ mathematical knowledge to represent seamlessly the context of the situation using mathematics in order to make the transition from a real model (reality) to a mathematical model (mathematics) as smooth as possible.

4.2.2. Horizontal and Vertical Mathematization

Mathematization is also used extensively in Realistic Mathematics Education (RME). RME is a mathematics curriculum and a theory of pedagogy that originates from the Wiskobas project (mathematics in primary school) founded by Edu Wijdeveld and Fred Goffree in 1968 in Netherlands. Soon after, Adrian Treffers joined Wijdeveld and Goffree, and in 1971, The Wiskobas project became a part of the IOWO institute (Instituut Ontwikkeling Wiskundeonderwijs) directed by Hans Freudenthal.

RME is the result of a mechanistic teaching approach in The Netherlands in the 1960s, where students learned step-by-step mathematical procedures as their teachers demonstrated these procedures to them. This resulted in students' inflexible use of their mathematical knowledge in problem solving situations (van den Heuvel-Panhuizen and Drijvers, 2014).

In order to promote students' flexible use of their mathematical knowledge in problem solving situations, RME aims to make mathematics in students' learning meaningful and uses "rich, "realistic"¹ situations ... [to initiate] the development of mathematical concepts, tools, and procedures and as a context in which students can in a later stage apply their mathematical knowledge, which then gradually has become more formal and general and less context specific" (van den Heuvel-Panhuizen and Drijvers, 2014, p. 521). It is important to note that *realistic situations* here refer to context situations or problems which students can imagine or relate to, including problems situated in reality or in the fantasy world. According to Freudenthal (1991), what is *real* to an expert mathematician is likely to be different from what is *real* to a novice mathematician:

The world of life is what is experienced as reality. ... For the expert mathematician, mathematical objects can be part of his life in quite a different way but for the novice. (pp. 41-42)

Freudenthal believes that it is important to keep mathematics relatable and relevant to learners. Under his direction, RME aims to use context problems as a starting point and as a medium through which students develop understanding of mathematical concepts. As students make sense of the context problems, they "organize the reality with

¹ The term "realistic" here originates from the Dutch expression "zich realiseren", meaning "to imagine".

mathematical means” (Frudenthal, 1973, p. 44). The act of structuring the contextual problem in terms of mathematics is called mathematization.

Treffers, in his 1987 dissertation, describes two types of mathematization processes, horizontal mathematization and vertical mathematization. Horizontal mathematization allows students to transition from reality into mathematics, and vertical mathematization allows students to build connections between mathematical concepts within mathematics:

In horizontal mathematization, the students use mathematical tools to organize and solve problems situated in real-life situations. It involves going from the world of life into that of symbols. Vertical mathematization refers to the process of reorganization within the mathematical system resulting in shortcuts by using connections between concepts and strategies. It concerns moving within the abstract world of symbols. The two forms of mathematization are closely related and are considered of equal value. (van den Heuvel-Panhuizen and Drijvers, 2014, p. 522)

In this context, horizontal mathematization is closely related to mathematization in the modelling sense. It allows modellers to organize the problem situation mathematically and takes modellers from reality into mathematics:

Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly; this is vertical mathematization. ... The distinction between horizontal and vertical mathematizing depends on the specific situation, the person involved and his environment. (Frudenthal, 1991, pp. 41-42)

In the context of mathematical modelling discussed in this thesis, modellers focus on the use of horizontal mathematization to help them build a mathematical model based on their real model as they recognize relationships between the various elements within the situation and those between the situation and the mathematics required to represent the situation, and organize or put additional structure on to the situation (Wheeler, 2001).

Mathematization brings out the mathematics in a situation and leads to the production of a representation of the original situation using mathematical languages. As such, mathematization is dependent on the context of the situation and the modeller’s knowledge and skills. Simply put, modellers interpret the context of the situation based on the extra-mathematical knowledge, EMK, they apply to the situation. From their

interpretation of the situation, modellers then draw from their mathematical knowledge and skills to generate a mathematical model. Blum and Borromeo Ferri (2009) refer to these mathematical knowledge and skills as “intra-mathematical skills” (p. 49).

I draw from Blum and Borromeo Ferri’s (2009) work to illustrate further the use of extra-mathematical knowledge and intra-mathematical skills in the process of mathematization. In their study, students were asked to work on *the lighthouse problem*² in small groups, which they needed to apply both their extra-mathematical knowledge and their intra-mathematical skills during the modelling process: students’ extra-mathematical knowledge allowed them to understand the situation, to draw from their experiences, and to apply non-mathematical knowledge and skills to the problem situation and to produce a realistic solution to the problem situation; and their intra-mathematical skills provided them with the mathematical skills and knowledge they needed to solve the problem. In this particular task, students’ understanding of the Earth as a spherical object and their knowledge about the Earth’s radius (extra-mathematical knowledge) led them to account for the Earth’s curvature. When combined with students’ choice to use the Pythagorean Theorem (intra-mathematical skills), a reasonable and realistic solution was produced. Without the application of both their extra-mathematical knowledge and intra-mathematical skills, it would have been impossible for students to produce a reasonable and realistic solution. In this particular case, without an understanding that they needed to take into account the Earth’s curvature (i.e. if student assume that the Earth is flat), students could assume that the ship will always see the lighthouse; and without their knowledge of the Pythagorean Theorem, they would not have been able to determine the minimum distance even if they understood the Earth’s curvature. Student’s intra-mathematical skills allowed them to pick the appropriate mathematical tool(s) and led them to a mathematical solution. As such, both extra-mathematical knowledge and intra-mathematical skills are crucial in the entire modelling process.

² The lighthouse problem goes as follows: “In the bay of Bremen, directly on the coast, a lighthouse called “RoterSand” was built in 1884, measuring 30.7 m in height. Its beacon was meant to warn ships that they were approaching the coast. How far, approximately, was a ship from the coast when it saw the lighthouse for the first time? Explain your solution” (Blum and Borromeo Ferri, 2009, p. 48).

Borromeo Ferri (2006) points out specifically that EMK is especially required in the process of constructing a real model and a mathematical model, and the process of interpretation and validation. Conversely, literature does not point out explicitly when modellers apply intra-mathematical skills in the modelling cycle. However, it could be reasonably assumed that intra-mathematical skills are needed in the process of creating a mathematical model, mathematical result, and real result.

4.2.3. Interpretation and Validation – From Mathematics to Reality

As the process of mathematization brings modellers from reality into mathematics, the process of interpretation, or in Mason & Davis' (1991) terms, de-mathematization, brings modellers from mathematics back into reality, and validation allows modellers to decide whether the generated solution is to be accepted or rejected. Interpreting and validating the generated solution against the original situation are two other major challenges students experience during the modelling process (Galbraith & Stillman, 2006). These challenges might have roots in students' difficulties to connect the generated solution with the original problem situation.

Borromeo Ferri (2006) suggests that the application of extra-mathematical knowledge, EMK, is crucial for students to make sense of their generated solution, just like how EMK is used to generate a real model and a mathematical model. For Borromeo Ferri, EMK enable students to transition between reality and mathematics.

Borromeo Ferri's use of EMK in the interpretation and validation processes is closely related to Dave Hewitt's (1992) metaphor of *train spotting* – to solve problems by looking for patterns and to predict solutions, to pay attention to the numbers involved in the problem but that move away from the original situation. As Hewitt points out, what students lack is not a fluency of their mathematical knowledge but a connection between their mathematical solution and the original situation, and the ability to recognize, access and apply this knowledge to solve everyday problems. This lack of connection results in problem solvers learning something about the patterns, but not about the original mathematical situation. This disconnection between their solutions and the original problem also contributes to the gap between reality and mathematics and their failure to make sense of their mathematical solution based on the original situation (Hewitt, 1992; Galbraith & Stillman, 2006).

A widely-cited example that has been used to demonstrate this disconnection between students' mathematical solution and the original situation is the bus problem given to students in the Third National Assessment of Educational Progress in the U.S. (Carpenter, Lindquist, Matthews, & Silver, 1983):

An army bus holds 36 soldiers, If 1128 soldiers are being bussed to their training site, how many buses are needed?

Results of the study indicate that roughly 70% of students correctly carried out the division that is required to produce a mathematical solution. However, more than half of these students gave the answer of 31 buses or 31 buses remainder 12, and less than a quarter of these students gave the answer of 32 buses. The results of this study, among others (Greer, 1997; Verschaffel, L., De Corte. E., & Lasure, S., 1994), show that students often are capable of producing a mathematical solution, but tend to disregard their mathematical solution from a realistic perspective.

These studies highlight the importance of the processes of mathematization and interpretation in the modelling cycle, as they enable students to apply the appropriate tools to solve a problem mathematically and to interpret their mathematical solution from a realistic perspective. These studies also demonstrate students' lack of fluidity during the transitions between reality and mathematics and the potential gap that exists between the two worlds. Since the modelling process draws on experiences and knowledge that originate from both reality and mathematics, it is crucial for students to integrate their knowledge and skills in reality and in mathematics to produce a realistic solution to the problem situation:

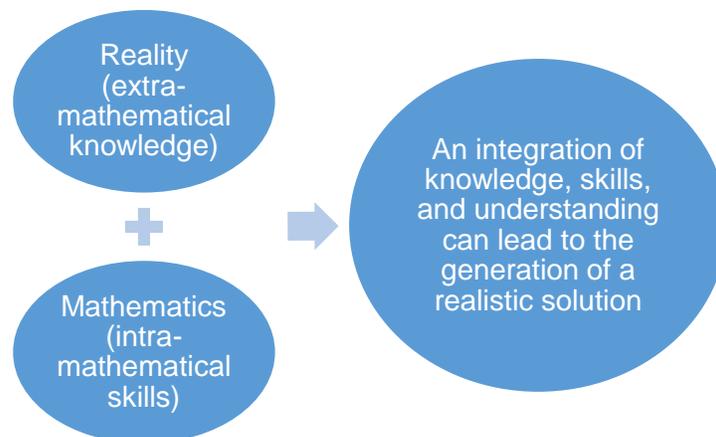


Figure 1. Modellers' success in the modelling process

And due to this integration of knowledge and skills in both worlds, it is “not possible to define neatly a border between mathematics and reality” (Grigoras, 2009, p. 2207). In other words, the flow from reality into mathematics should, ideally speaking, be fluidic. Unfortunately, studies show that this is often not the case. Galbraith and Stillman (2006) suggest that many students experience difficulties while transitioning between reality and mathematics during the mathematization, interpretation, and validation processes, where students need to represent context in reality in terms of mathematics or vice versa. These difficulties suggest that students may be interpreting reality and mathematics as two separate entities with a clear boundary between them.

In what follows, I provide readers with an overview of a selection of modelling cycles found in literature. The modelling cycles presented in this thesis are chosen to tell the story of researchers’ work to describe the process of mathematical modelling, beginning with Pollak’s (1979) modelling cycle. Growing out of Pollak’s (1979) work are Kaiser and Blum’s four-step modelling cycle, Galbraith and Stillman’s (2006) modelling cycle which aims to understand modellers’ difficulties and address the differences between novice and competent modellers’ modelling behaviors, and Blum and Leiß’s (2005, 2007) modelling cycle and Borromeo Ferri’s (2006) modelling cycle which provide additional details to modellers’ behaviors during the modelling process. On top of these modelling cycles which focus on modellers’ behaviors, there also exist Mason and Davis’s (1991) modelling cycle that highlights modellers’ thinking during the modelling process, and Lesh and Doerr’s (2003) proposal of model-eliciting activities (MEA) which are thought-revealing activities that evoke the construction of models. After a description of the above-mentioned modelling cycles, I provide readers with a comparison between these modelling cycles.

4.3. Modelling Cycles

4.3.1. Pollak’s Modelling Cycle

The first modelling cycle that describes the process to solve real-world situations could be traced back to Pollak’s (1979) work. Pollak suggests that modelling is a process that focuses on “the transition from real life situation into a mathematical problem” (Kaiser, 2005, p. 101). Pollak’s modelling process involves two worlds: mathematics and the rest of the world; and emphasizes the process which modellers represent the situation in

mathematical terms. The following is a diagrammatic representation of Pollak's modelling process:

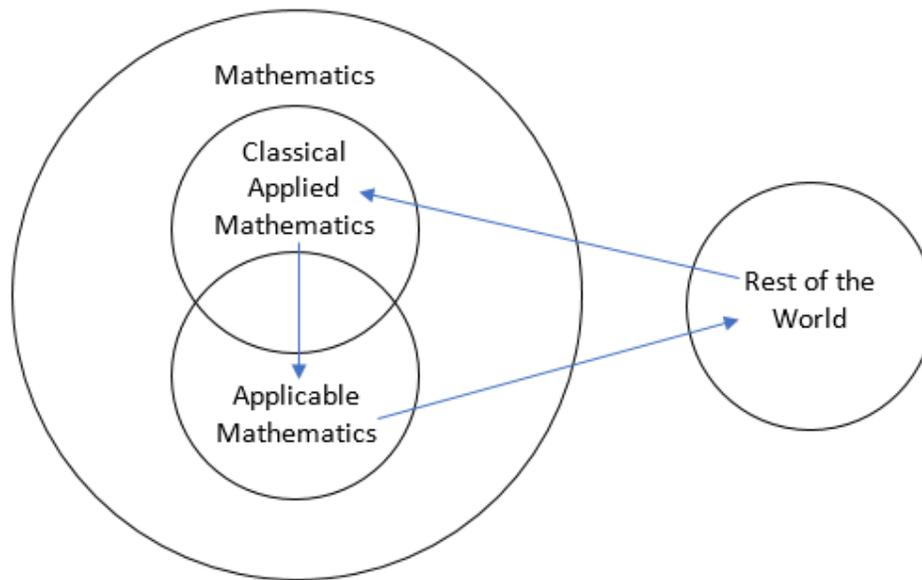


Figure 2. Pollak's (1979) modelling cycle

Pollak (2011) describes the process to create a model as a four-step process, where modellers first simplify or idealize the situation, create a mathematical model based on the idealized situation, apply their mathematical knowledge to determine a solution, and finally decide whether the solution is an acceptable one:

The real situation usually has so many “angles” to it that you can't take everything into account, so you decide which aspects are most important and you keep those. At this point, you have an idealized version of the real-world situation which you translate into mathematical terms. Now you have a *mathematical model* of the idealized question. Then you apply your mathematical instincts and knowledge to the model and get interesting insights, examples, approximations, theorems, and algorithms. You translate all this back into the real-world situation and you hope to have a theory for the idealized question. But you have to check back: are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning and try again. (p. 64)

In these descriptions, Pollak highlights that the process of mathematical modelling is done based on the modellers' interpretation of the situation, where they idealize the situation and highlight what they think is/are important, and their mathematical knowledge, which they apply to their mathematical model to produce a solution to the

situation. He uses words like *instincts* and *insights* to describe modellers' modelling experiences, and does not provide detailed descriptions of what these *instincts* and *insights* involve, where they come from, and how modellers select these to be applied to the situation. Finally, Pollak acknowledges that modellers may repeat the modelling cycle if they believe the solution produced is not acceptable. This also implies that modellers may repeat the modelling cycle in order to improve their solution and to determine a more suitable solution to the situation than the solution they currently have.

4.3.2. Kaiser and Blum's Modelling Cycle

Kaiser and Blum propose a modelling process that involves four stages (Blum, 1996; Kaiser, 1995; both as cited in Borromeo Ferri, 2006; and Kaiser, 2005): real situation, real world model, mathematical model, and mathematical results (Figure 3). In this representation, the modelling process begins with a real situation. Modellers then develop a real world model based on their understanding and interpretation of the situation. A real world model is a simplified and idealized version of the original situation. Afterwards, modellers proceed to create a mathematical model by representing the real world model using various mathematical formulae and equations. This is also when modellers transition from reality into mathematics. Afterwards, modellers use the mathematical model to obtain mathematical results, and finally apply these mathematical results to the original problem.

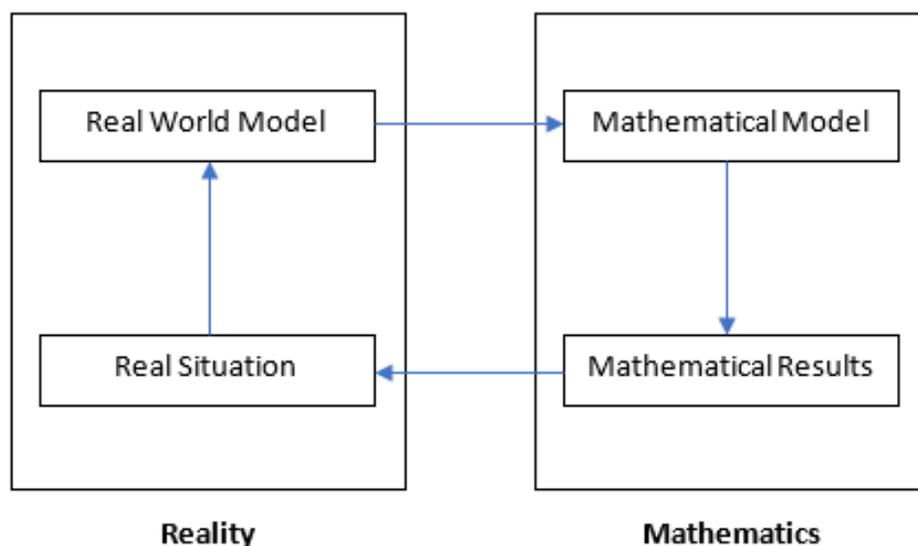


Figure 3. Kaiser and Blum's modelling cycle

This representation separates the modelling process into two worlds: reality and mathematics. Reality includes the real situation and the real world model, and mathematics contains the mathematical model and mathematical results. This representation provides us with a general understanding of the modelling process and suggests a direct transition from a real world model to a mathematical one. However, it does not clearly describe the processes which modellers build a real world model and a mathematical model. Similar to Pollak's modelling process, Kaiser and Blum's modelling process suggest that modellers may repeat the stages before arriving at a satisfactory solution to the real situation.

4.3.3. Galbraith and Stillman's Modelling Cycle

Galbraith and Stillman's modelling cycle begins with a messy real world situation, where modellers may need to tease the situation apart and to sort out the situation prior to proceeding to the next steps. As modellers clarify the problem, make assumptions to simplify the situation, and identify possible strategies and key elements to solve the problem, modellers create a real world problem statement to describe the original problem. Similar to a real world model, a real world problem statement is a simplified and an idealized representation of the problem based on students' interpretations and strategies chosen to solve the problem (Galbraith & Stillman, 2006).

Afterwards, modellers further simplify and idealize the situation and use mathematical symbols and language to describe and represent the situation. This leads to the construction of a mathematical model, which modellers work on mathematically to obtain mathematical results, interpret these results by contextualizing them in the original situation, and evaluate the validity of these results. The evaluation process is a critical one, as it allows students to decide whether to accept the solution or to revise the model. Galbraith and Stillman (2006) suggest that while the process of evaluation appears at the end of the modelling process, competent modellers constantly evaluate their work during the modelling process at two levels: locally and globally. Locally, modellers determine whether each of the results generated are compatible with the context of the original problem. Globally, modellers determine whether the model is successful in determining a complete solution to the original problem. Galbraith and Stillman suggest that if modellers choose to accept their solution, they then proceed to generate a report

to communicate their solution. Otherwise, modellers may repeat the modelling cycle to improve their results. The following diagram demonstrates this modelling cycle:

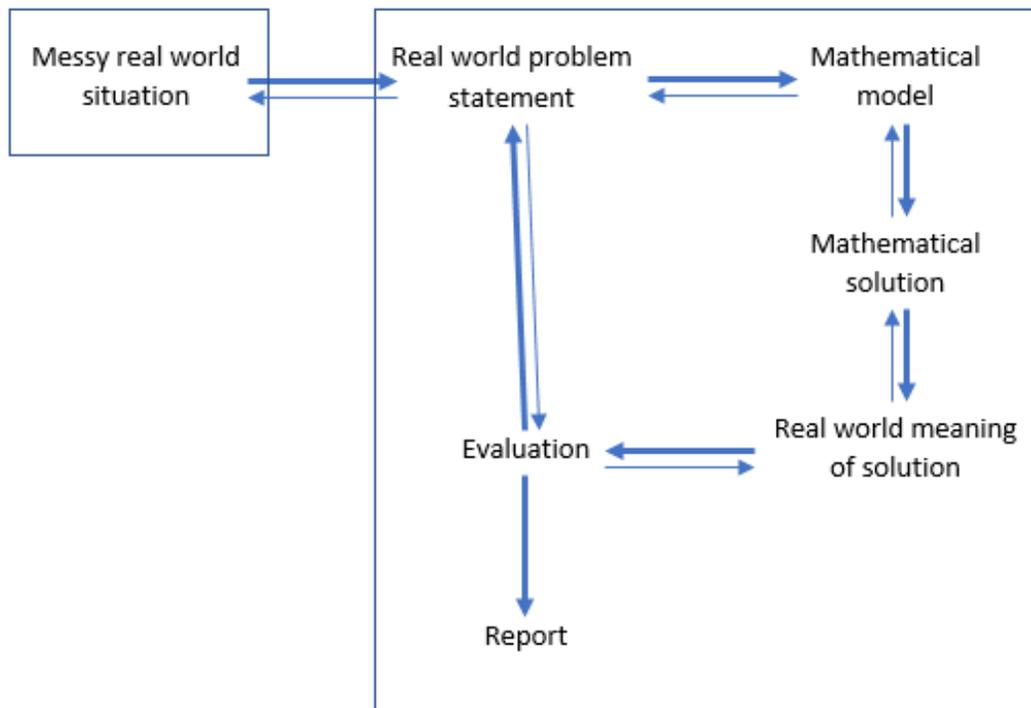


Figure 4. Galbraith and Stillman's (2006) modelling cycle

Galbraith and Stillman's modelling cycle focuses on students' transitions between the stages of the modelling cycle, and provides some descriptions to modellers' behaviors as they move between the stages within the modelling cycle: clarifying, interpreting, simplifying, and making assumptions about the situation, choosing strategies, etc. They also address various difficulties students may experience during each step in the modelling process, and emphasize the non-linearity of the modelling cycle, where students often move back and forth between stages during their modelling process rather than simply repeating the modelling cycle prior to finalizing the solution (see Figure 4). The non-linearity of this representation implies that ideally, modellers often monitor their progress and evaluate their results before the final step is reached.

4.3.4. Blum and Leiß's Modelling Cycle

Blum and Leiß (2005, 2007) suggest that while Kaiser and Blum's four-step model outlines important aspects of the modelling cycle, the modelling cycle may be described

using more detailed stages. They propose an elaborate modelling cycle which focuses on modellers' understanding and interpretation of the original problem situation, and describe the modelling behaviors which modellers may exhibit during their modelling processes. The following is a representation of Blum and Leiß's modelling cycle³:

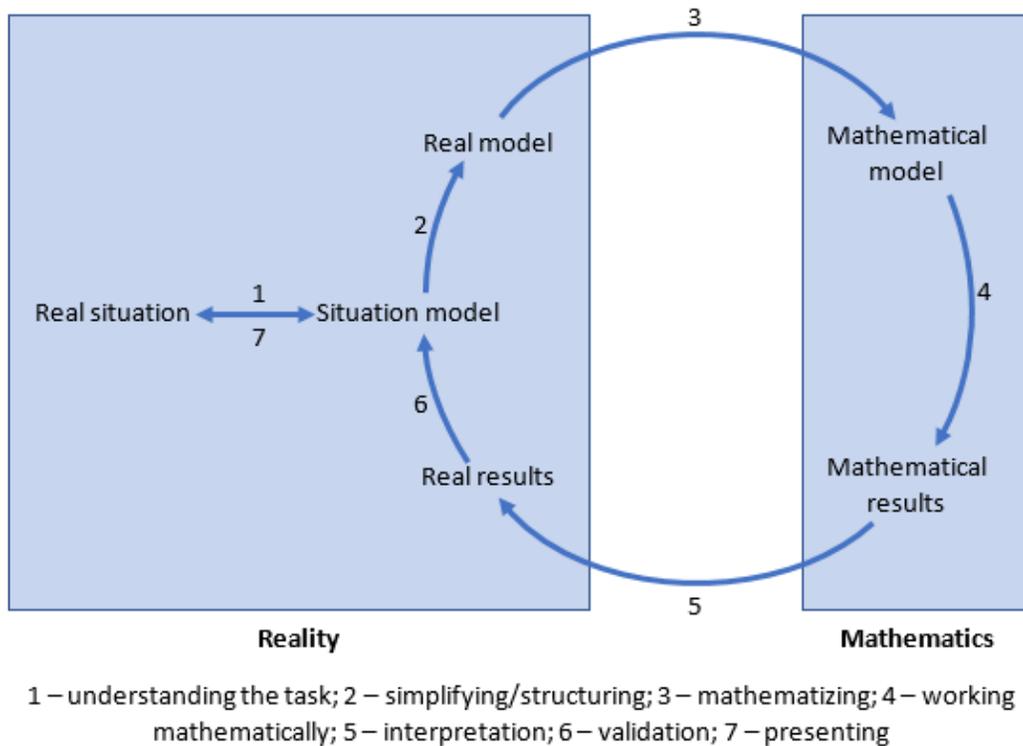


Figure 5. Blum and Leiß's (2005) modelling cycle

This representation begins with a problem situated in reality (real situation), emphasizes the importance to understand the problem, suggests the creation of a situation model prior to the construction of a real model, and highlights the movement between reality and mathematics, including mathematization and interpretation (Blum & Leiß, 2005). The term situation model has strong connections with word problems (see, for example, Kintsch & Greeno, 1985). A situation model describes a situation through the use of graphs, pictures, drawings, etc., contains information that is relevant to the original problem, and showcases students' understanding of the problem situation. In the case

³ Blum and Leiß's modelling cycle emerge from DISUM project at the University of Kassel. DISUM stands for "Didaktische Interventionsformen für einen selbstständigkeitsorientierten aufgabengesteuerten Unterricht am Beispiel Mathematik". The subjects of the DISUM project are mostly grade 9 students.

of Blum and Leiß's modelling cycle, the creation of a situation model requires more than simply knowing what the problem is asking; it also requires modellers to consider the various factors that play a role in the problem. Furthermore, they suggest that situation models are modellers' idiosyncratic constructions, which demonstrate modellers' own understanding and interpretation of the situation. In turn, situation models allow educators and problem posers to diagnose and assess modellers' understanding of the problem, and to provide proper interventions if necessary.

They also imply that modellers work in groups to solve modelling tasks, pointing out that during the modelling process, modellers need to clearly articulate their ideas with their group members and convince them of their interpretations and approach to the problem. As such, the initial steps of understanding the problem and the construction of the situation model are very cognitively demanding, because not only do modellers need to make sense of the problem, they also need to demonstrate to others their understanding, and to argue for their interpretation and convince others their approach is a suitable one.

After the initial construction of the situation model, modellers then simplify, idealize, and fine tune the situation model into a real model to represent the situation, where only the information modellers believe to be relevant to solve the problem is included. While real models may contain some mathematics to represent the situation, they still contain information relevant to the approach modellers choose to solve the problem situation. Next, modellers mathematize and transform the real model into a mathematical model that involves the use of equations, calculations, etc., and use it to generate mathematical results. This step also brings the real model from reality into mathematics, where the original real situation is represented in terms of formal mathematics, and the original context disappears.

Ideally speaking, modellers then interpret these mathematical results and transition them back into reality as real results, and validate these results as possible solutions to the original problem. However, modellers do not always validate their real results. Modellers may fail to connect their real results to the original situation in reality and may treat the process of validation as the process to make sure the mathematical results were calculated correctly using the mathematical model, rather than one that allows them to make sense of their solution based on the original problem situated in reality

(Borromeo Ferri, 2006). Maaß (2006) refers to this type of validation as superficial validation.

If the validation step shows that the solution is not a suitable one to the problem, modellers may repeat particular steps or the entire modelling cycle to improve the solution (Maaß, 2010). In reality, however, modellers often become easily satisfied when they reach a solution and therefore do not validate or reflect on their solution. The lack of validation and reflection prevent modellers from improving their solutions (Blum & Leiß, 2005; Borromeo Ferri, 2006).

4.3.5. Borromeo Ferri's Modeling Cycle

Borromeo Ferri (2006) describes the modelling cycle from a cognitive perspective and focuses on modellers' individual thinking processes. She proposes that these thinking processes are made explicit through modellers' actions during their modelling processes (Borromeo Ferri, 2007; 2010). She presents a modelling cycle similar to Blum and Leiß's (2006), but uses mental representation of the situation (MRS) in place of situation model because situation model is often associated with word problems. Since modelling problems are much more messy and complex than word problems, the term situation model does not best describe the process of the modelling cycle. Instead, the term mental representation of the situation (MRS) highlights the mental process modellers go through to understand the task that happens on an implicit level (Borromeo Ferri, 2007). MRS is closely connected with mental imagery and represents an individual's thinking process. Borromeo Ferri (2010) emphasizes that MRS includes "visual, verbal, auditive, or formal" (p. 104) representations, and is dependent on the individual's thinking style and experiences.

This MRS can be very different, for example depending on the mathematical thinking style of the individual: visual imaginations in connection with strong associations to own experiences; or the focus lies more in the numbers and facts given in the problem, which the individual wants to combine or relate. (p. 92)

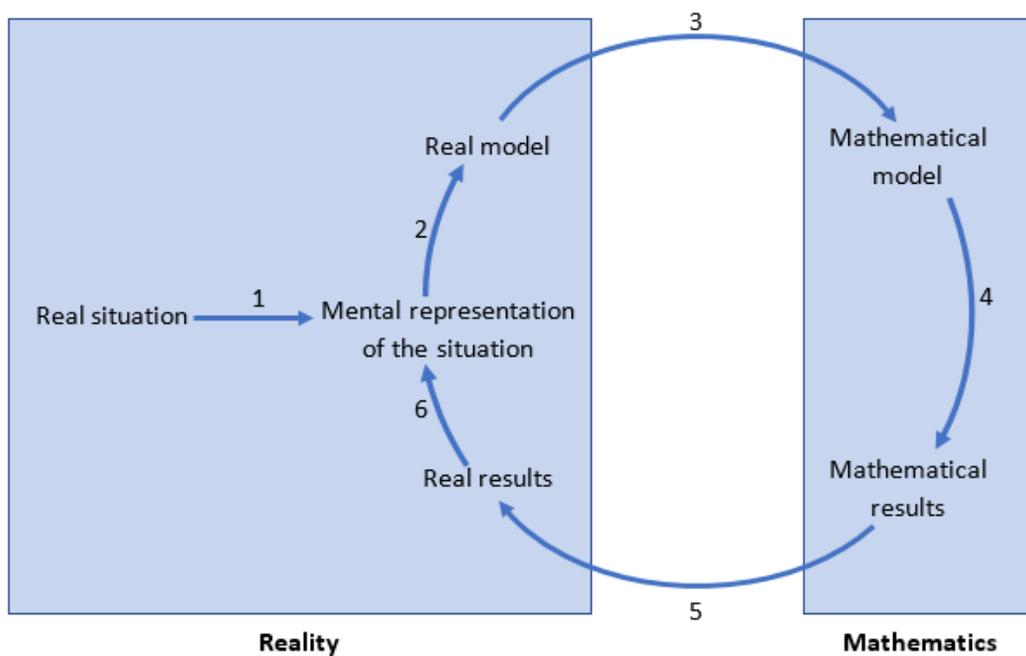
In the formation of a MRS, modellers may make connections with the real situation and simplify the real situation and discuss with each other the approach of the problem. Therefore, a MRS is a representation of modellers' thinking process, a demonstration of their understanding and interpretation of the situation, and an illustration of the direction

in which they take to solve the problem. After modellers create a MRS, they make further assumptions of the situation and draw on their experiences to create a real model to represent the situation. Borromeo Ferri (2006) refers to these experiences as extra-mathematical knowledge, EMK. EMK plays an essential role in the development of the real model and the mathematization process. EMK includes any knowledge or experiences that originate from outside of students' mathematical experiences. The EMK Borromeo Ferri describes here seem to be closely related to the French term, *connaissance*⁴, which is derived from the Latin word, *conoscere*. *Connaissance* refers to *knowledge in situation*, or knowledge that is acquired through experience (Margolinas and Drijvers, 2015). In Borromeo Ferri's (2006) terms, EMK may have little or no clear connections to mathematics, but plays an important role in the modelling process (Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2006). EMK enables students to consider the problem situation from a realistic perspective, and allows them to produce a realistic model to represent the situation. In the process of building a real model, modellers make decisions based on their assumptions about and their approach to the problem. Similar to Blum and Leiß's (2006) modelling cycle, a real model here may use some mathematics to represent the situation, but also contain the original context. Borromeo Ferri (2006) suggests that a real model is strongly connected to the MRS and "is mostly built on an internal level of the individual. This also means that the level of external representations (sketches or formulae) can represent a real model as well. But this really depends on the verbal statements of the individuals while making an external representation" (p. 92). A real model "contains essential features of the original situation, but is on the other hand already so schematized that (if at all possible) it allows for an approach with mathematical means" (Blum & Niss, 1991, p. 38). In other words, a real model, whether it is an internal or external representation, should contain a plan to approach and mathematically solve the original situation.

Afterwards, modellers mathematize the real model into a mathematical one. The process of mathematization demands modellers to apply extra-mathematical knowledge (Borromeo Ferri, 2006). Modellers then use the mathematical model along with their

⁴ The French language distinguishes between two types of knowledge, *connaissance and savoir*. Both terms are derived from Latin, *conoscere and sapere*, which refer to knowledge in situation and institutional knowledge. *Savoir*, or institutional knowledge, is closely related to vertical mathematization in Realistic Mathematics Education (RME), while *connaissance*, or knowledge in situation, is closely related to horizontal mathematization in RME.

intra-mathematical skills to produce mathematical results, and interpret and then validate these results by comparing them to their understanding of the original situation, or their MRS. Validation can happen unconsciously (intuitive validation), or consciously (knowledge-based validation). Intuitive validation is a type of verification process based on modellers' feelings, or reasons that s/he cannot clearly articulate. Knowledge-based validation is a type of validation process based on modellers' EMK (Borromeo Ferri, 2006), where modellers apply their EMK to compare their real results against their real situation to determine whether to accept or reject the solution. While EMK is used in multiple stages of the modelling cycle, Borromeo Ferri puts EMK only in the first phases of her modelling cycle, as she believes that "this kind of extra-mathematical [knowledge] is already activated at the beginning and then used again at the end" (2010, p. 104). In other words, EMK is a non-separable part of the mathematical and real results. The following is a diagrammatic representation of the modelling cycle proposed by Borromeo Ferri (2006):



1 – understanding the task; 2 – simplifying/structuring the task, the use/need of extra-mathematical knowledge (EMK) is task dependent; 3 – mathematizing, EMK is strongly needed; 4 – working mathematically, individual mathematical competencies; 5 – interpreting; 6 – validating

Figure 6. Borromeo Ferri's (2006) modelling cycle

While this diagrammatic representation seems to suggest the linearity of the modelling process, Borromeo Ferri (2007) points out that modellers follow individual modelling routes during which they may move back and forth between modelling stages and may jump between stages, and may repeat the modelling cycle a few times prior to the generation of a satisfactory realistic solution to the real situation.

4.3.6. And so on

Other than the aforementioned modelling cycles, there exists many other similar diagrammatic representations of the modelling process (see for example, Greefrath, 2011; Greefrath et al., 2011), and I will not present and discuss modelling processes that are similar to those already mentioned in this thesis. For the remainder of this chapter, I present two additional modelling cycles that have a different perspective than the aforementioned modelling cycles. Afterwards, I highlight the similarities and differences between all of these modelling cycles.

4.4. Lesh and Doerr's Model-Eliciting Activities

Model-eliciting activities (MEA) originate from Lesh's work on modelling activities. MEAs are thought revealing activities that evoke the construction of models. They are open-ended problem solving activities that are contextualized in a real world setting. MEAs are guided by 6 principles (Lesh et al., 2000): the model construction principle (MEAs are thought revealing activities), the reality principle (MEAs allow students to make sense of the situation based on their lived-experiences), the self-assessment principle (MEAs require students to make conscious decisions about the purpose of the situation, the appropriateness of the generated solution, the need to refine or improve the generated solution, etc.), the construct documentation principle (MEAs reveal students' thought about the problem situation, including the patterns they notice and the relations they examine, and the paths students take to arrive at the generated solution), the construct shareability and reusability principle (MEAs lead to the development of general models that are shareable and reusable), and the effective prototype principle (the solution of a MEA should be a useful prototype for students to solve other MEAs that are based on similar situations). These activities allow students with various

mathematical knowledge and abilities to interpret and approach problems in meaningful ways.

Lesh and Doerr (2003) describe the development of models in model-eliciting activities as a four step interacting process:

(a) *description* that establishes a mapping to the model world from the real (or imagined) world, (b) *manipulation* of the model in order to generate predictions or actions related to the original problem solving situation, (c) *translation* (or prediction) carrying relevant results back into the real (or imagined) world, and (d) *verification* concerning the usefulness of actions and predictions. (p. 17)

Lesh and Doerr's work provides descriptions of modellers' behaviors during the modelling process. First, modellers may focus on specific aspects and structural relationships of the system, express concerns for insufficient information, interpret and think about the problem in different ways, and later on expand their focus to other parts of the system, and eventually, the entire system. They may combine various data, quantify qualitative data, and simplify and reduce the amount of information by focusing on specific relationships, patterns, and trends while ignoring other information. As modellers progress in solving the task, they develop more sophisticated ways of thinking about the problem, begin to look at the data as a whole, and investigate various relationships, patterns, or trends in the data. Afterwards, modellers organize data and information in meaningful ways and create suitable and useful mathematical models in attempt to solve the problem, and test and refine their models to create suitable solutions to the problem. During this process, modellers may develop a sequence of models which demonstrate their thinking and learning process. They refer to this as multiple modelling cycles.

Eventually, as modellers arrive at their conclusions, they take into account the various possibilities that may affect the way data can be processed. Their concluding solution may include supplementary procedures to gather additional information, or a series of telescoping procedures to organize data into groups and apply a different set of procedures to each group. While suggesting these stages as the process of solving model-eliciting activities, Lesh et al. (2000) also recognize that students' behaviors may not closely follow these stages. Nonetheless, these behaviors provide researchers with

a possible framework to describe modellers' actions during the modelling process. This process can be summarized in the following diagram:

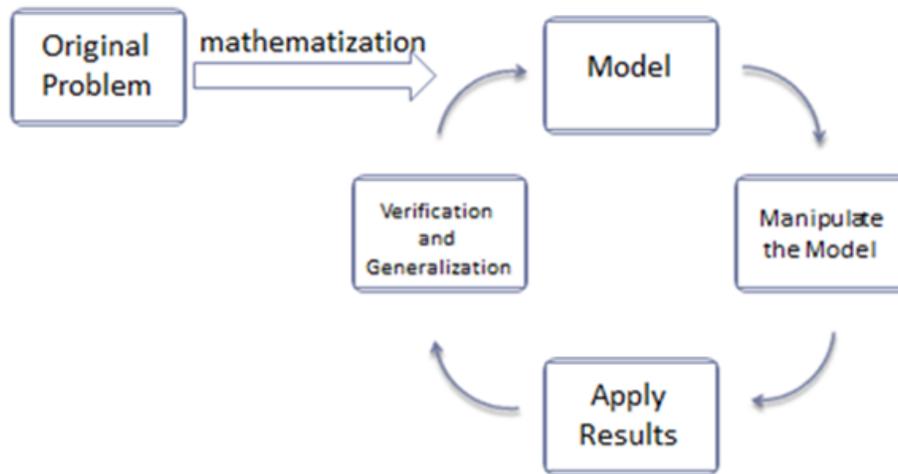


Figure 7. A general process of model-eliciting activities

Unlike the aforementioned modelling cycles, Lesh and Doerr's model eliciting activities do not acknowledge reality, the modelling world, and the transitions between these worlds. Rather, they emphasize the difference between MEAs, where students "make mathematical descriptions of meaningful situations" (Lesh & Doerr, 2003, p. 4) through the process of mathematization, and traditional word problems, where "students make meaning of symbolically described situations" (ibid.) through the process of mathematization. They define 'mathematization' as the process which allows students to make sense of a situation "by quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities" (p. 5); and focus on the products of mathematization – conceptual systems which allow for mathematical reasoning. In this sense, the purpose of MEAs is similar to modelling as vehicle, which focuses on the learning of mathematics.

4.5. Mason and Davis' Modelling Cycle

Mason and Davis (1991) propose a modelling cycle that highlights modellers' thinking during the modelling process. In this proposal, they argue that researchers tend to overlook the process during which modellers imagine and understand the situation but

focus on transitioning the problem situation from reality into mathematics and presenting the problem situation and solution using mathematical languages. In order to highlight the thinking process modellers go through during their modelling process, they argue that three worlds (instead of two) are involved in the modelling process. These three worlds include: the material world, the world of mental imagery, and the algebraic-mathematical world.

The idea of the three worlds originates from Jerome Bruner's three forms of representation, where three parallel systems are used to process and represent information, "one through manipulation and action, one through perceptual organization and imagery, and one through symbolic apparatus" (Bruner, 1966, p. 28). Enactive representations refer to representations via physical actions. Iconic representations "[depend] upon visual or other sensory organization and upon the use of summarizing images" (Bruner, 1966, pp. 10-11). Symbolic representations are arbitrary representations which "are almost always highly productive or generative in the sense that a language or any symbol system has rules for the formation and transformation of sentences that can turn reality over on its beam ends beyond what is possible through actions or images" (p. 11). Bruner believes that these three forms of representations can be used to describe an individual's developmental hierarchy.

Rather than taking Bruner's three representations as a developmental hierarchy, Mason and Davis (1991) draw attention to how enactive representations can support iconic representations, which in turn support symbolic representations:

The point of the enactive apparatus is to support the iconic, the mental imagery, the inner sense which need not have any imagistic or pictorial connections. It may, for example, be largely kinesthetic. He recommended that apparatus be gradually withdrawn, first so that it is still visible and in range for use but only if needed, then simply visible but not manipulable, then taken away altogether. Throughout such a development, pupils are encouraged to adopt the standard language patterns associated with what they are doing, and most importantly, encouraged to say in their own words (though with increasing use of technical terms) what they are doing, thinking, and what the apparatus is illustrating. (p. 72)

They believe the process of modelling involves the interaction of these three worlds. The modelling situation begins in the material world, which is the physical world we live in. As modellers begin to imagine or to think about the problem, they move into the world of mental imagery.

The process of recognising, refining and formulating the actual question takes place inside people, not in the material world. As soon as thinking begins, the modelling shifts to the world of mental imagery. (p. 51)

The world of mental imagery provides a space for modellers to think about and imagine the situation, make sense of the problem, and create a representation based on their understanding and interpretation of the problem situation. The world of mental imagery can be considered an “inner world of imagery and imagination” (p. 52). It includes

anything and everything that happens inside you when you are thinking, planning, considering, and reflecting. For some it may be predominantly visual, for others predominantly verbal, and for others predominantly visceral; for most there will be elements of all of these. ... [It includes] all inner experiences, including inner pictures, inner sensations, inner-positioning, inner-posturing, a fuzzy sense-of, and so on. (Mason, 2002, pp. 75-76)

The world of mental imagery also allows modellers to transition from the material world into the algebraic-mathematical world. As modellers imagine the situation, they also make plans for the situation, reflect on the situation, and eventually set up a model that describe the situation. The formation of a mental imagery requires modellers to de-construct the problem situation, acquire a deep understanding of the situation, and then re-construct the problem situation and eventually represent the situation in an external medium in terms of a mathematical problem.

The mathematical problem is situated in the algebraic-mathematical world. It generally uses symbols and/or diagrams to represent the problem situation, but may involve the use of other physical objects (p. 52). Once modellers create a mathematical problem for the situation, they solve the mathematical problem to generate a mathematical solution for the situation. This process also happens in the algebraic-mathematical world. After a mathematical solution is generated, modellers interpret the solution. This can be referred to the process of de-mathematization and happens in the world of mental imagery (p. 52). The process of de-mathematization leads modellers from the algebraic-mathematical world back to the material world. Finally, modellers apply the solution to the problem situation to validate their solution. If the results are satisfactory, modellers generate a report for their solution. The following is a representation of Mason and Davis' modelling cycle:

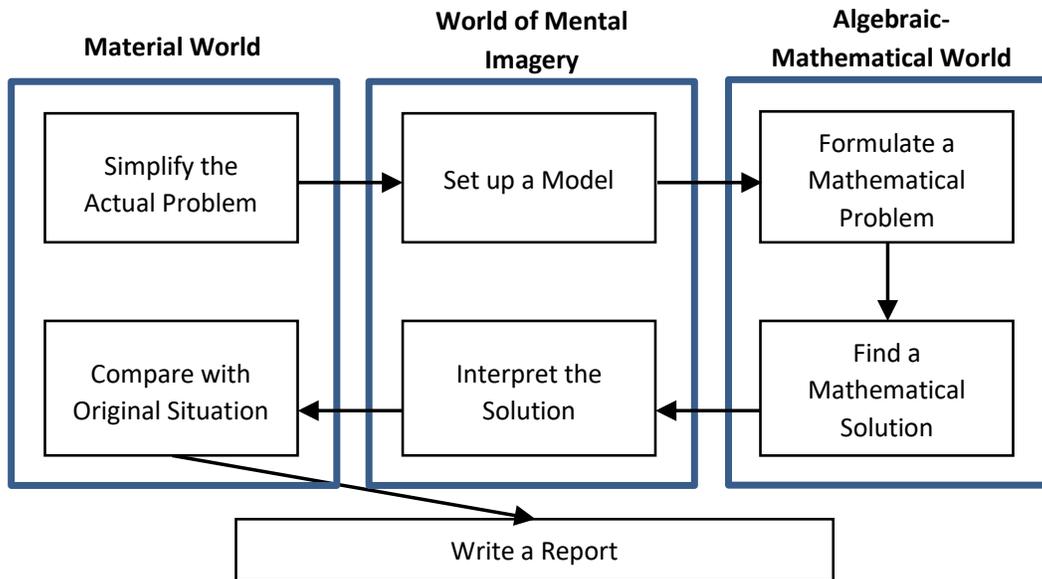


Figure 8. Mason and Davis' (1991) modelling cycle

On top of the thinking process, Mason and Davis also highlight the non-linearity process of their modelling cycle. As modellers think about the problem, they may need to go back to the original problem in order to gain a deeper understanding of the situation. Also, as modellers are setting up for a mathematical problem, they may find that the problem involves too many variables and need to go back to simplify or idealize the problem, or to restate the problem. Modellers may also find that their mathematical solution does not fit the problem and need to go back and rethink and reformulate the problem, which lead them to repeat some of the steps in the modelling cycle, or even repeat the modelling cycle itself.

4.6. Modelling Cycles as Idealized Diagrammatic Representations of the Modelling Process

So far in this chapter, I have discussed a number of modelling cycles that describe modellers' behaviors during their modelling process. While modelling cycles are useful tools to describe the modelling process, it is important to keep in mind that they are "highly idealized, artificial and simplified" (Arleback, 2009, p. 45) descriptions of the modelling processes. In reality, it is likely for modellers to go back and forth between steps to improve on their understanding of the situation and to modify and refine their models in order to produce reasonable solutions (Borromeo Ferri, 2006). Furthermore,

modellers often follow individual modelling routes, where modellers may not follow the phases of the modelling cycle in order, nor do they go through all the phases of the modelling cycle (Arleback, 2009; Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2010). Modellers may begin the modelling “process during a certain phase, according to their preferences, and then [go] through different phases several times or only once, focusing on a certain phase or ignoring others” (Borromeo Ferri, 2006, p. 91). Based on these descriptions, a “real authentic modelling process is better described as haphazard jumps between different stages and activities” (Arleback, 2009, p. 45). As such, while modelling cycles are great tools to describe modellers’ modelling process, it is important to keep in mind that they only represent idealized modelling routes, and individual modellers’ modelling routes may vary based on their thinking styles and their extra-mathematical knowledge (Borromeo Ferri, 2007). Borromeo Ferri (2007, 2010) distinguishes between three mathematical thinking styles: visual thinking style, analytical thinking style, and integrated thinking style. Visual thinkers prefer:

distinctive internal pictorial imaginations and externalized pictorial representations as well as [...] the understanding of mathematical facts and connections through holistic representations. The internal imaginations are mainly affected by strong associations with experienced situations. ... [They] argue preferably in the realistic context of the problem and hardly come off the reality. Their translation process is strongly influenced by dynamical figures and by putting themselves mentally into the real situation. The mathematical models are mainly pictorial and directly reflect the real situation. (Borromeo Ferri, 2010, pp. 105-106)

Visual thinkers tend to be reality-based and produce visual arguments. They also tend to move back and forth between reality and mathematics. Conversely, analytical thinkers prefer:

internal formal imaginations and [...] externalized formal representations. They are able to comprehend mathematical facts preferably through existing symbolic or verbal representations and prefer to proceed rather in a sequence of steps. ... [They] easily take themselves out of the realistic context of the problem and prefer arguing formally while translating into or working within the mathematical world. They mainly focus on given numbers, etc., of the problem. (p. 106)

While analytical thinkers prefer to move into and remain in mathematics as soon as possible, they tend to remain in reality more when they have strong EMK in connections with the task (Borromeo Ferri, 2007). Finally, integrated thinkers “combine visual and

analytic ways of thinking and are able to switch flexibly between different representations or ways of proceeding” (Borromeo Ferri, 2010, p. 106).

4.7. A Comparison of the Discussed Modelling Cycles

As I reflect on these aforementioned modelling cycles, I notice that these descriptions seem to be related to Pólya’s (1945) work on problem solving, which he describes the process in four steps: understanding the problem, devising a plan, carrying out the plan, and looking back. *Understanding the problem* is related to the first few steps in all modelling cycles, such as MRS and situation model, where modellers interpret and reflect on the real situation and develop an understanding of the situation. In the process modellers also restate the problem in the form of a real model. Pólya’s next step, *devising a plan*, is related to the processes of mathematization and the development of a mathematical model, during which modellers determine the appropriate mathematical tools to solve the problem. Afterwards, modellers *carry out the plan* and determine mathematical results and interpret these results as their real results. Finally, Pólya suggests problem solvers to *look back* at their work, which is very comparable to the validation stage in the modelling cycle. The relation between modelling cycles and problem solving is not a surprise, as modelling tasks are in general problems situated in reality and modelling cycles describe the process of finding a solution to these problems. What is interesting here is the applicability of Pólya’s work on problem solvers’ general behaviors in modellers’ modelling processes.

During my reflection I also notice various similarities and differences between the aforementioned modelling cycles, including the various modelling stages and the emphasis of each modelling cycle. Table 1 is a comparison of the modelling cycles proposed by Kaiser (1995) and Blum (1996), Blum & Leiß (2006), Borromeo Ferri (2006), Pollak (1979) and Kaiser (2005), Lesh & Doerr (2003), and Mason & Davis (1991).

Table 1. A comparison of the various aforementioned modelling cycles

	Pollak (1979)	Kaiser (1995) and Blum (1996)	Galbraith and Stillman (2006)	Blum and Leiß (2006)	Borromeo Ferri (2006)	Lesh and Doerr (2003)	Mason and Davis (1991)
Worlds (the number of worlds and the stages involved in these worlds)	2 worlds: The rest of the world; Mathematics (classical applied mathematics and applicable mathematics)	2 worlds: Reality (real situation and real world model); Mathematics (mathematical model and mathematical results)	2 worlds: Reality (messy real world problem); Mathematics (real world problem statement and mathematical model); It is unclear whether interpretation, evaluation, and reporting belong to mathematics or reality.	2 worlds: Reality (real situation, situational model, real model, and real results); Mathematics (mathematical model and mathematical results)	2 worlds: Reality (real situation, mental representation of the situation, real model, and real results); Mathematics (mathematical model and mathematical results)	2 worlds: Reality (original problem and final solution); Mathematics (mathematical model and mathematical results)	3 worlds: Material world (specify the actual problem and compare with original situation); Mental Imagery (set up a model and interpret the solution); Algebraic-mathematical world (formulate a mathematical problem and find a mathematical solution)
Note: Most modelling cycles suggest the concept of two worlds: reality/rest of the world and mathematics. Mason and Davis' modelling cycle suggests three worlds: material world, the world of mental imagery, and the algebraic-mathematical world							
Begin with a problem situated in reality	✓ (the problem is situated in the rest of the world)	✓	✓	✓	✓	✓	✓ (the problem is situated in the material world)
Note: All modelling cycles begin with a problem situated in reality/the rest of the world/the material world.							

	Pollak (1979)	Kaiser (1995) and Blum (1996)	Galbraith and Stillman (2006)	Blum and Leiß (2006)	Borromeo Ferri (2006)	Lesh and Doerr (2003)	Mason and Davis (1991)
A stage between the problem situated in reality and the real model	None, see mathematical model	None, see real world model	None, see real model	Situation model – an idiosyncratic construction by modellers	Mental representation of the situation (MRS) – happens on an implicit level, an idiosyncratic construction by modellers.	None based on the descriptions	None, this modelling cycle refers to this step as set up a model which happens in the world of mental imagery
Note: Blum and Leiß (2006) and Borromeo Ferri (2006) suggest a situation model/MRS as a step between the original situation and the real model.							
Real Model	None, see mathematical model	Created as the first step in the modelling cycle	Replaced with a real world problem statement	Modellers simplify and fine tune the situational model into a real model.	Modellers simplify and fine tune the MRS into a real model.	None based on the descriptions	None, this modelling cycle refers to this step as set up a model which happens in the world of mental imagery
Note: Most researchers (other than Pollak (1979), Kaiser (2005), and Lesh and Doerr (2003)) suggest the construction of a real world model (or a real world problem statement) prior to the construction of a mathematical model. Mason and Davis (1991) refers to this process as “set up a model”.							
Mathematization	Term not used, implied	Term not used, implied	Yes, turns a real world problem statement in reality into a mathematical model in mathematics	Yes, turns a real model in reality into a mathematical model in mathematics	Yes, turns a real model in reality into a mathematical model in mathematics	Yes, modellers make sense of a situation through the application of the mathematical knowledge and skills	Yes, modellers set up a model in the world of mental imagery and formulate a mathematical problem
Note: Not all researchers use the term mathematization to describe the process of using mathematics to represent the situation but it is implied that the same process exists in all modelling cycles. Researchers suggest mathematization and the construction of the mathematical model as one of the most important steps in the modelling cycle.							

	Pollak (1979)	Kaiser (1995) and Blum (1996)	Galbraith and Stillman (2006)	Blum and Leiß (2006)	Borromeo Ferri (2006)	Lesh and Doerr (2003)	Mason and Davis (1991)
Mathematical Model	No distinction between situational, real, and mathematical models. The transition from the real situation into a mathematical problem is the core of modelling.	Yes, modellers manipulate the mathematical model to obtain mathematical results.	Yes, modellers generate mathematical results using the mathematical model.	Yes, modellers work on the mathematical model and generate mathematical results.	Yes, modellers work on the mathematical model and generate mathematical results.	Yes, these are conceptual systems which allow for mathematical reasoning.	Yes, modellers generate mathematical results from the mathematical problem
Interpretation and Validation of Mathematical Results	Unclear if this step exists.	Mathematical results are applied to the original problem	Modellers interpret and evaluate the validity of the mathematical results, and decide whether to accept the results as possible solutions, or to refine their model.	Modellers translate the mathematical results back to reality and validate these results as possible solutions to the problem.	Modellers translate the mathematical results back to reality and validate these results as possible solutions to the problem.	Modellers translate the mathematical results back to the reality and validate these results as possible solutions to the problem.	Modellers interpret their mathematical solution in the world of mental imagery (de-mathematization) and compare their results with the original situation in the material world.

	Pollak (1979)	Kaiser (1995) and Blum (1996)	Galbraith and Stillman (2006)	Blum and Leiß (2006)	Borromeo Ferri (2006)	Lesh and Doerr (2003)	Mason and Davis (1991)
Emphasis	This modelling cycle emphasizes the importance of mathematization.	This modelling cycle outlines the general framework of the modelling cycle.	This modelling cycle emphasizes students' transition between phases.	This modelling cycle emphasizes the cognitive aspect of modelling and that the problem solvers' interpretation of the problem affects the possible solutions.	This modelling cycle emphasizes the cognitive aspect of modelling and the mental process problem solvers go through in order to grasp an understanding of the task, and the use of EMK and intra-mathematical knowledge to produce a realistic solution.	This model emphasizes the cognitive aspect of modeling, focusing on students' understanding and interpretation of the problem.	This modelling cycle emphasizes modellers' thinking processes and that modellers' understanding of the task affects the way they approach the solve the problem.

4.8. Borromeo Ferri and Mason and Davis' Modelling Cycles

The tasks used in this thesis are closely related to the work of Kaiser and of Mason and Davis. As I reflect on the modelling cycles that are derived from Kaiser's work, I find Borromeo Ferri's modelling cycle provides me with the most details in modellers' modelling behaviors. As such, I will direct readers' attention to Borromeo Ferri's (2006) modelling cycle and Mason and Davis' (1991) modelling cycle.

The discussion that follows has two goals: to provide readers with a comparison between these two modelling cycles, and to clarify some of the terms that appear in these two modelling cycles.

Borromeo Ferri's (2006) modelling cycle could be considered an excellent representative among researchers who follow Kaiser's work. Her work looks deeply into mathematical modelling from a cognitive perspective (Kaiser & Sriraman, 2006), and provides a detailed analysis of modellers' progress in all stages of the modelling cycle.

Mason and Davis' (1991) modelling cycle also offers a detailed analysis but approaches the modelling process from a slightly different perspective. Instead of focusing on the products modellers produced, Mason and Davis also highlight modellers' thinking processes through the inclusion of the world of mental imagery.

4.8.1. Reality, material world, and the beginning of the modelling cycle

A problem task situated away from mathematics marks the beginning of both modelling cycles. Borromeo Ferri's modelling cycle begins in *reality*. *Reality*, or *real world*, which Pollak refers to as *the rest of the world* (Pollak, 2012, pp. viii-xi), includes "everything that has to do with nature, society or culture, including everyday life as well as school and university subjects or scientific and scholarly disciplines different from mathematics" (Blum, 2002, p. 230). This points to everything "outside mathematics, ... [including] everyday life and the world around us" (Blum & Niss, 1991, p. 37). While Blum and Niss (1991) do not elaborate any further, it could be inferred that "everyday life" includes things that happen in one's life and includes one's daily experiences, which Borromeo Ferri (2006) refers as one's extra-mathematical knowledge (EMK).

Mason and Davis' (1991) modelling cycle begins in the *material world*. It may be thought of as “a world of confidently manipulable ‘objects’ which may be material objects in the physical world, but may be images and symbols” (Mason et al., 2010, p. 238). Mason and Davis choose the term *material world* over *reality* because *reality* is in the eyes of the beholder. To them, the term *material world* much better describes where the situation is grounded.

Both *reality* and *material world*, although termed differently, both refer to the physical world modellers live in. It is also where modelling problems are situated and where modelling is initiated. Having said that, however, Mason and Davis' (1991) *material world* has certain advantages over the term *reality*, as one's *reality* may be influenced by his/her experiences and interpretation. In other words, *reality* may be somewhat subjective. Conversely, *material world* takes the subjectivity out of the equation and focuses on the objects in the physical world, thus allows the *material world* and therefore the problem task to remain objective.

4.8.2. Reality, the world of mental imagery, and modellers' interpretation of the modelling problem

Whether it is *reality* (in Borromeo Ferri's terms) or the *material world* (in Mason and Davis' terms), the modelling problem is situated in a context that is away from mathematics. From here onward, Borromeo Ferri's (2006) modelling cycle moves to a mental representation of the situation (MRS) and a real model prior to the process of mathematization. As *reality* refers to the world modellers live in, Borromeo Ferri includes the mental process which modellers go through to arrive at a real model in *reality*. It is not until modellers mathematize their real model into a mathematical one then they leave *reality* and enter mathematics. Conversely, Mason and Davis' (1991) modelling cycle removes modellers' thinking process from the *material world*. As the *material world* carries with it a sense of objectivity, modellers' thinking process does not happen in the *material world*. Rather, this thinking process occurs in the world of mental imagery. The world of mental imagery is where modellers recognize and express applicable and relevant relationships within the problem. It “constitutes a world of experience in which models are created” (Mason & Davis, 1991, p. 9), and allows interactions between the *material world* and the algebraic-mathematical world.

In comparing both modelling cycles, I find that the world of mental imagery in Mason and Davis' (1991) modelling cycle and Borromeo Ferri's (2006) mental representation of the situation (MRS) and real model to be closely related, as these are the stages during which modellers reflect on the problem task based on their understanding, and apply their knowledge from outside of mathematics to the task as they interpret and make assumptions about the situation. Borromeo Ferri uses the term EMK to describe such knowledge. Mason and Davis (1991) did not mention the use of EMK in their modelling cycle. However, they emphasize on modellers' thinking which in turn relies on modellers taking into consideration their experiences and knowledge from the material world. These experiences and knowledge are similar to what Borromeo Ferri (2006) refers to as EMK. As such, it could be said that both modelling cycles highlight modellers' knowledge from outside of mathematics as critical contributors to their success in solving the tasks.

4.8.3. Mathematics and the algebraic-mathematical world

Borromeo Ferri's MRS and real model, and Mason and Davis' *set up a model* eventually lead to the production of a mathematical model (or mathematical problem) in mathematics (or the algebraic-mathematical world). Modellers then work with the mathematical model/problem to generate a mathematical result/solution. Following Pollak (1979) and Kaiser's (2005) work, Borromeo Ferri uses *mathematics* to describe the world in which modellers apply their mathematical knowledge to generate mathematical results to their mathematical model. Mason and Davis (1991) use *algebraic-mathematical world* to describe the world in which these processes occur. The term algebraic-mathematical world highlights the importance or value of algebra in the field of mathematics, and suggests that algebra is critical in engaging students to think mathematically.

4.8.4. Interpretation, Validation, and the process of de-mathematization

After the generation of a mathematical result/solution, both Borromeo Ferri (2006) and Mason and Davis' (1991) modelling cycles suggest the processes of interpretation and validation. Borromeo Ferri's modelling cycle brings modellers back to reality directly from mathematics, and Mason and Davis' modelling cycle goes through the world of

mental imagery in which the process of de-mathematization, a process similar to Borromeo Ferri's interpretation, occurs. After the process of validation, Borromeo Ferri's modelling cycle comes to an end if the results are deemed acceptable, but Mason and Davis' modelling cycle suggests that modellers also include a written report to communicate their work with others.

4.8.5. Summary

There are similarities and differences between Borromeo Ferri's (2006) and Mason and Davis' (1991) modelling cycles. Both modelling cycles emphasize the importance of modellers' understanding, thinking, and reflecting on the problem situation prior to the production of a mathematical representation of the situation. These processes lead modellers from reality or the material world, where modelling is initiated, to mathematics or the algebraic-mathematical world, where mathematical models are produced. Both modelling cycles also emphasize the importance of interpretation and validation of the mathematical result/solution, which lead modellers away from mathematics or the algebraic-mathematical world, where mathematical results/solutions are generated, back to reality or the material world, where modellers make decisions to repeat or to complete the modelling cycle.

Both modelling cycles highlight modellers' thinking processes and their thinking styles. Borromeo Ferri (2006) recognizes modellers' thinking styles (visual, analytical, and integrated) affecting the way they process and represent the problem situation. Mason and Davis (1991) recognize modellers may use visual, verbal, visceral, or a mixture of these illustrations to represent the situation (Mason, 2002).

The critical difference between the two modelling cycles is what is being emphasized. Mason and Davis' (1991) modelling cycle highlights modellers' thinking along with the actions modellers take during the modelling process. They emphasize words such as *simplify, think, recognize, refine, formulate, plan, consider, reflect, set up, find, interpret, compare*, etc. in their description of modellers' transition between the three worlds. These emphasis are evident in their their discussion of the modelling process.

Borromeo Ferri (2006) highlights the product of modellers' thinking process in her modelling cycle, and uses MRS, real model, etc. in her modelling cycle. These products

lead modellers from reality to mathematics and back. This is not to say that Borromeo Ferri does not value the actions modellers take in the modelling process. She also uses words such as *understand*, *simplify*, *structure*, etc. to describe modellers' actions, and recognizes that some of the modelling procedures happen at an internal level. However, her emphasis is put on the external representations modellers generate rather than the inner workings during the process.

Borromeo Ferri (2006) and Mason and Davis' (1991) work play a crucial role in the descriptions of modellers' behaviors during the modelling process. They highlight mathematical modelling as a process which modellers use their mathematical knowledge and skills to solve problems situated in reality. Both processes put a strong emphasis on modellers' reflection during the modelling process, and the need for modellers to draw on their experiences outside of mathematics to produce a reasonable solution for the problem situation.

4.9. Summary

Since Pollak's (1979) descriptions of the modelling process, researchers have investigated into the modelling process and modellers' behaviors during the modelling process, and describe the idealized modelling process through the use various diagrammatic representations. These diagrammatic representations allow researchers and educators alike to gain a deeper understanding on the process of solving modelling problems and the aspects involved in solving these problems. While researchers may emphasize different things in their modelling processes, modelling processes in general include a realistic problem, students' understanding and interpretation of the problem, the mathematization process which lead to the representation of the problem through mathematics, the use of mathematics to solve the problem, and an interpretation and validation process. Growing out of researchers' work on the modelling process are descriptions on modelling competencies. In the next chapter I discuss modelling competencies that are related to modellers' abilities to carry out the modelling cycle and those that are not explicitly connected to the modelling cycle but also play a critical role in modellers' overall modelling process.

Chapter 5.

Modelling competencies

... if we want students to develop applications and modelling competency as one outcome of their mathematical education, applications and modelling have to be explicitly put on the agenda of the teaching and learning of mathematics. (Niss, Blum, & Galbraith, 2007, p. 7)

Literature suggests two major directions in defining modelling competencies. While some researchers suggest that modelling competencies are the abilities to follow and carry out each step of the modelling cycle, others suggest that modelling competencies run deeper than these steps (Maaß, 2006, 2010). In this chapter, I first look at these descriptions of modelling competencies. This is followed by a discussion on the development of these competencies by utilizing the modelling cycle, and the tasks that can be used to target the development of specific competencies.

5.1. Modelling Competencies and the modelling cycle

Niss et al. (2007) define mathematical modelling competencies as, “the ability to identify relevant questions, variables, relations or assumptions in a given real situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation” (p. 12). Similarly, Blum and Kaiser (Maaß, 2006) relate modelling competencies to the modelling cycle and modellers’ abilities to carry out specific stages of the modelling cycles. They further break down modelling competencies as the following five sub-competencies:

- 1) The competency to understand the real problem and to create a real model based on the problem:
This includes the competencies to “make assumptions for the problem and simplify the situation; to recognize quantities that influence the situation; to name them and to identify key variables; to construct relations between the variables; to look for available information and to differentiate between relevant and irrelevant information” (Maaß, 2006, p. 116).
- 2) The competency to create a mathematical model using mathematical equations based on the real model:
This includes the competencies “to mathematize relevant quantities and their relations; to simplify relevant quantities and their relations if

necessary and to reduce their number and complexity; to choose appropriate mathematical notations and to represent situations graphically” (ibid.).

- 3) The competency to determine the solutions to the mathematical equations in the mathematical model:
This includes the competencies “to use heuristic strategies such as division of the problem into part problems, establishing relations to similar or analog problems, rephrasing the problem, viewing the problem in a different form, varying the quantities or the available data etc.; to use mathematical knowledge to solve the problem” (ibid.).
- 4) The competency to interpret the mathematical results produced in the original situation:
This includes the competencies “to interpret mathematical results in extra-mathematical contexts; to generalize solutions that were developed for a special situation; to view solutions to a problem by using appropriate mathematical language and/or to communicate about the solutions” (ibid.).
- 5) The competency to validate the solution based on the original situation:
This includes the competencies “to critically check and reflect on found solutions; to review some parts of the model or again go through the modelling process if solutions do not fit the situation; to reflect on other ways of solving the problem or if solutions can be developed differently; to generally question the model” (ibid.).

These descriptions suggest that modelling competencies are strongly tied to the modelling cycle, as they are the skillsets modellers need to carry out the steps of a modelling cycle (Maaß, 2010). These modelling competencies are also closely related to Galbraith and Stillman’s (2006) detailed descriptions of students’ difficulties during the modelling process. Rather than highlighting the skills required to complete the modelling process, Galbraith and Stillman analyze the difficulties students face at every stage of the modelling cycle and the skills students need to overcome these difficulties.

Similarly, Ludwig and Xu (2009) suggest strong connections between modellers’ modelling competencies and the modelling cycle, and devise a scale to measure students’ mathematical modelling competencies (levels 0-5) according to their abilities to follow the modelling cycle to produce a solution. Level 0 describes the students’ inability to understand the situation and to produce a solution. Level 1 describes the students’ ability to understand the situation but their inability to create a real or mathematical model to represent the situation. Level 2 describes students’ ability to create a real model, but their inability to represent this real model using mathematics languages.

Level 3 describes students' ability to create both a real and a mathematical model, but their inability to solve the mathematical model. Level 4 describes students' ability to generate mathematical results based on their real and mathematical model. Level 5 describes students' ability to generate and validate their mathematical results from a realistic perspective.

While this scale evaluates modellers' abilities to follow and carry out the steps in a modelling cycle, it does not highlight the difficulties modellers may experience during the modelling process, such as the process of mathematization and validation, as suggested by Galbraith and Stillman (2006). Also, this scale assumes the linearity of the modelling process, where a higher level on the scale is dependent on modellers' success in completing a prior step in the modelling process, and does not take into consideration of individual modelling routes (Borromeo Ferri, 2007).

5.2. Modelling Competencies: a complex construct

Niss et al.'s (2007) work, Blum and Kaiser's (Maaß, 2006) work, and Ludwig and Xu's (2009) work as discussed in 5.1 strongly relate modelling competencies to modellers' performance in following the steps of the modelling cycle to produce a proper solution to the problem. While these are excellent indicators of modellers' modelling competencies, Maaß (2006) acknowledges the importance of motivation in modelling competencies and suggests that there also exist competencies which may not have explicit connections to the specific steps in the modelling cycles but are required to successfully carry out the modelling process. Maaß (2006) argues that mathematical modelling competencies should be more than the ability to follow and complete the steps of a modelling cycle. Modelling competencies should also include

metacognitive modelling competencies; competencies to structure real world problems and to work with a sense of direction for a solution; competencies to argue in relation to the modelling process and to write down this argumentation; and competencies to see the possibilities mathematics offers for the solution of real world problems and to regard these possibilities as positive. (p. 139)

Metacognitive modelling competencies refer to one's ability to reflect on his/her modelling process and strategy. This includes declarative metacognition – one's ability to make judgements and to determine the proper strategy or strategies to solve the

problem by applying their knowledge about modelling cycles; procedural metacognition – one’s ability to plan, monitor, and validate one’s actions, strategies, and solutions during the modelling process, and to analyze one’s actions and mistakes in a productive manner; and motivational metacognition – one’s willingness and motivation to use metacognition (Sjuts, 2003, as cited in Maaß, 2006). In order for a modeller to regulate his/her own modelling process, s/he needs to have a fairly good understanding of the modelling process and experiences with modelling. Schoenfeld (1992) argues that one of the major differences between expert and beginner problem solvers is their use of metacognition. While beginners often proceed without structure, experts often review their strategies and therefore are able to reach a solution more effectively than beginners. We can extend Schoenfeld’s (1992) work to mathematical modelling: expert modellers apply their metacognition during their modelling process much more than novice modellers and are therefore able to reach a solution more effectively than beginner modellers.

A sense of direction refers to an understanding of what the problem is asking and what it is that the modellers are trying to solve. It serves to direct modellers to a specific goal when working on the modelling problem. Without this understanding modellers can easily lose track of what they are doing and fail to produce a solution to the problem. While this is similar to some of the early stages of the aforementioned modelling cycles, the understanding discussed here happens at a meta-level. It allows students to structure their thoughts and the direction to which they approach the problem.

Competencies in arguing in relation to the modelling process refer to the abilities for modellers to use the information provided in the modelling problem. Maaß (2006) argues that without these abilities, modellers may base their entire work on their experiences and bypass the modelling process. These students do not recognize the importance and relevance of the information provided in the modelling problem and they do not use these information to guide their modelling process.

Finally, Maaß (2006) views modellers’ attitudes towards modelling and mathematics as an important trait in modelling competencies. She differentiates between modellers’ attitudes towards modelling and context-free mathematics and distinguish between four categories of modellers: reality-distant modelers, mathematics-distant modelers,

uninterested modelers, and reflecting modellers, and relate their attitudes to the difficulties they possibly experience during the modelling process.

Reality-distant modellers demonstrate interest in context-free mathematics but lack interests in real-world problems. Their lack of interest in these real-world problems prevents them from engaging with these modelling problems from a realistic perspective and they are likely to experience trouble in the construction of the real model, and the interpretation and validation of their mathematical results and real results.

Mathematics-distant modellers are interested in the context of the modelling problem but demonstrate negative attitudes towards mathematics. These modellers lack mathematical skills, which Maaß refers as their “mathematical capacities” (p. 136). They engage with the modelling problem and are able to construct a real model to represent the problem and to validate their real results. However, their lack of mathematical skills prevents them from successfully constructing a mathematical model, generating mathematical results, and to interpret their mathematical results.

Uninterested modellers lack interests in mathematics and the context of the modelling problems. Difficulties are likely to arise in all parts of the modelling process. Finally, reflecting modellers hold positive attitudes towards mathematics and modelling problems. These modellers engage with the problem situation are able to successfully construct a real model to represent the problem situation, to apply their mathematical skills to build a mathematical model and to generate mathematical results and to interpret their mathematical results, and to successfully validate their real results against the original situation. These modellers experience minimal difficulties in the modelling process.

Maaß's (2006) four types of modellers have strong ties to the modelling cycle and the modellers abilities to carry out the steps in the modelling cycle to determine a real result. Based on Maaß's descriptions, reality-distant modellers and mathematics-distant modellers are uninterested in exploring the context of the problem from a realistic or a mathematical perspective but may not have completely removed the problem from reality or mathematics. As such, they may apply minimal extra-mathematical knowledge or intra-mathematical skills to solve the problem and therefore still participate in mathematical modelling. The same goes for uninterested modellers who may apply

minimal extra-mathematical knowledge and intra-mathematical skills to produce a solution. However, if reality-distant modellers completely remove the problem situation from reality and interpret it as a mathematical problem and apply only their intra-mathematical skills to solve the problem, they have arguably not participated in mathematical modelling and therefore are not modellers. The same goes for mathematics-distant and uninterested modellers.

In summary⁵, modelling competencies include more than the modellers' ability to carry out the modelling cycle and to follow the steps in a modelling cycle to produce a real result. Modelling competencies is a complex construct which includes modellers' "abilities and skills to conduct modelling process adequately and in a goal-oriented way; as well as the willingness to put these abilities and skills into practice" (Maaß, 2006, p. 139).

5.3. Developing Modelling Competencies

Literature describe the possibilities to develop these aforementioned competencies by utilizing the modelling cycle holistically and atomistically (Blomhøj & Jensen, 2003; Brand, 2014; Maaß, 2006). A holistic approach emphasizes students' complete mathematical modelling experiences in the development of their modelling competences:

If some processes are not apparent in the students' activities, one might expect that they will miss important sub-competences when it comes to mathematical modelling in new contexts. If students always work with pre-structured problems, they cannot be expected to develop competences in structuring a complex domain of enquiry. Moreover, full-scale mathematical modelling may give authenticity to the students' work and this might be a motivating factor. (Blomhøj & Jensen, 2003, p. 128)

Conversely, an atomistic approach believes that "full scale mathematical modelling ... limits the time and effort spent on mathematization and analysis of the mathematical system compared with the time spent on investigating the real-life problem at hand and

⁵ Maaß (2006) acknowledges that linguistic plays a role in modelling competencies and her analysis does not illustrate a complete list of modelling competencies. However, the linguistic role is not a focus in this thesis. As such, I will not explore its role in students' modelling competencies.

on structuring the real life complexity into an object of mathematical modelling” (ibid.). The atomistic approach also emphasizes the process of mathematization, solving the mathematical equations found in mathematical models, and interpreting the mathematical results in the mathematics classroom, as these stages are closely associated with mathematics. Furthermore, the atomistic approach allows for a focus on students’ mathematization processes as this is often where students experience difficulties during their modelling processes (Blomhøj & Jensen, 2003; Maaß, 2006).

In comparing the holistic and the atomistic approaches, Brand (2014) finds that both approaches are effective in developing students’ modelling sub-competencies as outlined by Blum et al. (2007) and Blum and Kaiser (Maaß, 2006) for students enrolled in the *Gymnasien*, or high performance, track in Germany. Conversely, the holistic approach is more effective for students enrolled in the *Stadtteilschulen*, or comprehensive, track. There is a lack of study in comparing the effectiveness of the holistic and atomistic approach in developing metacognitive modelling competencies as suggested by Maaß (2006).

As I reflect on both approaches, I find that the holistic approach may be more suitable in developing metacognitive modelling competencies than the atomistic approach. This approach to modelling competencies draws students’ attention to the difficult and critical stages of the modelling cycle such as mathematization. As the mathematization process is built based on students’ reflection and understanding of the original situation, these reflection and understanding serve as the foundation of the mathematization process. Therefore, I believe these processes are inseparable and it would be unwise to focus solely on the mathematization process but ignore the others. Also, focusing solely on mathematization or any other stages of the modelling cycle seems to defeat the purposes of mathematical modelling as discussed in Chapter 2. As well, the holistic approach seems to concern modellers’ modelling competencies at a meta-level away from the specific steps of the modelling cycle, and is therefore more suitable for the purpose of this study than the atomistic approach.

5.4. Tasks to address specific goals or needs

In order to address students' modelling competencies, whether it is competencies closely related to the modelling cycle or competencies as a complex construct, it is crucial to use tasks that could slow down students' modelling process.

Slowing down the modelling process serves two purposes. Firstly, it allows students to demonstrate their modelling competencies: to reflect deeply on the situation from a realistic perspective; to make realistic assumptions about the situation and to determine and apply the necessary EMK to the situation; to reflect on and consider various realistic possibilities prior to settling on a particular approach and to use mathematics to represent the situation; and to reflect deeply on their mathematical results based on the original situation prior to deciding whether to accept or reject their solution as a reasonable solution to the situation. Secondly, slowing down the modelling process allows researchers and educators alike to observe and understand students' modelling behaviors, including their difficulties. From an education perspective, this understanding allows researchers and educators to identify students' difficulties and the specific skills or competencies that could help students with their modelling processes, and thereby prepare for possible interventions.

To slow down students' modelling processes, the tasks need to present a realistic and complex situation that requires students to reflect deeply on the situation, to consider the situation from a realistic perspective, and to pull from their extra-mathematical knowledge prior to deciding on a specific approach to solve the problem. I refer to these as rudimentary mathematics complex tasks (RMCTs).

RMCTs bring out students' modelling competencies and highlight the difficulties related to their modelling competencies. The complexity of the situations in RMCTs slows down the modelling process by ensuring that the mathematical operations required to solve the problem are not trivial or obvious, that students are obligated to think deeply about the situation prior to the decision of a mathematical operation that would lead to mathematical results, to ask questions about both the original situation and their mathematical results, and to verify their results from a realistic perspective by comparing it to the original situation. As such, the context of RMCTs should be believable, imaginable, and relatable to students. However, as each student brings with them into

the classroom a wide range of extra-mathematical knowledge, it is important to point out that this study only pays attention to the way students handle the RMCTs during their modelling process, rather than how the context of the RMCTs relate to each student.

Also, RMCTs should require students to work out a solution on their own and should not allow students to search directly for the answer through the use of the internet. As we live in a technological world, students are able to access a lot of information at their fingertips. In order for students to think deeply about the presented situation, the solution to the situation should not be readily available through a few clicks with the mouse or the keyboard.

Conversely, the situations presented in RMCTs should not be overly complicated that students cannot be certain whether they got a reasonable answer or not. Students need to be able to verify their solutions based on their mathematical knowledge and what they have learned about the situation. Therefore, RMCTs should not have a heavy demand on students' intra-mathematical skills. Rather, RMCTs should allow students to use their well-worn mathematical tools, or their rudimentary mathematical skills, to solve the task. This way, students shift their focus away from the mathematics required to solve the task to the task itself. The level of difficulty of mathematical skills are audience specific. For example, while multiplication and division skills may be considered rudimentary to secondary school students, these skills may not be as rudimentary to elementary school students who have not quite mastered them yet.

By minimizing mathematical demands, RMCT's allow students' modelling behaviors to emerge. In other words, they are a form of modelling tasks that focus on *mathematical Modelling* rather than *Mathematical modelling*. *Mathematical modelling* may use modelling tasks to bring out the mathematics and focus on the mathematical aspect of modelling. This form of modelling has strong ties to vertical mathematization and modelling as vehicle. Conversely, *mathematical Modelling* has strong ties to horizontal mathematization and modelling as content. It removes the stress of mathematics, brings forth students' natural modelling behaviors during their modelling process, require students to apply both their extra-mathematical knowledge and intra-mathematical skills to solve the problem situation, and highlights the steps leading up to mathematization and the validation process. In other words, *mathematical Modelling* puts a strong

emphasis on students grappling with reality and reason rather than using their mathematical knowledge and skills to find a solution.

5.5. Summary

Rudimentary mathematics complex tasks, RMCTs, are a specific type of modelling tasks that promote *mathematical Modelling*. As they are problems situated in reality and involve complex situations, it could be assumed that the process to solve these tasks is similar to that of a modelling cycle. As such, this thesis employs RMCTs to investigate the *mathematical Modelling* process using a modelling framework.

Chapter 6.

A framework for the thesis and research questions

Theory plays a powerful role in educational research, in that it operates both explicitly and implicitly in research text, often structuring what research is able to say. (de Freitas & Walshaw, 2016, p. 4)

This study aims to investigate the process of *mathematical Modelling* and approach mathematical Modelling using phenomenology as a research methodology and I focus on students' experiences with RMCTs and the decisions students made while working on the tasks. As I want to look deeply into how students work on RMCTs, I chose to use mathematical modelling cycles as a framework to help me understand students' handling of these tasks and their experiences in solving these tasks.

6.1. A Framework for the thesis

As discussed, both Borromeo Ferri's (2006) and Mason and Davis' (1991) modelling cycles provide detailed descriptions of students' modelling behaviors. As I focus on *mathematical Modelling*, it is important for me to be able to describe students' modelling behaviors focusing on the steps taken and the difficulties students experience moving from the modelling task to a mathematical model, and from their generated mathematical results to their real results. For the purpose of this thesis, I use Borromeo Ferri's (2006) modelling cycle as my framework of analysis. After studying various frameworks, I find that her (2006) modelling cycle is most helpful in describing students' modelling behaviors during the modelling process, as it provides me with the tools and the language to describe these processes and difficulties effectively.

While I believe students spend a lot of time and effort thinking about the real situation in order to set up a model that represents the situation as described in Mason and Davis' (1991) proposal, I also agree with Borromeo Ferri (2006) that students first spend time thinking about and understanding the real situation to generate a mental representation of the situation (MRS), and then simplifying and idealizing the situation based on their understanding to construct a real model that represents the situation. Borromeo Ferri (2006) separates MRS from real model in order to highlight the importance of and

students' difficulties with understanding the situation and setting up a real model. Both processes play crucial roles as they lead to mathematization and the construction of a mathematical model. Borromeo Ferri's use of MRS to highlight the difficulties modellers experience in attempts to understand the problem situation comes from a "constructivist view that tasks as part of the world cannot be captured in an objective way" (Maaß, 2010).

Borromeo Ferri's (2006) modelling cycle emphasizes both students' thinking process and their articulation of their understanding in terms of MRS and real model. According to this model, students' lived-experience, or their EMK, also plays an important role in affecting their approach to solving the problem situation. I agree with Borromeo Ferri in the use of EMK because modellers' approach to the problem situation is dependent on their interpretation of the situation, their skillsets, and their lived-experiences. From this perspective, Borromeo Ferri's use of the term *reality* is a better fit than the term *material world* as modellers will always bring in their interpretation and their EMK to solve a problem which the context is based in the world they live in.

6.2. Research Questions

This thesis aims to research on two big questions surrounding students' *mathematical Modelling* behaviors. In doing so, this thesis employs RMCTs to focus on the transitions from the real situation to the MRS, the real model, and the mathematical model; the transition from mathematical results to real results; and the process of validation.

Research Question 1

Research done around modelling tasks often involves settings where students are made aware of the modelling cycle (Blum and Borromeo Ferri, 2009; Brand, 2014; Stender and Kaiser, 2015). This allows researchers to shift the focus from the modelling cycle to promote students' modelling competencies. This thesis does not aim to promote students' modelling competencies. Rather, this thesis aims to investigate the natural modelling behaviors of students who have no prior experience of mathematical modelling or knowledge of the modelling cycle. This also prevents students from treating the modelling cycle as modelling heuristics which they follow to generate a real result to the task. Specifically, this thesis uses RCMT's to investigate students' natural

mathematical Modelling behaviors by slowing down the aforementioned processes within the modelling cycle. The examination of students' *mathematical Modelling* behaviors and the identification of students' challenges allows for a means to bridge reality and mathematics and to smooth the transitions between these two worlds and the procedures leading up to the transitions between these two worlds. More specifically, I aim to investigate:

What actions do students take to produce a real model based on the real situation? What are the challenges students experience during these processes that require them to engage with the problem from a realistic perspective?

What actions do students take to interpret and validate their solution and to determine whether their solution is acceptable? What are the challenges students experience during the interpretation and validation processes?

Research Question 2

The second research question pays attention to students' modelling competencies, including the role of EMK, during students' modelling processes.

Modelling literature paints a picture of extra-mathematical knowledge (EMK) using broad strokes. EMK, in general, include any knowledge or experiences that originate from outside of students' mathematical experiences. EMK is deeply intertwined with the modelling cycle, and it directly affects the quality of the real result (Borromeo Ferri, 2006). Modelling literature also suggest that students naturally apply EMK during their modelling processes to interpret real situations, to create real models, and to interpret and validate mathematical and real results against the real situations (Borromeo Ferri, 2006, 2010).

These descriptions found in modelling literature contribute to our understanding of EMK and the crucial role of EMK in students' modelling processes. However, these descriptions lack details and do not help educators and researchers alike to identify students' difficulties during the modelling process and to explicitly articulate students' EMK related competencies. As such, my second research question aims to identify the facets of EMK as a complex construct and to shed light on students' application of EMK as they work on RMCTs. In particular, I aim to investigate:

What triggers students to activate EMK in their modelling process? What are the characteristics of EMK in students' work on RMCTs?

As EMK is based on students' non-mathematical knowledge and/or experiences, EMK should be audience specific. As such, participating students could be idiosyncratic according to their non-mathematical knowledge or experiences, and their EMK may vary. It follows then some students may come with sufficient EMK to approach and solve a task from a realistic perspective, while others have insufficient EMK. It is therefore also important to investigate:

How do students overcome their insufficient EMK? How do these actions shape their modelling process and affect the quality of their solution?

To answer these questions, I first draw readers' attention to the setting of this study in Chapter 7. Afterwards, I present the data (Chapter 8–Chapter 11) and provide an analysis and a discussion on the data (Chapter 12–Chapter 16).

Chapter 7.

The setting of the study

To answer the research questions of this study as outlined in the previous chapter, I use two RMCT's that allow me to focus on *mathematical Modelling*, where the mathematics involved in these tasks are rudimentary to the participating students, yet the context which these tasks are based on are believable, relatable, and complicated.

7.1. The tasks

Two RMCTs were used for the data collection process in this thesis: The *Door Project*, and the *Design a New School* task. These two tasks were chosen based on the relevance of the situations in students' lives and their experience. Both tasks revolve around students' home and school lives. As such, it could be assumed that all participating students have some EMK to begin the modelling process.

Door Project

The *Door Project* is a RMCT I created as a means to develop students' *mathematical Modelling* skills. It was the first of the two tasks administered as it is the simpler of the two tasks. Each group of students were given a printed copy with the following instructions:

I have recently renovated my home office. As a final touch, I'm going to repaint and decorate my office door. The plan is to paint and then decorate the door with Starbucks gift cards. How much will this project cost me?

Students are expected to have some of the necessary EMK and all of the intra-mathematical skills to solve the *Door Project*. Mathematically, students need to apply basic arithmetic skills in the *Door Project*, which could be considered rudimentary to the group of participating students. Realistically, the *Door Project* involves an everyday object (an office door) that is familiar to all students and requires students to paint the door and decorate it using Starbucks gift cards. It is assumed that all participating students have some experiences with painting. This assumption was made based on the local curricula of the art courses students were taking (BC Ministry of Education, 1995; BC Ministry of Education, 2017). It was also assumed that students knew what a

Starbucks is, as there are at least 3 Starbucks in the school neighborhood, students brought Starbucks drinks to class sometimes, and most students were able to name a number of drinks from the Starbucks menu without looking it up. Also, being a Starbucks customer is not a pre-requisite to the *Door Project*. Students could approach the problem without any knowledge of Starbucks as students only need to work with Starbucks gift cards. A sample Starbucks gift card was also provided to students as they worked on the task.



Figure 9. A typical classroom door and a Starbucks gift card.

Using the classification scheme provided by Maaß (2010), the *Door Project* could be described as a modelling task that contains a personal situation (situation), involves extra-mathematical modelling (cognitive demand), uses text to represent the situation (types of representation), and requires a low level of mathematical skills (mathematical content). Under Liljedahl's (2010) numeracy tasks, the *Door Project* may be described as a crossover between a planning task and an estimation across a large number of variables task, as the task requires students to generate a design and to estimate the cost of the design.

Design a New School

The second task used in this study is the *Design a New School*⁶ task. Similar to the *Door Project*, a printed copy of the *Design a New School* task was given to each group

⁶ The task is taken from: <http://www.peterliljedahl.com/teachers/numeracy-tasks>

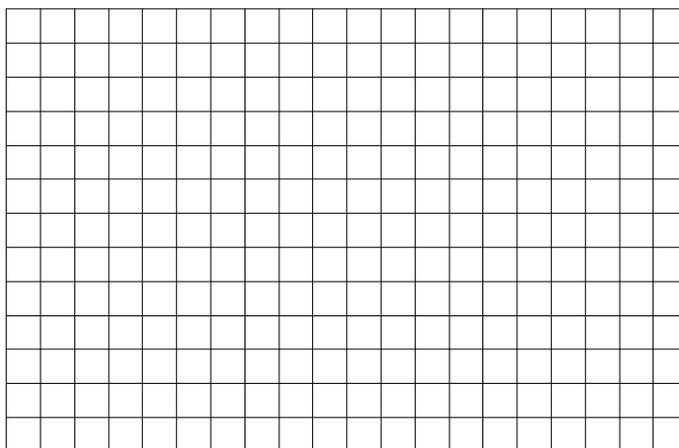
of students. The *Design a New School* task is used as a second task in this study as it is more complex than the *Door Project*. The *Design a New School* task involves multiple aspects surrounding students' daily lives (as compared to two aspects in the *Door Project*). It can be described as a numeracy task, more specifically, a planning task (Liljedahl, 2010) that requires students to spatially plan and design a middle school. It could also be described as a modelling task that has missing data (data), contains a personal or educational situation (situation), involves extra-mathematical modelling (cognitive demand), and requires a low level of mathematical skills (mathematical content) using Maaß's (2010) classification scheme. Similar to the *Door Project*, the *Design a New School* task requires students to perform basic arithmetic skills in the process of solving the task. As such, students were expected to have all the necessary intra-mathematical knowledge to solve the *Design a New School* task. The *Design a New School* task is set in a setting which students are familiar with – the school grounds. Given that participating students were enrolled in grade 8 or 9 at the time of the study, it was assumed that they had at least 7 years of school experiences. The school students attended at the time of the study also contains all the features mentioned in the *Design a New School* task. As such, it was also assumed that their familiarity with the local school environment could be used towards producing a realistic solution for the task, and that they have some of the required EMK to solve the task. The following is a summary of the *Design a New School* task:

Your city is getting a new 11000m² middle school. It is going to be built on a lot (200m x 130m) just outside of town. Besides the school, there will also be an all-weather soccer field (100m x 75m), two tennis courts (each 15m x 27.5m), and a 30 car parking lot on the grounds.

The following requirements must be met:

- all fields, courts, buildings, and parking lots must be no closer than 12.5m to any of the property lines.
- any leftover property will be used as green space – grass, trees, shrubs.
- good use of green space is an important part of making the school grounds attractive.

To help you with your design and layout, you have been provided with a scaled map of the property (every square is 10m x 10m). Present your final design on a copy of this map. Label all structures and shade the green space.



7.2. Participants and Data Collection

The data for this thesis were collected in a Mathematics 8 (age 12-13, n=28) and a Mathematics 9 (age 13-14, n=26) classroom in a secondary school in western Canada. With few exceptions, Mathematics 8 and Mathematics 9 are considered part of the *common* stream in that all students, irrespective of ability, take the same course and sit in the same classrooms. In grade 10, students may choose to remain in the *common* stream or shunt themselves into the Workplace and Apprenticeship stream if they do not see themselves as continuing onto post-secondary education or are aiming to enter into a trades programs. In grade 11, the common stream splits again into a Pre-Calculus stream and a Foundations stream. Although students completing either of these

streams are eligible for admission into academic post-secondary programs, only students completing the pre-calculus stream are eligible for entrance into the sciences.

The two classes in which the research was conducted were exceptions to this *common* stream in that it belonged to an enriched and accelerated program for students who are motivated and eager to learn and who have been recommended by their grade 7 teacher. The program covers the regular Mathematics 8, 9, and 10 curricula in two years as opposed to three. This allows the students to finish the pre-calculus stream by the end of grade 11 and frees them up to take more elective mathematics and science courses in their grade 12 year. Also, these two classes encourage and expects students to think, to work collaboratively with each other, to learn together, and to construct understanding and knowledge through various individual and group activities. What I describe in this thesis is what happens in these two classes on a regular basis.

Students in these two classes worked on RMCTs on a regular basis (approximately one task per month). On top of RMCTs, students also did group projects, solved open ended problems, played mathematical games, etc. on a regular basis. Some of these activities have relevance in the content, and all of the activities have relevance in the curricular competencies as discussed in Chapter 3. Students were aware that these activities were in place to promote their mathematical and critical thinking skills along with their flexibility (as opposed to fluency) in mathematics. As such, when students submitted their solutions at the end of each activity, they were given feedback to their work but their work were not assigned a mark. This is done to encourage students to focus on developing the aforementioned skills rather than finding a correct answer to the activities.

As this was the grade 9 students' second year in the enriched and accelerated program, they had one year of experiences with solving RMCTs amongst other open ended problems. As for the grade 8 group, it is not possible to know whether they had seen similar tasks in their previous years.

Data were collected while students worked on the *Door Project* and the *Design a New School* task in the grade 8 and the grade 9 Mathematics class. Both tasks were administered to students within the same academic year. Students were randomly assigned to groups of 2-4 and given the relevant tasks to work on during a 75 minute

class. No additional directions were provided other than the instructions given in the *Door Project* and the *Design a New School* task. While the students worked on these tasks I circulated naturally through the classroom and engaged in conversations with the students – sometimes prompted by me and sometimes prompted by the students.

These conversations were audio recorded and transcribed. At the same time, photographs of students' work were taken and students' finished work was collected. These coupled with field notes summarizing the interactions as well as observed student activity, allowed me to build cases for each group of students. Each of these cases is a narrative of their modelling experience punctuated by significant moments of activity and emotive expression. These cases constitute the data.

Given that natural and unscripted nature of my movement through the room, not all of the cases are equally well documented. Regardless, each of these cases were analyzed separately through Borromeo Ferri's (2006) modelling cycle.

7.3. Data Analysis

In this thesis, I use Borromeo Ferri's (2006) modelling cycle as my framework of analysis. Borromeo Ferri's modelling cycle is an idealized representation of modellers' expected behaviors and actions, and the products of their actions and decisions. It provides me with the language and the tool to investigate students' natural *mathematical Modelling* behaviors and their EMK-related competencies.

7.4. My Role in this Study

My role in this study was dual – I acted as both the participating students' teacher and as a researcher. My dual role allowed me to create the classroom setting that most fitted my research: it allowed me to observe what students made of the tasks without teaching them the modelling process.

As a researcher, I aimed to investigate the research questions as stated in Chapter 6 and to contribute to the modelling research community. I aim to investigate students' natural *mathematical Modelling* behavior, so I kept my interference to a minimum. As I classroom teacher, I understand the importance to maintain a balance between

“(minimal) teacher’s guidance and (maximal) students’ independence” (Blum and Borromeo Ferri, 2009, p. 52). When taking both the researcher and the teacher role into consideration, I provided students with the opportunities to work on RMCT’s in groups with minimal interference, I asked students thought-provoking questions, I answered students’ questions with suggestions and/or additional questions but never direct answers, and I only provided students with help and interfered with their modelling process when they were on the edge of becoming frustrated or giving up.

Chapter 8.

The door project

In what follows I present the work of four groups of students. These cases were chosen because they are the most complete and comprehensive of all of the cases. They also represent the spectrum of modelling processes analyzed. I experienced data exhaustion after I analyzed four groups of students' work. Pseudonyms are used to protect students' identities. In the reported transcripts, I use *T* to indicate lines spoken by me, and various pseudonyms to indicate lines spoken by the students.

Prior to describing students' modelling process, it is worth mentioning that many students had a Starbucks gift card with them when they were presented with the *Door Project*. Students also assumed that the length and width of a Starbucks gift card are the same as any other gift card and their student identification card. During their modelling process, students either used a Starbucks gift card, a gift card from another company, or their own student identification card to help them with making decisions regarding how to approach and solve the *Door Project*. In the following descriptions and discussions of students' modelling processes, a gift card is regarded as a two-dimensional object, or a rectangle, with a length and a width. Finally, students switched between metric and imperial units of measurements during their modelling process. I report these measurement units in this thesis as used by the students as they worked on the task.

8.1. Amanda, Andy, and Anna (Group A-D)

After reading the instructions, Amanda, Andy, and Anna quickly measured the width and the height of the classroom door using a gift card as a unit of measurement. They lined up the short edge of the gift card to the width of the door and the long edge to the height of the door (Figure 10).

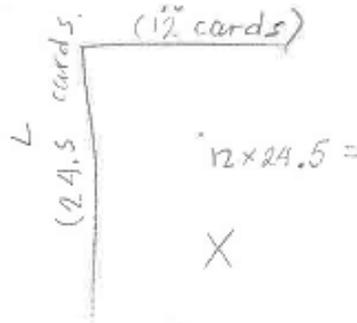


Figure 10. Group A-D's measurement of the classroom door.

They fitted the short edge of a gift card 12 times along the width and 24.5⁷ times along the height of the door. The three of them explained that if they multiply 12 by 24.5, they would get the number of gift cards that could fit on the face of the classroom door, and if they then multiply the product by the cost per gift card, which they termed *x*, it would result in *y*, the cost of the *Door Project* (Figure 11). They did their calculations by hand instead of a calculator and rounded their result to the nearest ten, but made a mistake in the calculation⁸.

Amanda: OK. Since we don't know the price we will use *x*. OK. 24 times, 24.5 times 12 ... (pause and calculates by hand) Sooo... that would be...

Andy: Two hundred and eighty-eight dollars and 50 cents?

Amanda: No what? No that's the amount of gift cards we need

Andy: Oh yeah, Oh yeah, Oh yeah oh my God ... You wanna estimate? So 288,

Amanda: So like, how...

Anna: Two hundred and ninety

⁷ Andy made a mistake with the measurements. The classroom door is 36" (91.44cm) wide and 80" (203.20cm) tall. The gift card measures 3.375" (~8.57cm) wide and 2.125" (~5.39cm) long. Based on these measurements, Andy should be able to fit more than 16 gift cards (short edge; $36" \div 2.125" = 16.94$ (round to 2 decimal places)) across the width of the door, but no more than 24 gift cards (long edge; $80" \div 3.375" = 23.70$ (round to 2 decimal places)) along the height of the door.

⁸ Their mistake in their calculation did not affect their progress nor their final solution.

$$\begin{array}{r}
 12 \\
 194.5 \\
 \underline{48} \\
 240.5 \\
 \underline{288.5} \approx 290
 \end{array}$$

$290X = 290 \times X = Y \leftarrow \text{price (unknown)}$
 \approx amount of cards needed

Figure 11. Group A-D's plans and calculations

Afterwards, the group verified the way Andy measured the door and questioned the orientation which the gift cards were placed (they referred to this as a *vertical* placement). Soon enough, they ran into a non-mathematical idea of aesthetic. Anna was concerned about the orientation of the gift cards. Since most cards have a horizontal design (see Figure 12), Anna believed that placing the cards vertically on the door would make the design look awkward.



Figure 12. Horizontal and vertical placement of a Starbucks gift card

- Amanda: You put the cards this way (palms face each other, about one hand width apart) not this way (palms face each other, about two hands width apart)
- Andy: (Nodes)
- Amanda: Yea..... high five!
- T: So which way did you put the cards?
- Amanda: This way (palms facing each other, about one hand width apart to indicate the cards are placed vertically). It makes sense.
- Anna: But the design's like this, and it will go side way.
- Amanda: Huh?

Andy: Ooh!

Anna: Coz the design is like this (palms face each other, about two hands width apart).

Amanda: Ooooooo... no, it'll be like, I dunno I guess you can tile it make it look like (turns and looks at the teacher)?

T: I dunno, I dunno what you wanna do...

Amanda: Hah.

As the group discussed the horizontal and vertical placement of the gift cards for the next ten minutes, they sent Andy to re-measure the dimensions of the door. Andy aligned the gift card vertically with the door, just like he previously did. Based on these new measurements, the width of the door is 17 gift cards (short edge) wide and the height is 24 gift cards (long edge) tall. At the end of their discussion, Anna believed that the cards should be placed horizontally on the door, but Amanda and Andy ignored her and insisted to place the gift cards vertically on the door. Without resolving this tension, the group pushed forward and looked up the possible cost for each gift card, and quickly determined a solution based on the number of gift cards used and the cost per gift card.

Anna: If you have an opinion on having the cards like length wise or width wise what would you prefer?

T: Why don't you take a look at the gift cards? (points at gift card) Which one do you prefer?

Amanda: (laughs) It's for you!

T: What do the gift cards look like?

Anna: Well do you want them this way (palms face each other, one hand width apart) or do you want them this way (palms face each other, two hands width apart).

T: Well I want it to look nice... would you go vertical or horizontal?

Anna: Horizontal!

T: Okay?

Anna: I think we are done.

T: You think you are done? What are we doing here?

Amanda: So we, when put the cards on the door, we think, we are thinking of, we

T: So

Amanda: Vertical? Yeah. So we did that after that, and then, the bottom line we filled the door.

T: Ok, so how many... how many cards?

Andy: Each card is five dollars, and we

Amanda: Vertical. This is the price (points at paper), so we just put

T: Wait a minute, four hundred and eight times five is not two forty.

Andy: Two thousand forty.

Amanda: Wa?

Andy: Two thousand forty.

Amanda was certain that she has found the solution to the problem and was very confident about the answer. As the group continued to discuss their solution with me, I pointed out to the group that their current solution costs too much money and they should reconsider their approach to the problem. This is when Amanda realized that they were asked to decorate rather than to cover the door with gift cards, and that they were required to paint the door as well.

Andy: Two thousand forty.

Amanda: Wa?

Andy: Two thousand forty.

T: That's a lot of money for a door.

Andy: Yea...

T: I mean that's too much money for a door. See if you can lower the cost. That's too much. And you haven't painted the door yet.

Anna: Oooo how much is the cheapest amount.

Amanda: You don't have to cover the whole door?

T: Well I dunno, you decide.

Amanda: The paint?

Anna: Wait, it says, huh, paint the door, and then decorate the door.

Amanda: Oooohhh...

These realizations led Amanda, Anna, and Andy to reconsider their approach. They decided that they should first use gift cards to create a design on the face of the door, and consider painting the door thereafter. Using the information they previously gathered, they created a 13 by 24 grid to represent the door and used this grid as a guide to generate a possible design (Figure 13)⁹. After an approximately twenty minutes discussion, the group settled on a design and determined the number of gift cards required, the cost of the gift cards, and the cost of the paint required for the project.

Amanda: We got, a hundred and sixty-four dollars.

T: A hundred and sixty-four dollars? Can you show me please?

Amanda: Uh, well, we make a "C" for coffees, or cupcakes, or cats, or anything that starts with a "C".

T: A "C", okay?

Amanda: Like for coffee.

T: Okay, yeah! So how many cards do I need for a "C"?

Amanda: Twenty-eight. We counted twenty-eight cards, and we times it by five, so, like, five dollars because the cards cost five dollars.

T: Okay, so five dollars a gift card.

Amanda: So a hundred and forty, and then, we times, wait, wait no, wha??? Oh yea, we times twenty-eight by five, so a hundred and forty plus twenty-six, is the cost of one paint can.

Anna: For the door?

T: For the door?

⁹ It is unclear why they created a grid with only 13 columns while their measurements showed they could fit up to 17 cards across the door.

Anna: Yea.

T: How did you figure that out?

Amanda: Home Depot-¹⁰

T: Homedepot.ca, I like that.

Amanda: So it's a hundred and sixty-six, in total.

T: In total?

Amanda: Yup.

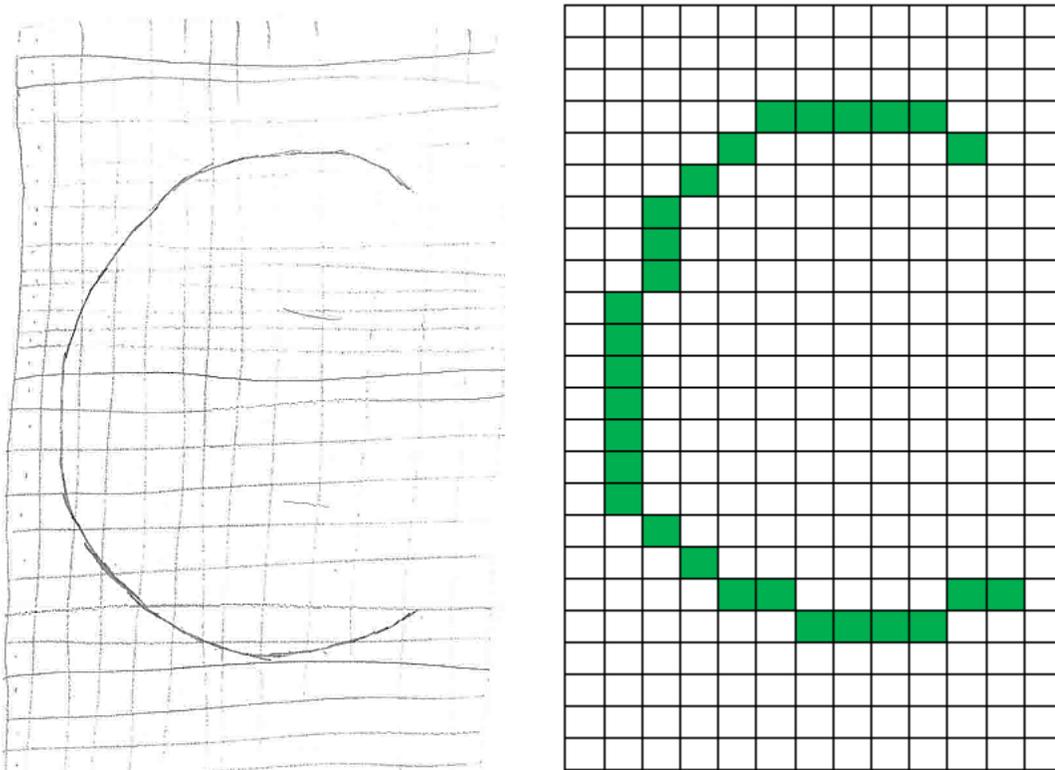


Figure 13. Group A-D's design and an illustration of their design.

As Amanda, Andy and Anna discussed and verified their solution with me, the three of them became less confident with their solution and changed their answer from “a hundred and sixty-six dollars” to “two hundred dollars total” to “under two hundred” dollars. To push the group forward to consider the various tools that were not mentioned in the question, I hinted to the group the need of painting tools, to which they

¹⁰ Home Depot is an American based company that sells home improvement and construction products.

immediately argued that a brush was readily available and it was not necessary to budget a brush into the cost. I rejected the idea, and hinted they also needed adhesives to secure the gift cards on the door.

Amanda: So it's a hundred and sixty-six, in total.

T: In total?

Amanda: Yup.

T: Okay.

Amanda: I'm scared.

T: Okay. So that would cover the entire project?

Anna: Yes.

Amanda: So... two hundred dollars total.

T: Two hundred dollars total?

Amanda: No, like, under two hundred! So...

T: Under two hundred?

Amanda: Kind of, kind of, ...

T: Okay, show me the entire details. ... so me what colour of paint I'm getting, do I need ... br...

Amanda: Oooo the brush!!!

T: Ahh... the brush...

Amanda: I thought, I thought you already have them!

T: I don't have a brush!

Amanda: If you are renovating your entire house, you would need a paint brush so you would have a paint brush ready for that project!

T: So am I supposed to recycle all my old paint brush?

Amanda: Yes! You could wash it!

T: Uhhh... Oops. And how am I getting the cards on [the door]?

Amanda: Oooooo

Helen¹¹: You need glue!

Amanda: Oooooo

After working on painting tools and adhesives, the group showed their work to me once more in the form of a detailed budget, including: a can of green paint (\$26), a paint brush (\$1.25), 28 gift cards (\$140), and a roll of tape (\$1.25); and concluded that the project would cost them \$168.50. During the discussion, Anna pointed out that they did not include sales tax in their solution. Instead of modifying their work, the students removed sales tax from their solution by assuming they “live in a tax free country” or “Oregon, USA”¹².

Anna: Did you add tax?

Amanda: (whining tone) why did you...

T: Ooo did you add tax?

Amanda: Okay... let’s say we live in a tax-free country?

Andy: Oregon, USA.

In the end, Anna accepted Andy and Amanda’s proposal and removed tax from their work. They further modified their solution by replacing the paint brush with a paint roller, and added a paint tray and glue to their budget. They found the cost of these tools from the home depot website, determined the final cost of the project, and submitted what seems to be a reasonable budget along with a design of the door. They made a mistake in this final calculation – they did not include one of the items on the list (either the roll of tape or the bottle of glue, which costs \$1.25) in their final budget. Students’ submitted budget is a cleaned up version of their work which includes the cost of each item

¹¹ Helen is a student in the same class but in a different group. She overheard the conversation and answered my question.

¹² Oregon, USA is a state in the USA that does not have sales tax.

required but very little details of these items (

$$\begin{array}{l} 28 \text{ gift cards} = \$140 \\ 1 \text{ green paint can} = \$26 \\ 1 \text{ paint roller} = \$5.99 \\ 1 \text{ tape} = \$1.25 \\ 1 \text{ paint tray} = \$0.98 \\ 1 \text{ glue} = \$1.25 \end{array} \quad \begin{array}{r} 140 \\ + 26 \\ \hline 5.99 \\ + 1.25 \\ \hline .98 \\ \hline \$174.22 \end{array}$$

Figure 14).

$$\begin{array}{l} 28 \text{ gift cards} = \$140 \\ 1 \text{ green paint can} = \$26 \\ 1 \text{ paint roller} = \$5.99 \\ 1 \text{ tape} = \$1.25 \\ 1 \text{ paint tray} = \$0.98 \\ 1 \text{ glue} = \$1.25 \end{array} \quad \begin{array}{r} 140 \\ + 26 \\ \hline 5.99 \\ + 1.25 \\ \hline .98 \\ \hline \$174.22 \end{array}$$

Figure 14. Group A-D's budget for the "Door Project" – Final Submission

In what follows I discuss students' modelling process with a focus on the modelling stages using Borromeo Ferri's (2006) modelling cycle as my unit of analysis.

8.1.1. A discussion on group A-D's modelling process

Amanda, Andy, and Anna began their modelling process by reading the instructions provided (RS1¹³). The three of them initially believed that they needed to completely cover the face of the office door with gift cards and never considered any other possibilities. Their initial modelling behavior also suggests that they focused solely on

¹³ The letters in this abbreviation refers to the stage of the modelling cycle, the number represents the step in the overall modelling process. In this case, RS1 refers to the real situation, step 1.

the gift cards and neglected all other aspects of the *Door Project*. Their understandings and interpretations, or lack thereof, formed their first MRS (MRS2¹⁴).

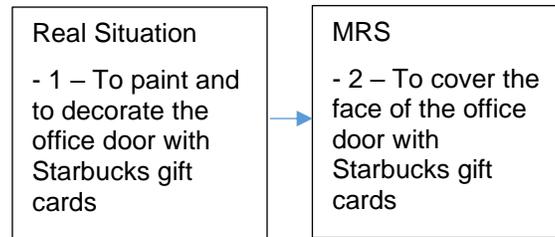


Figure 15. Group A-D's initial understanding of the problem

Their MRS and their initial modelling behavior suggest that they were treating the problem as a mathematical word problem. As discussed in Chapter 3, mathematical word problems are well-defined problems commonly found in mathematics textbooks, which could be solved using basic arithmetic operations (Maaß, 2010).

As such, the three of them made assumptions about the situation to extract numerical values from the situation: they assumed that the office door has the exact same dimensions as the classroom door and sent Andy to take measurements. It is possible that Amanda, Andy, and Anna did not realize doors come in different sizes because they have little EMK in this area. It is also possible that they made the assumption because the classroom door was readily available and easily accessible. Assuming the office door to have the same dimension as the classroom door also allowed them to simplify and to access the real situation, an important step in Borromeo Ferri's (2006) modelling cycle prior to creating a real model (RM3¹⁵).

Using the measurements they took, the three of them developed a plan to determine mathematically the number of cards required and the cost for the *Door Project* (MM4¹⁶). Their calculations led them to 290 gift cards (MR5¹⁷). Instead of proceeding with their plan to determine the cost of the project, they first verified the orientation the gift cards to be placed on the door and re-measured the classroom door. This process led them to

¹⁴ MRS stands for the mental representation of the situation.

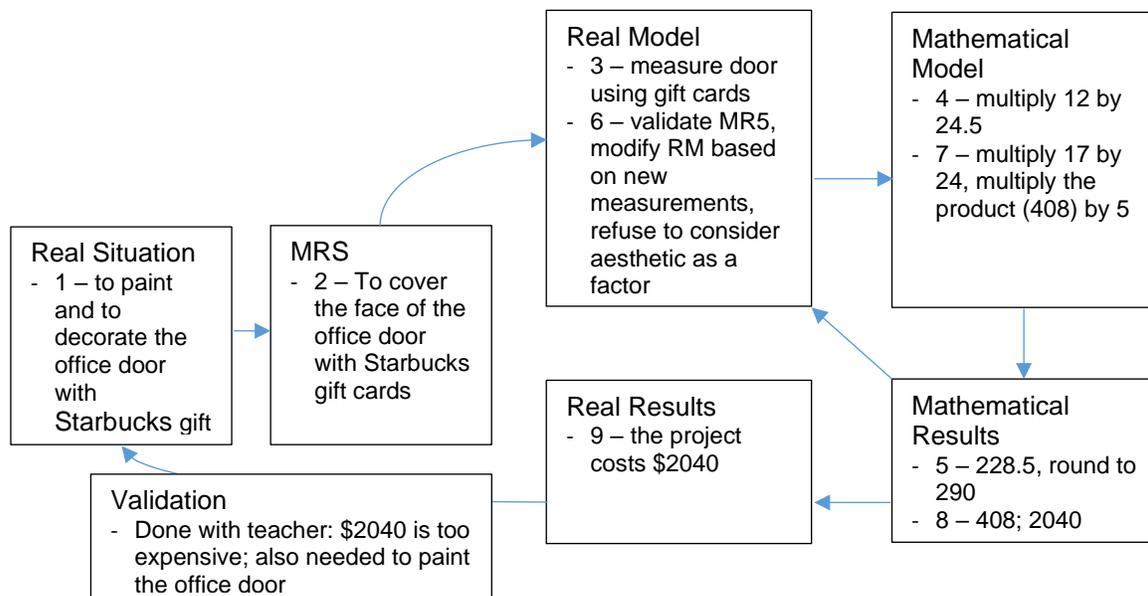
¹⁵ RM stands for real model.

¹⁶ MM stands for mathematical model.

¹⁷ MR stands for mathematical results.

discover a mistake in their original measurements. Their validation at this stage of the modelling cycle seems to go against Galbraith and Stillman's (2006) claim that most novice modellers do not validate their work during their modelling process. This is also different from what modelling cycles in general suggest, that validation happens at the end of the modelling cycle. Conversely, the three students' actions are in line with Borromeo Ferri's (2006) work on individual modelling routes, where "the individual starts this process during a certain phase, according to his/her preferences, and then goes through different phases several times or only once, focusing on a certain phase or ignoring others" (p. 91).

As Amanda, Andy, and Anna modified their real model as a result of their validation (RM6), Anna drew on her EMK regarding Starbucks gift cards and added the idea of aesthetic to the group's real model. She questioned whether the current orientation of the gift cards would result in a visually pleasing solution. However, Anna was unable to convince Amanda and Andy to change the orientation of the gift cards. Anna's concern with aesthetic shows that she has activated her EMK, and Amanda and Andy's unresponse of her comments suggests that they have not yet activated their EMK, or that they were not willing to consider aesthetic as a contributing factor in their approach. Without resolving the tension, the group rebuilt their mathematical model (MM7), generated a mathematical result (MR8), and interpreted the result in terms of the cost for the project (RR9¹⁸).



¹⁸ RR stands for real results.

Figure 16. Group A-D's first modelling cycle (gift cards)

Their first modelling cycle (MC1¹⁹, steps 1-9) took gift cards into consideration. Their work shows that they had a partial understanding of the question as they ignored painting the door, interpreted the problem as a mathematical word problem, and took minimal realistic perspectives into consideration.

Afterwards, Amanda, Andy, and Anna called me over to discuss their results. This validation process is different from what modelling literature describe (for example, see Borromeo Ferri, 2006; Galbraith & Stillman, 2006; Blum & Leiß, 2005, 2007). Instead of validating their results by making a decision whether to accept or reject their real results on their own, Amanda, Andy, and Anna relied on my approval. Calling me over to discuss the results suggests that the three students were not confident with their work and that they wanted my assurance that they got the *correct answer* prior to submitting their work. However, they did not receive this assurance. Rather, the students found out their real results were incomplete and had room for improvement. These realizations pushed them to re-read the question and re-evaluate their approach (RS10), and to re-enter the modelling cycle (MC2, steps 10-15). As their understanding of the situation deepened, they developed a new MRS (MRS11). They engaged with the problem from an increasing realistic perspective and applied EMK in their approach. They completely rebuilt their real model (RM12) and developed a design based on a grid they created. The 13 by 24 grid is a visual representation of students' original understanding and interpretation of the problem (to tile the gift cards), and the *C* they drew represents their new understanding of the problem situation (to decorate the door using gift cards).

I find it extremely interesting that Amanda, Andy, and Anna created a grid and used it as their guide to generate a design while they could have put the cards anywhere they want on the door. For example, when they arranged the letter *C* on the door, they could have placed cards horizontally or diagonally on the door, or arrange the cards around parts of an arc of an imaginary circle or oval, rather than lining up the cards vertically on a grid. The group did not explain their approach to create their door design on a grid.

¹⁹ MC1 stands for modelling cycle 1.

This is not to say that there is something wrong with the approach they used for the decoration portion of the problem. Rather, it seems that their initial approach to tile the face of the door influenced their subsequent approach. It is possible that they simply took their initial real result and built on it by turning this initial result, where all cards lined up perfectly on the door, into a grid and used it to guide them to decorate the door. Also, the grid they used resembles the Cartesian plane or a piece of graph paper, something they were familiar with at the time of the study. However, it is unclear if the group saw the resemblance when they created the design and whether they saw or understood the mathematical relations between the grid they used and the Cartesian plane or a piece of graph paper.

Other than the newly created design, group A-D also included a can of green paint in this second version of their real model. They claimed that the information came from the Home Depot website. Their search of paint from the Home Depot website demonstrates another application of their EMK (their understanding and ability to find paint at Home Depot and information of paint at their website). However, they did not include any information about the can of paint other than the cost (\$26) in any of their work. A quick search on the Home Depot website shows that it is likely that Amanda, Andy, and Anna rounded the price of the can of paint they chose to the nearest dollar, and that the can of paint they chose was approximately 1L in volume. This is a nice example of Borromeo Ferri's (2006) suggestion that a real model is "mostly built on an internal level of the individual". In other words, Amanda, Andy, and Anna's external representation of their real model is not a full representation of their interpretation and understanding of the situation.

To summarize, Amanda, Andy, and Anna's second real model (RM12) contains a plan to decorate the door (the letter C drawn on a 13-by-24 grid) and a can of green paint. Afterwards, the group proceeded to create a mathematical model (MM13) and determined the cost of the gift cards based on the number of cards required and added that to the cost of a can of paint (MR14). They interpreted their mathematical result as the cost of the project (real result, RR15). The resulting real result is a more realistic one than their first. Prior to concluding their second modelling cycle, group A-D validated their results with me, just as they did in their first modelling cycle.

Their second validation with me was proven to be fruitful as well, because it helped them to recognize ways to improve their work further. However, the group was resistant to my comments and the changes they needed to make surrounding the additional tools they required to complete the project. Amanda tried to argue her way out regarding painting tools, but was eventually convinced that they should further engage with the problem from a realistic perspective. The validation process triggered the recognition of additional EMK and led the group to enter the modelling cycle for the third time (MC3, steps 16-20). Their realization of tools that were not mentioned in the problem situation added to their MRS (MRS16), and guided them to make changes to their real model (RM17).

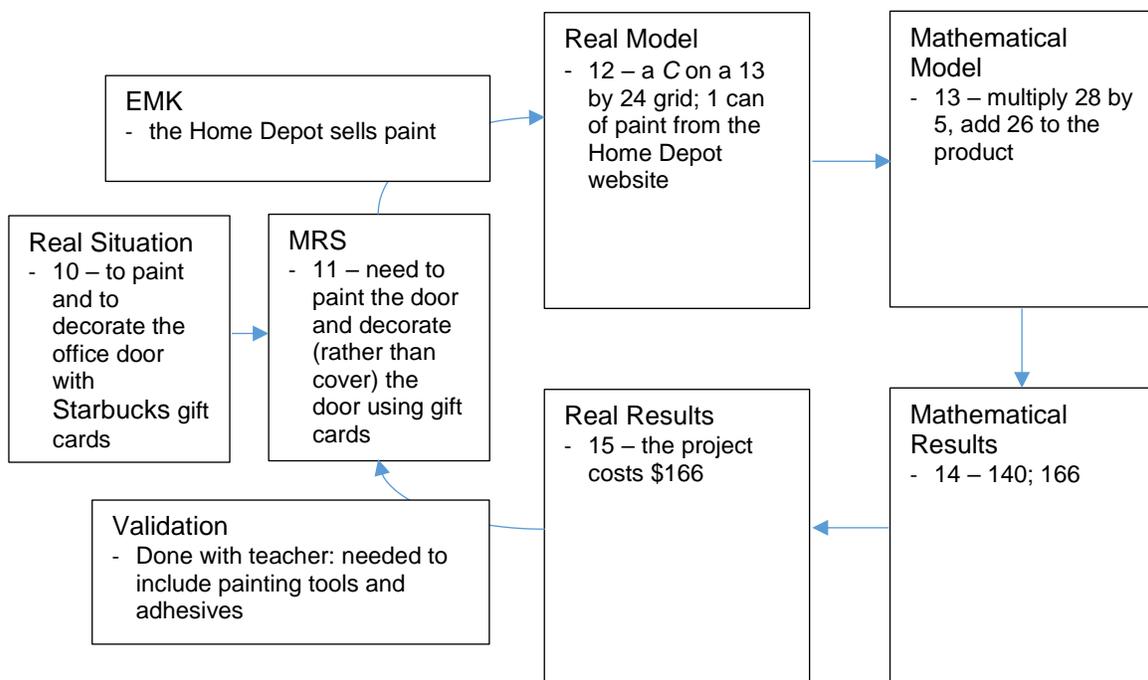


Figure 17. Group A-D's second modelling cycle (gift cards and paint)

The repetition of their modelling cycle is in line with Stender and Kaiser's (2016) study, where students completed the modelling cycle a few times to generate an increasingly complete and realistic real result to a modelling problem, and with modelling literature in general (for example, see Borromeo Ferri, 2006; Galbraith & Stillman, 2006), where the modelling cycle represents an iteration process until an appropriate result to the problem situation is determined.

Group A-D did not include painting tools and adhesives in their first and second modelling cycles. Based on the courses they were taking at the time of the study, it could be argued that they all have some experiences in painting and they understand they needed adhesive to glue one thing on another. However, they did not relate these experiences and understanding to the *Door Project*, as if these experiences existed in isolation from the problem. Their omission of such tools might be related to their poor understanding of the problem along with their initial interpretation of the *Door Project* as a mathematical word problem, where they aimed to tile the face of the door with gift card. This interpretation prevented them from activating their EMK and limited the EMK they applied to solve the problem. As seen in their validation processes during both MC1 and MC2, group A-D only activated their EMK when they were reminded of these possibilities.

During their third modelling cycle (MC3), Amanda, Andy, and Anna made modifications to their real model (RM18) and added a paint brush and a roll of tape. Afterwards, they modified their mathematical model by including the cost of these tools in their calculations (MM19) and determined a mathematical result which included the cost of 28 gift cards, a can of paint, a paint brush, and a roll of tape (MR20). The process of modifying their mathematical model, determining the mathematical result, and interpreting the mathematical result in terms of the problem situation happened very smoothly and naturally. This may be a result of the nature of RMCTs. As the mathematics required to mathematically solve the problem is rudimentary to students, students naturally followed the path to generate a mathematical model, a mathematical result, and a real result once they built their real model.

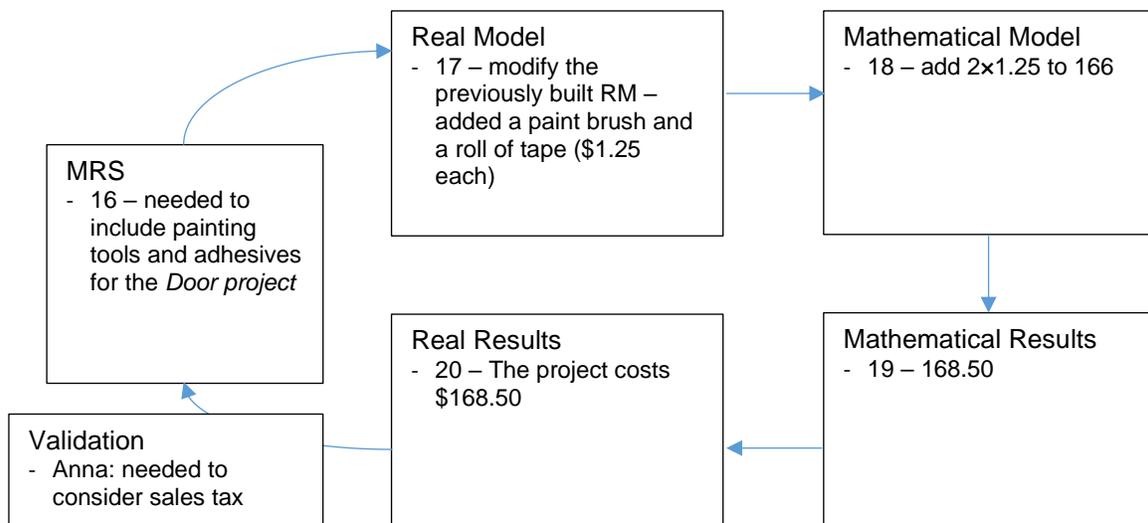


Figure 18. Group A-D’s third modelling cycle (tools)

Upon producing their third real results, Amanda, Andy, and Anna validated their work with me once more. By now they approached the problem from a much more realistic perspective than they initially did. They created a design instead of tiled the door with gift cards, and included a can of paint, a paint brush, and a roll of tape in their work. During this final discussion, Anna attempted to add sales tax, another realistic aspect, to the situation (MRS21). This forced the three of them to reflect on their work and to enter the modelling cycle once last time (MC4, steps 21-25). Instead of making further modification to their work, Amanda and Andy chose to avoid sales tax and made assumptions to the situation (RM22). However, they did not stop there. Although the three of them refused to incorporate sales tax into their work, they made minor changes to their real model (RM22), mathematical model (MM23), mathematical result (MR24), and therefore their real result (RR25).

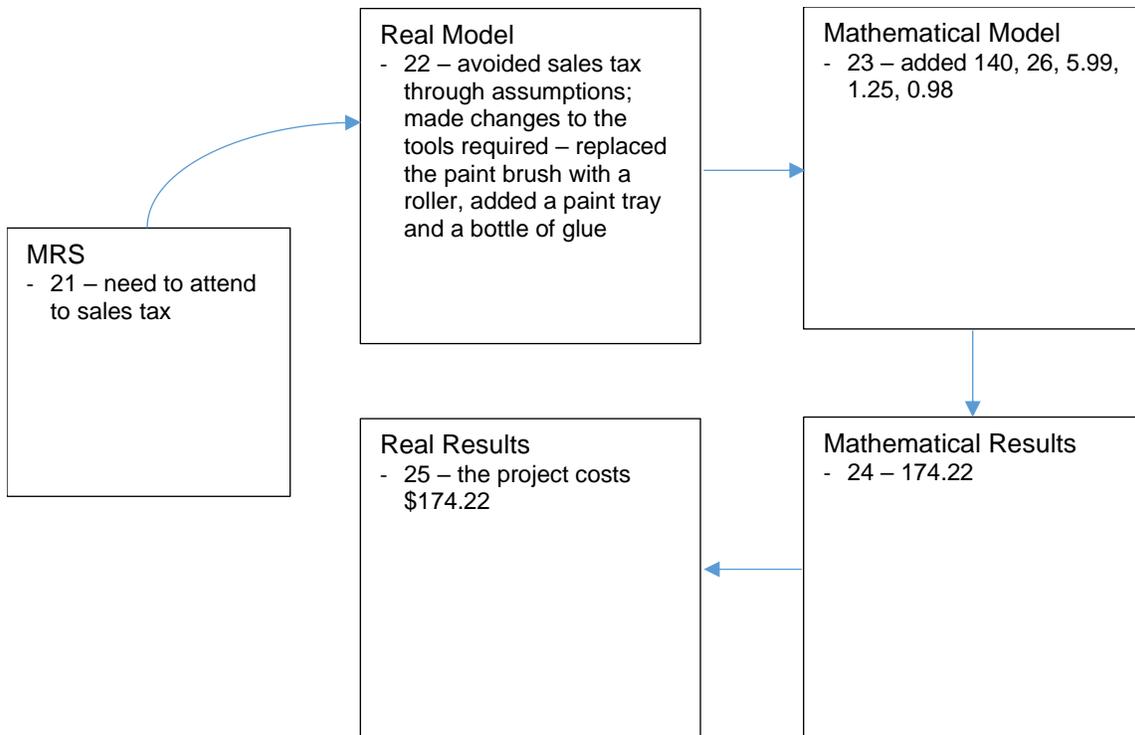


Figure 19. Group A-D’s fourth modelling cycle (sales tax)

Students’ assumptions about sales tax further demonstrate the importance of their EMK in the modelling process. Immediately after Anna pointed out the problem with sales tax, Amanda suggested that they “live in a tax-free country”, and Andy quickly pointed out the possibility of “Oregon, USA” as a solution to their sales tax problem. Amanda and Andy’s immediate reactions to Anna’s question suggest that they understood excluding sales tax in their calculations contribute to the inaccuracy of their work, and a possible solution to resolve this tension is to make assumptions about where they make their purchase. A brief conversation with Andy showed that he has fond memories of Portland, a city in Oregon – he went there with his family a few times, where they shopped a lot and did not pay sales tax on their purchases. Andy’s ability to draw from his past experiences, which forms a part of his EMK, allowed them to make assumptions about the situation and to reach a possible solution without much effort.

In summary, Amanda, Andy, and Anna went through the modelling cycle four times prior to the generation of what they believed to be a real result. They first generated what they believe to be a correct result without a deep understanding of the problem situation. They engaged with the *Door Project* from a mathematical perspective, extracted numerical values from the question, and generated an answer to the problem. As their

understanding of the situation deepened through various discussions among themselves and with me, they re-entered the modelling cycle a few times and engaged with the problem at an increasingly realistic perspective, which eventually resulted in a much more realistic result as compared to their initial work. The following arrow diagram is a visual representation of the steps taken by Amanda, Andy, and Anna throughout their entire modelling process. The red arrows in the diagram represent their first modelling cycle (steps 1-9), the green arrows represent their second modelling cycle (steps 10-15), the blue arrows represent their third modelling cycle (steps 16-20), and the purple arrows represent their fourth modelling cycle (steps 21-25).

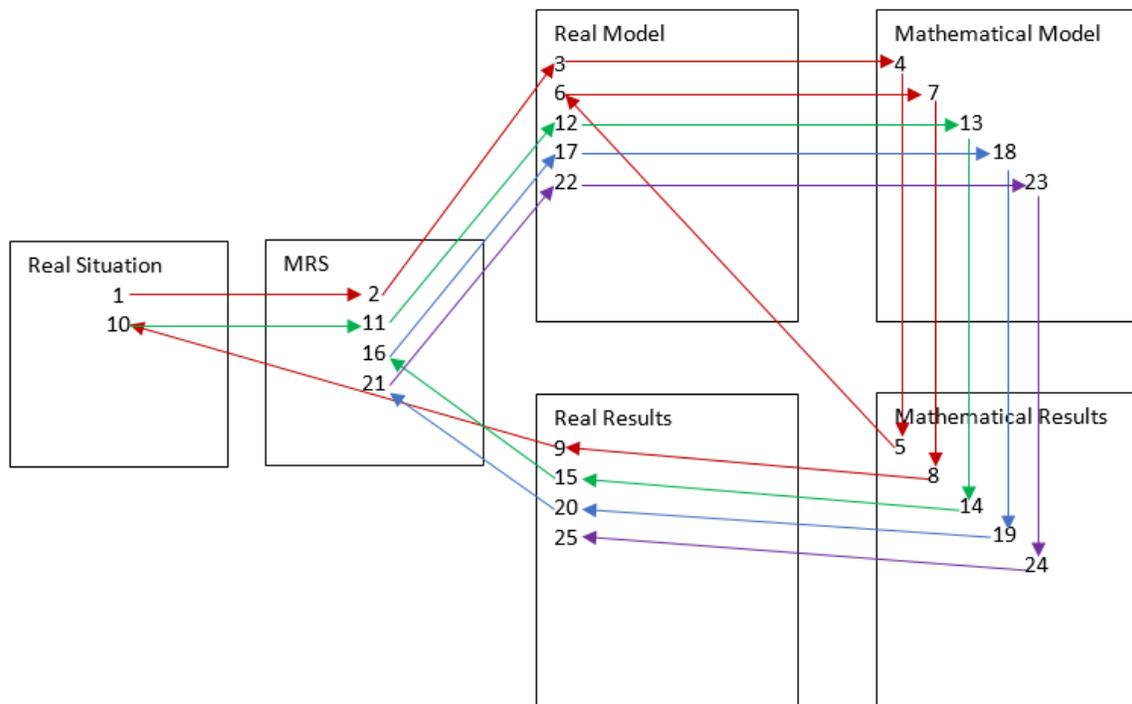


Figure 20. A diagrammatic representation of group A-D's modelling process.

8.2. Barry, Bella, and Betsy (Group B-D)

Similar to group A-D, Barry, Bella, and Betsy assumed the importance of the gift card and the office door and quickly measured the dimensions of a gift card using a ruler and recorded them as 8.5 and 5.4 (no units reported). They also assumed the office door in the question has the same dimensions as the classroom door. However, instead of measuring the classroom door using a metre stick or a ruler, the three students estimated the width and the height of the classroom door to be 1m and 2.5m, but

reported these measurements in their written work as 100cm and 250cm²⁰. They then determined the area of the face of the door as 25000 (no units reported) and the area of a gift card as 45.90 (no units reported).

Barry, Bella, and Betsy then did a quick online search to confirm the minimum cost per gift card. They decided to tile gift cards on both the front and the back sides of the door and very quickly calculated the number of gift cards required and the cost of the project by dividing the total area of both sides of the door (50000) by the area of a gift card (45.9²¹), rounded the quotient up (from 1089.3 to 1090), and multiplied it by 5 (\$5 per gift card). They interpreted the product, 5450, as the cost of the project, and reported \$5450.00 as their solution to the problem (see Figure 21).

$$\begin{array}{r}
 8.5 \\
 \times 5.4 \\
 \hline
 340 \\
 4250 \\
 \hline
 459.0
 \end{array}$$

$$2 (250\text{cm} \times 100\text{cm}) =$$

$$\begin{array}{r}
 50,000 \\
 45.9
 \end{array}$$

$$\begin{array}{r}
 1089.3 \\
 \hline
 1090 \\
 \times 5 \\
 \hline
 \$5450.00
 \end{array}$$

Figure 21. Group B-D’s first real result to the door project

Similar to group A-D, these students then called me over to discuss their work and asked for my opinion regarding the cost per gift card. The group presented their argument to spend over \$5000 because they needed to “fill the WHOLE door with Starbucks gift cards”. During the process, they also assumed that the gift cards would

²⁰ The classroom door is 36” (0.9144m) wide and 80” (2.032m) tall, as compared to group B-D’s estimated measurements of 1m wide and 2.5m tall.

²¹ The group changed the reported value from 45.90 to 45.9.

tile perfectly on both sides of the door, and ignored the cost of painting and additional tools.

Barry: We are wondering, what, what amount of gift cards do you want?

T: I dunno?

Barry: Like, five dollar gift cards or like, ten dollars?

T: Say five.

Barry: So, we have to make the... if you were to buy five dollars gift cards, you would approximately five thousand four hundred fifty dollars.

T: Why?

Barry: Because, you have to fill the WHOLE door with Starbucks gift cards, and then in order to do that, you'll have to buy each one individually and would add up to five thousand four hundred fifty dollars.

In order to steer students to reconsider their approach, I gently complained about the cost of the project. In response to the complaint, the three students suggested to change the parameters of the question by replacing actual gift cards with hand drawn or photocopied ones to lower the cost rather than to approach the problem differently. They also claimed that it "only [requires] five thousand four hundred and fifty loonies" to pay for the project. While the three of them tried their best to argue their way out of the situation, I refused to accept their argument as a reasonable solution. I suggested for them to reread the question and sent them to discuss possible approaches with students in other groups.

T: (whiny tone) I don't wanna spend so much...

Betsy: Just stick it with paper.

Barry: Turns out...

Betsy: Does it have to be real gift cards?

Barry: (laughs)

Bella: Can you photocopy it?

T: (whiny tone) I don't like that...

- Bella: Why?
- Barry: What if... yea, yea okay, uh...
- Bella: Why can't you photocopy it, cut it out, and like...
- Barry: Okay what if I say it like this. You will only need five thousand four hundred and fifty loonies?

After roughly five minutes of discussion, the group did not make progress and refused to approach the problem differently. As they reread the problem, they reckoned that they needed paint for the project and made two assumptions: they could purchase paint at the dollar store, and a small bottle of paint would suffice. They then called me over to validate their approach and results. Instead of directly answering their questions, I suggested for them to go to the Home Depot website for additional information. Towards the end of this conversation, Bella made some random guesses in hopes that I would "accidentally" give the answers away. In response, I gently hinted that the amount of paint required is related to the size of the door.

- Bella: How am I supposed to know what paint you are gonna use?
- Barry: Uhh...
- T: You have a smart phone? (all three students took out their phones) You wanna try Home Depot?
- Barry: Okay Home Depot.
- Betsy: You just randomly buy paint?
- Bella: Okay a hundred dollars.
- T: How big is my door?

Barry, Bella, and Betsy worked on paint for another ten minutes, but were unable to determine the amount of paint required for the project. As I engaged in a conversation with the group, they asked me about the colour scheme of the office, at which point Bella raised the question about the cost of the paint again.

- Bella: Are you okay with two hundred dollars for paint?
- T: Two hundred dollars for paint? What kinds of paint...
- Bella: White.

T: Two hundred dollars of white paint?

Bella: Yeah.

T: How much are we buying?

Bella: (no response)

Barry, Bella, and Betsy carried on with their discussion afterwards, focusing on the number of gift cards, and the amount and type of paint they needed for the project. During this time, there were a lot of intergroup discussions in the classroom. Students from different groups shared with each other ideas of their designs and possible ways to lower the cost of the gift cards, and useful information such as the minimum cost per gift cards, where to find discounted paint and painting tools, etc. At one point, Edward²² came over and explicitly explained to Barry that *to decorate* means to create a design in attempts to convince Barry that they did not need to tile any of the surfaces of the office door with gift cards. Barry, Bella, and Betsy seemed to understand the idea of *to decorate* and that their approach resulted in an expensive door. However, they maintained their original idea and approach and refused to interpret and approach the problem differently.

Barry: I think it's reasonable to spend five thousand-

T: Five thousand!

Barry: Five thousand four hundred and ninety-three dollars, on a door, because, as far, you have already renovated the room, so, why not finish off the door? ... I think the door needs to be expensive, coz it's a, it's a portal to your, your amazing room.

In the end, the group submitted their original design that uses close to 1100 gift cards and costs almost \$5500. They did not include any painting tools or adhesives in their work, and approached the problem mostly from a mathematical perspective.

²² Edward is Barry's good friend who was working on the *Door Project* in a different group.

The whole thing would take around \$5493 if everything were to go as planned.

The decoration would cost \$5450 because you would need 1090 cards to fill both sides of the door.

The paint would cost around \$43 to paint it all up found in Home Depot.

Figure 22. Group B-D's submitted work

8.2.1. A Discussion on group B-D's Modelling Process

After reading the instructions (RS1), Barry, Bella, and Betsy were quick to assume the importance of the dimensions of the door and that of a gift card, and aimed to tile gift cards on the front and the back sides of the office door. Their initial interpretation led them to focus only on gift cards and ignored paint and other aspects of the problem. This interpretation formed their first MRS (MRS2).

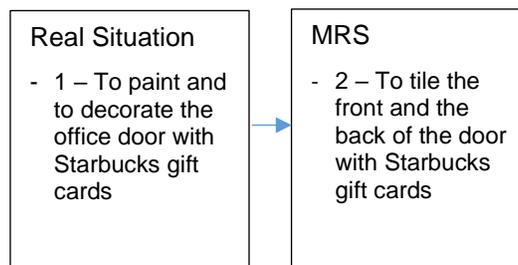


Figure 23. Group B-D's understanding of the problem

Similar to Group A-D, Barry, Bella, and Betsy assumed the classroom door has the same width and height as the office door in the question, and used their measurement and estimation skills to determine the dimensions of a gift card and that of the classroom door.

The information Barry, Bella, and Betsy presented in their real model contains context from the real situation, and represents their understanding and interpretation of the real situation (to tile the front and back sides of the door using gift cards) and their plan to achieve a solution (to determine the number of gift cards required by comparing the area of a gift card to the area on the door to be covered with gift cards) (RM3). Using this real

model, the three students then built a mathematical model (MM4) and determined a mathematical result (MR5). They interpreted this as the cost of the project (RR6).

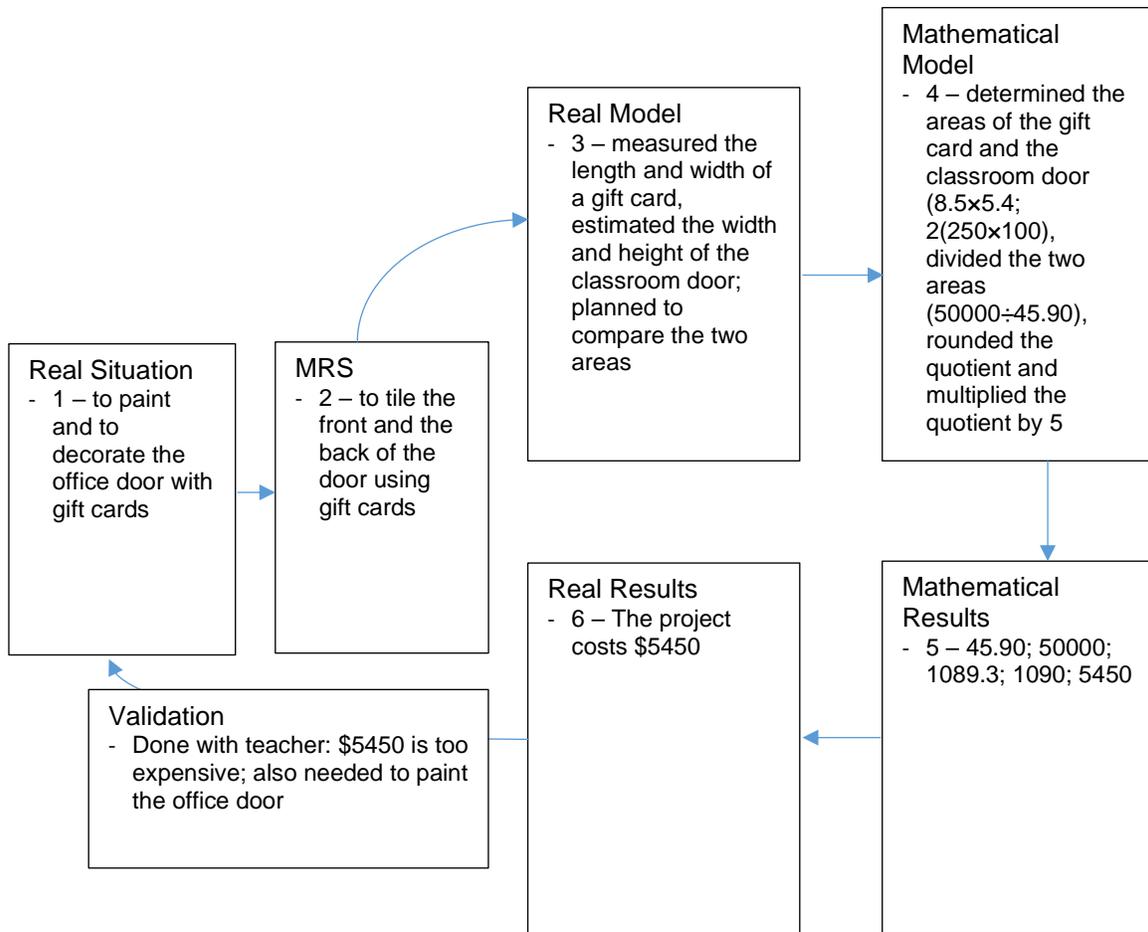


Figure 24. Group B-D's first modelling cycle (gift cards)

Group B-D's first modelling cycle (MC1, steps 1-6) demonstrates their poor understanding of the overall problem as they focused only on gift cards and ignored the painting aspect of the problem. Their work also suggests that the three of them approached the problem mostly from a mathematical perspective and had not activated their EMK. There was no indication that they thought about how the gift cards would line up on the door and how that would affect the aesthetics of the final product. And there was no indication that they thought about the tools required to complete the *Door Project*.

Barry, Bella, and Betsy's approach is similar to that of a mathematical word problem, such as the bus problem (Carpenter, Lindquist, Matthews, & Silver, 1983). The bus problem requires students to determine the number of busses required to move a group of soldiers. Students would typically divide the number of soldiers by the number of seats on each bus to determine the number of buses required. Those who reflected on the meaning of the solution would also round the resulting quotient up to the closest whole number to determine the actual number of busses required. Barry, Bella, and Betsy approached the problem exactly like the bus problem: they extracted numerical values from the question through measurements and estimations, determined the areas involved, divided the area of the door to be tiled with gift cards (50000) by the area of a gift card (45.9), rounded resulting quotient (1089.3) up to the nearest whole number (1090), multiplied it by 5, and interpreted the resulting product, 5450, as the cost of the project. This approach allowed them to determine a possible answer to the problem situation quickly, but demonstrated that they engaged with the problem from a minimal realistic perspective.

While their decision to round 1089.3 gift cards up to 1090 gift cards demonstrates their understanding in counting gift cards as whole objects, for them to accept 1090 gift cards as a solution demonstrates that they had not reflected on their solution deeply. From a mathematical perspective, 1090 has a prime factorization of $2 \times 5 \times 109$. This means that if they were to tile each side of the door evenly without cutting the gift cards into pieces, each side would have 5 rows of 109 gift cards or 109 rows of 5 gift cards. The layout of the gift cards simply does not make sense. While it is possible to tile the door where gift cards are not lined up perfectly in equal rows or columns, the lack of a design suggests that the three students had made the assumption, without reflecting deeply, that they could tile 545 gift cards on each side of the door with no gaps between the cards, or that they could cut the cards into pieces to completely tile the door.

While Barry, Bella, and Betsy interpreted the problem situation mostly from a mathematical perspective, it is interesting to note that they decided to cover both the front and the back of the door with gift cards. They did not choose to cover only one side of the door (the front or the back), nor did they choose to cover all six sides of the door. From a realistic perspective, it would not be logical to put gift cards on any sides other than the front and/or the back side of the door. However, it is also possible that they only viewed the door as a 2-dimensional object which only has a front and a back.

Also, while using 1090 gift cards and spending over \$5000 on a door is a possible solution, it is not necessarily a realistic solution. It is a possible solution because one can certainly decide to spend however much money to decorate his/her office door. However, it is not necessarily a realistic solution because economically speaking, why would someone put over \$5000's worth of unused gift cards on a door as decorations?

The validation process was unsuccessful in pushing the group to reconsider their approach. At its best, the validation process led them to briefly consider the cost of the project and looked for ways to lower the cost without changing their approach (hand-drawn or photocopied gift cards). Also, they ended up re-reading the question (RS7), which led them to realize that their solution was incomplete. They re-entered the modelling cycle (MC2, steps 7-12) and considered paint for the project (MRS8).

As they rebuilt their real model (RM9), their EMK led them to consider the possibility to purchase paint from a dollar store and to consider the aesthetic of their solution. Their conversations and decisions suggest that they had a fair understanding of the products a dollar store carry, probably from experiences in getting materials for school projects, and they struggled with the amount of paint they needed for the *Door Project*. In the end the group chose a can of paint from the Home Depot website, but did not include any additional information about the chosen can of paint. Similar to Group A-D, it is likely that Barry, Bella, and Betsy rounded the cost of the paint to the nearest dollar. A quick search on the Home Depot website shows that a can of paint with a similar cost has an approximate volume of 4L. This amount is sufficient to cover an area of 27m^2 to 37m^2 , or 300-400 square feet (single coat). Using their measurement as a guideline, the amount of paint purchased was well above the required amount. This is possibly a result of the students' inexperience in painting, where they needed to determine the amount of paint required for the job. After modifying their real and mathematical models to include paint (MM10), Barry, Bella, and Betsy quickly used the information to determine a mathematical result (MR11) and interpreted this as their real result (RR12). Afterwards, they discussed their work with me once more, and Edward, a student from a different group, attempted to reason with them the unnecessary to tile any sides of the door with gift cards. Despite the effort, these three students were stuck in their initial way of interpreting and approaching the problem. They tried their best to argue for their interpretation of the real situation, refused to approach the situation from any other

perspective, and submitted a solution that includes 1090 gift cards along with a can of paint.

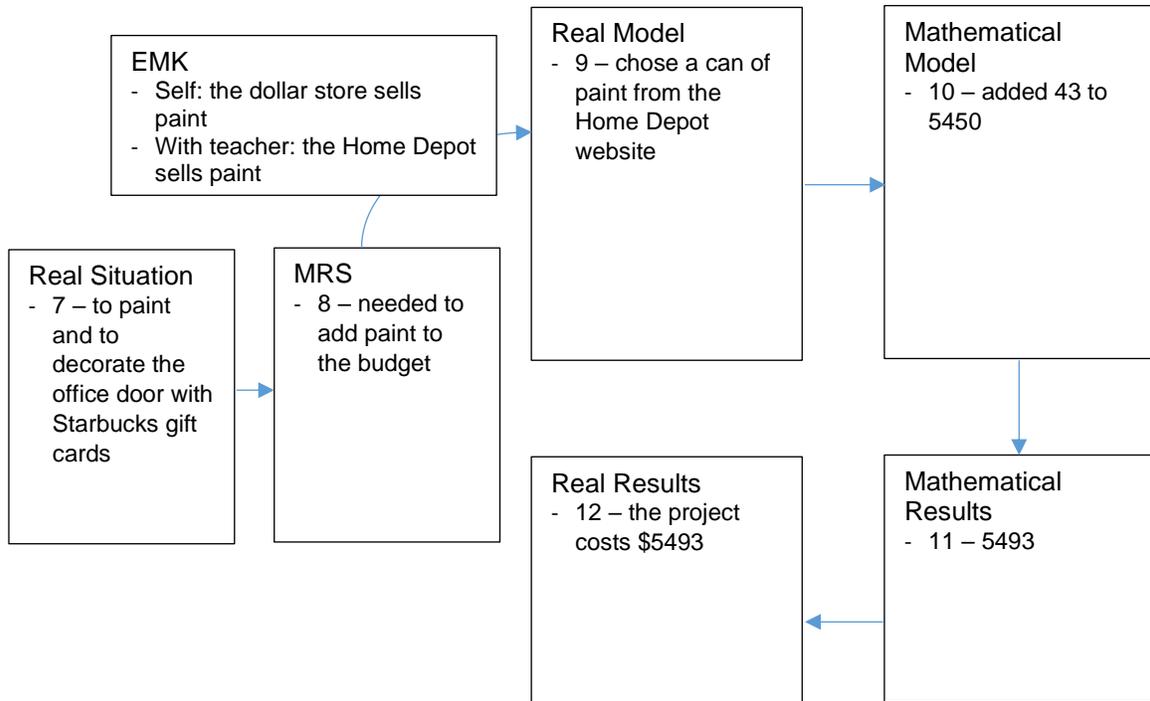


Figure 25. Group B-D’s second modelling cycle (paint)

Group B-D’s modelling process is singular. Unlike other groups, they interpreted the *Door Project* mostly from a mathematical perspective and insisted on a mathematical approach despite the various discussions I and Edward had with them. When asked to explain their reasons behind their work during a post-task interview, the three students insisted that what they did was necessary and it was what the question asked of them to do. They claimed that “you have to cover the door with gift cards”, “we need to cover the entire door with gift cards”, and “you need to assert your wealth with gift cards”. The following diagram summarizes Barry, Bella, and Betsy’s entire modelling process.

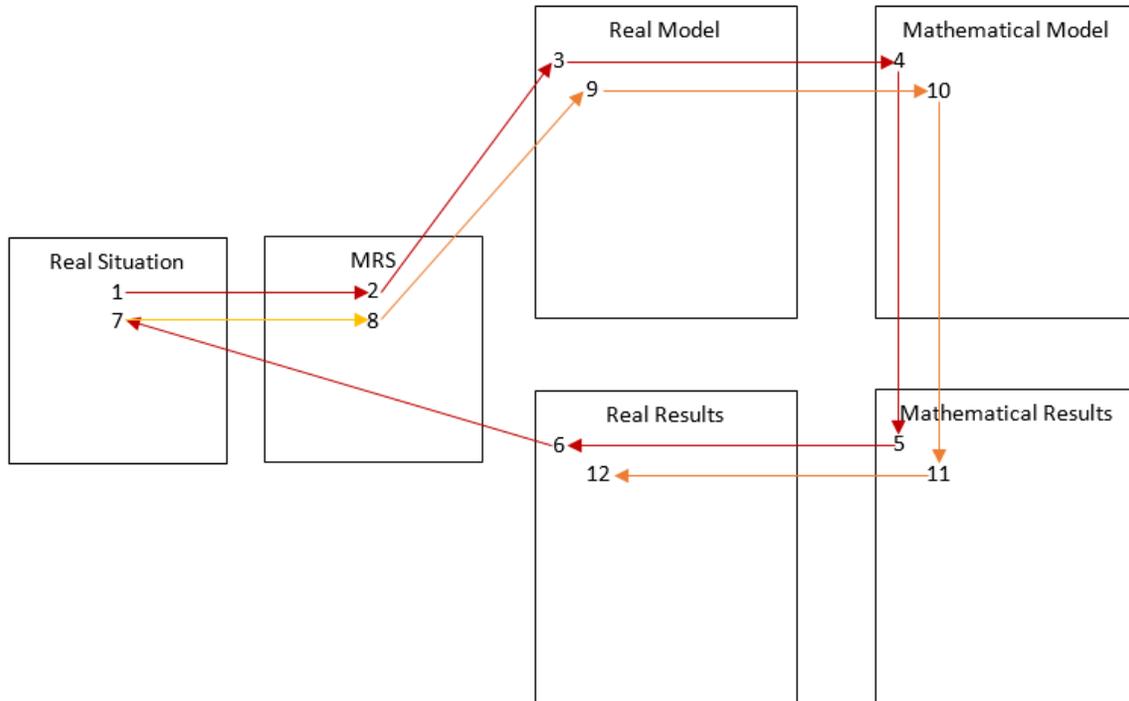


Figure 26. A diagrammatic representation of group B-D’s modelling process.

In summary, Barry, Bella, and Betsy completed the modelling cycle twice (see Figure 26). The red arrows (steps 1-6) represent their first modelling cycle and the yellow arrows (steps 7-12) represent their second modelling cycle). They were engaged with the problem from a mathematical perspective and interpreted the problem as one that asked them to tile gift cards on the front and the back sides of the door. They did not draw from their EMK to create a decorative design, to include painting tools and adhesives in their solution, and did not extend their thinking to consider cost as a factor of their solution.

8.3. Carmen and Carol (Group C-D)

Carmen and Carol quickly measured the width and the height of the classroom door using a metre stick and the length and the width of a gift card using a ruler after reading the instructions. Afterwards, they put these measurements aside and moved on to paint. They took out their smart phones and went on the Home Depot website. At this very early stage, Carmen and Carol did not have a specific idea about the type and colour of paint they wanted or needed. They wanted to find the “average cost” of paint, but did not explain what “average cost” is and the reasons behind their approach.

Carmen: [to Carol] Can you find, like, the average paint cost?

Carol: Um...

After a few minutes of searching, Carmen and Carol backed away from paint and decided they would need some sort of adhesives. They considered a number of possibilities, but did not draw any conclusion to what they would use (Figure 27).

Carmen: There are so many possibilities here that it makes me so angry. There are so many possibilities. Paint, glue, tape, um...

Carol: Super glue, there's also duct tape.

Carmen: Duct tape... I'm not writing down all the different types of glue. You said paint, glue, tape, super glue, it's all just glue, [Carol]!

Carol: Okay, okay, stop it!

The consideration of adhesive naturally led them to think about gift cards, more specifically, the cost of these gift cards. Carmen and Carol wanted to keep the cost of gift cards as low as possible. They considered asking their friends for used gift cards, and walking into Starbucks and pick up gift cards from the counter.

Carmen: Um... You could just like, ask all your friends for the Starbucks gift cards when they are done with them.

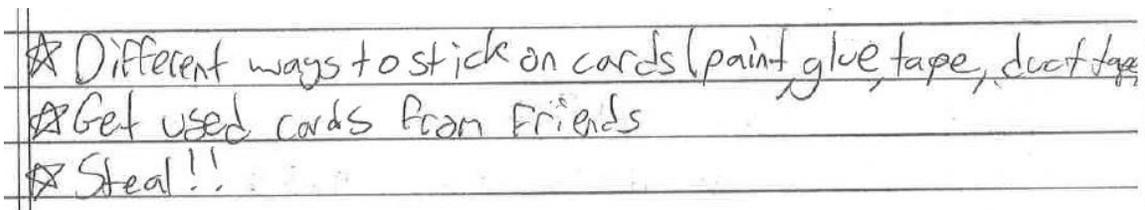
Carol: [laughs]

Carmen: We can recycle the cards.

Carol: Okay... what if you just walk into Starbucks and stole... [laughs]

Carmen: That's instead of buying, paying like

T: [Carol]! That's called stealing!



★ Different ways to stick on cards (paint, glue, tape, duct tape)
★ Get used cards from friends
★ Steal!!

Figure 27. Group C-D's ideas about the problem

Carmen and Carol did not remain on the topic of gift cards for long. Rather, they moved on to discuss with each other the number of gift cards they needed to tile the face of the door. Carol calculated the area of the face of the door and that of a gift cards based on the measurements they took at the very beginning, and divided the area of the door by that of a gift card, and got 385.02673797cm^2 .

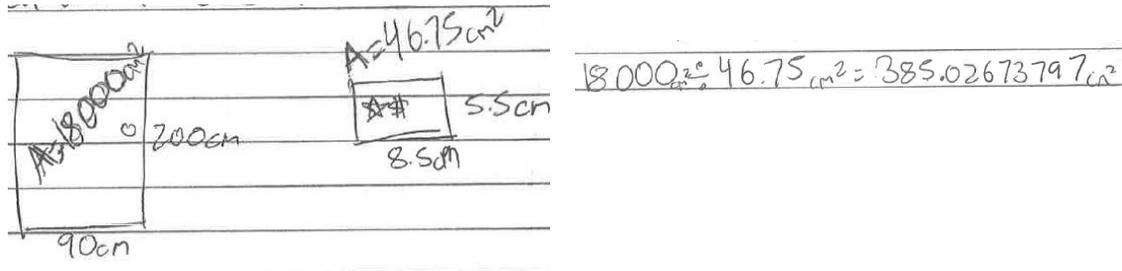


Figure 28. Group C-D’s calculations regarding the door and a gift card.

Carol rounded the quotient down to the nearest whole number and interpreted the resulting value, 385, as the number of gift cards required “to cover the entire door with gift cards”, to which Carmen pointed out that they “[didn’t] necessarily want to cover the entire door with gift cards that are right beside each other” because they wanted to “see the pretty paint”, and constructed a rectangle to visually represent the door.

Carol: So, I took this number, which is the door.

Carmen: The area of the door.

Carol: And then I divide it by the area of the gift card.

T: Okay, and what do you get?

Carol: That.

T: 385.02673797 Oh my. What’s that mean?

Carmen: It means...

Carol: It means...

Carmen: 385 gift cards. However, the thing is, if we don’t necessarily want to cover the entire door with gift cards that are right beside each other, you could leave spaces in between them then you can see the pretty paint.

T: Good point.

- Carmen: So let's take off some gift cards.
- Carol: And the other side of the door too.
- Carmen: We should construct a door (draws a rectangle to represent the door)
- Carol: It's an excellent idea.

The pair then worked on a design. Their conversation quickly moved from putting multiple gift cards on the door to putting one gift card on the door, and four Starbucks cups at the four corners of the door. The pair also suggested to reuse Starbucks cups to decorate the door (Figure 29).

- Carol: Do we want to keep spaces between the cards?
- Carmen: You can paint whatever you want. You can put one gift card right here (draws a small rectangle on the door).
- Carol: Okay, let's draw this properly.
- Carmen: One card, right in the middle, here, I can't draw, surrounded by Starbucks cups! [laughs] Actually, you take a Starbucks cup, flatten it, like using two thick books
- Carol: Or you can just cut it in half.
- Carmen: And then you take it, and stick it there, stick it there, stick it there... (slams her left hand on the four corners of the rectangle she constructed)
- Carol: That's a brilliant idea!
- T: Can you draw me a design of that, and tell me how much.
- Carol: It's like, free, for those Starbucks cups?
- T: The cups are free, okay?
- Carmen: Go through the Starbucks trash [laughs] reduce, reuse, recycle! Go be like, hey, um, I'm decorating my house with Starbucks cups, do you mind if I um, help save the planet and take some Starbucks cups from your garbage can so I can reuse them?

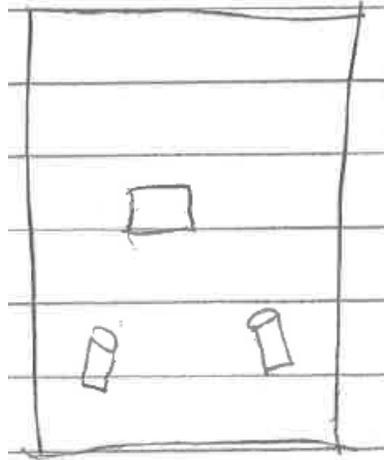


Figure 29. Group C-D's initial design (partially done).

After a few minutes' worth of work, Carmen and Carol developed some concrete ideas for their design on the front of the door. The pair maintained to put four Starbucks cups at the four corners of the door, and decided to draw a picture of me wearing a crown, sipping my latte while sitting on a Starbucks gold card.

Carol: We have a crown!

Carmen: Here's Ms. Liu sitting on a gift card, drinking her Starbucks cup, probably a sign that says "my office, do not enter".

Carol: And wearing a crown.

T: That's me!?

Carol: (laughs)

T: So... so I'm holding on to my Starbucks gift card

Carmen: And drinking coffee. Wait, [refers to drawing] you are not wearing clothes yet, Ms. Liu.

T: What!? I'm naked!?

Carmen: We haven't drawn it yet, you are this stick figure!

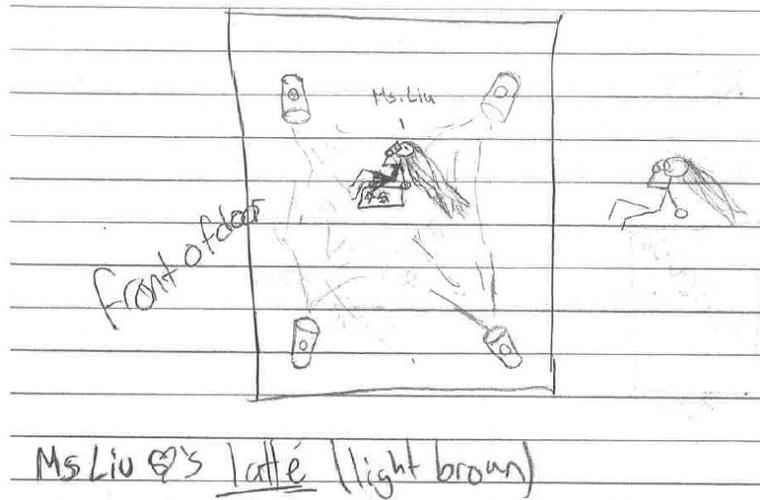


Figure 30. Group C-D's design of the front side of the office door.

Meanwhile, they also determined the colours they would need: some brown paint for the coffee and the hair, olive green paint for the background, white paint for the cups, and gold for everything else.

- ~~light~~ brown (coffee) & hair
- olive green (door)
- gold (everything and anything) - ms liu's dress
- white

Figure 31. Group C-D's list of colours required for their design

Shortly after, the pair created a design for the back side of the door: a big coffee cup with my face in place of the Starbuck's logo on the cup. They also labeled all the colours they needed to make the design work (Figure 32).

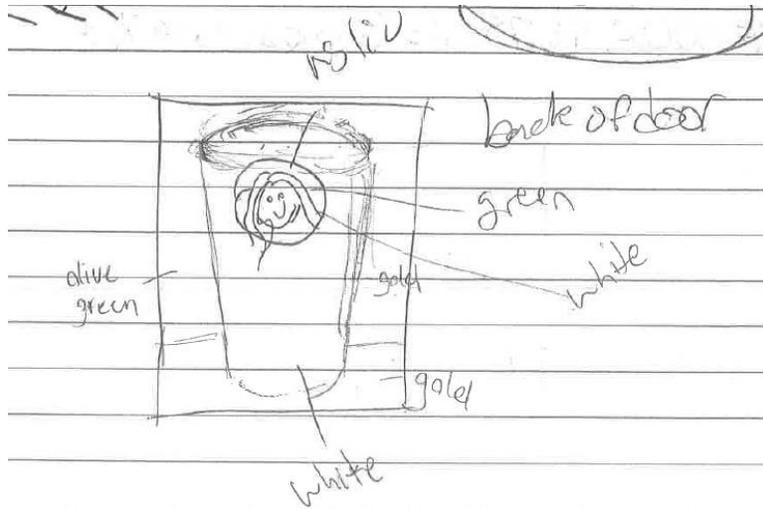


Figure 32. Group C-D's design for the back side of the door.

After creating both designs, Carmen and Carol discussed with each other where to get paint, and decided to take paint from arts class for home use. They argued that it was okay to do so since they were only taking a small amount of paint. When I rejected their idea, the pair used their smart phones to look for paint.

- Carmen: We'll just take some paint from the art room!
- T: You can't do that! You can take paint for home use! You gotta go buy them!
- Carmen: Just saying! We are not painting our garage!
- T: I know, but you can buy smaller cans of paint, right?
- Carol: No, those are expensive. There's this Chinese website...
- T: If you can find it on there that's fine, um, screenshot that and send it to me.
- Carol: Okay. Wait, can I go to, um, a website, it's like
- T: You can do whatever it is that you want, but find me the appropriate stuff that I need.
- Carol: Okay.

The pair focused on keeping the cost low. Carmen proposed to get paint from Michael's, a local arts and craft store, but Carol decided that Michael's was too expensive for the project and proposed to look at a website that she knew of. Eventually, they went on the Home Depot website again and found that "the Home Depot just lowered their prices on

paint”, and decided to purchase all their paint there: 5 cans of olive green paint that costs \$18.70 each, 1 can of brown paint that costs \$19.88, 3 small cans of gold paint that costs \$7.20 each, and 1 can of white paint that costs \$21.61. The cost of paint is \$156.59 without tax.

Carmen and Carol did not explain how they arrived at the number of cans they wanted for each colour, and they did not include any additional information in their work.

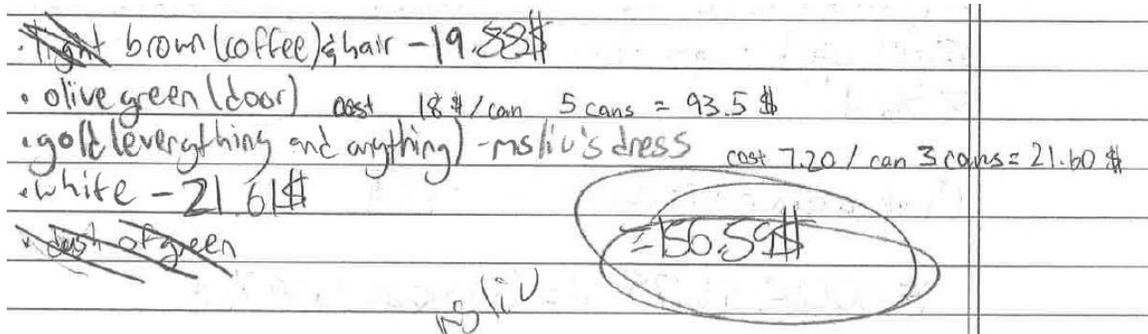


Figure 33. Group C-D’s budget for paint

Carmen and Carol stated that they needed 5 cans of olive green paint that costs \$18 per can. The calculated cost was \$93.5. However, 18×5 is only \$90 instead of the calculated \$93.5. It is unclear how the pair arrived at their calculated cost.

After Carmen and Carol determined the cost for paint, they decided they have completed the problem and found the solution. They circled their answer (see Figure 33) and submitted their work.

8.3.1. A discussion on group C-D’s modelling process

Carmen and Carol began their modelling process by carefully reading the instructions (RS1). They then paid attention to both gift cards and painting the door, and applied their EMK since the beginning of their modelling process. At this stage, they understood the problem as two-fold: paint the door and tile gift cards on the door (MRS2). As the pair built a plan to solve the problem (RM3), they looked into the various aspects of the problem: the dimensions of the classroom door and a gift card (RM3a), paint (RM3b), adhesives (RM3c), and free gift cards (RM3d). Similar to groups A-D and B-D, Carmen and Carol first considered the importance to know the measurements of the door and a gift card, and assumed that the office door has the same measurements as the

classroom door and took measurements of the classroom door and that of a gift card (RM3a). Meanwhile, they also recognized they needed to purchase paint and wanted to determine the “average cost” to paint the door (RM3b). They did not define “average cost”, nor did they explain how it would help them with their approach. It is possible that they were trying to get a feel of how much paint would cost them. It is also possible that they were hoping to use the average cost of paint, if such a thing exists, as the cost of paint for their project. Their work here demonstrates that the pair might have insufficient EMK regarding painting and had not reflected deeply on the problem, as they did not mention the relationship between the cost of paint, the volume of paint, and the type of paint they wanted for the project in their search. As such, they did not ask insightful questions and did not locate any useful information with regards to the average cost of paint from their google search.

Since they did not go very far with their work, the two backed away from paint, and generated a list of adhesives they could possibly use for the project (RM3c). Afterwards, the pair moved on to gift cards (RM3d). They brainstormed a few ways to get free gift cards and focused on keeping the cost low, planned to tile the gift cards on the face of the door, and built a mathematical model (MM4) accordingly: they divided the area of the face of the door by that of a gift card, and rounded the result (MR5) down to the nearest whole number. They interpreted this as their real result (RR6), to which Carmen was immediately dissatisfied with and rejected their approach to tile the door with gift cards. In this scenario, Carmen validated their real result and applied EMK to recognize the possibility to improve the real result. Although Carmen did not explain clearly the reason(s) behind her dissatisfaction with the real result, it is possible that Carmen had trouble grappling with the number of gift cards involved in this solution and recognized that 385 cards may not be a reasonable number of gift cards used to decorate a door. Carmen’s comment provided the pair with a realistic perspective to the *Door project* (MRS7), and led them to rebuild their design.

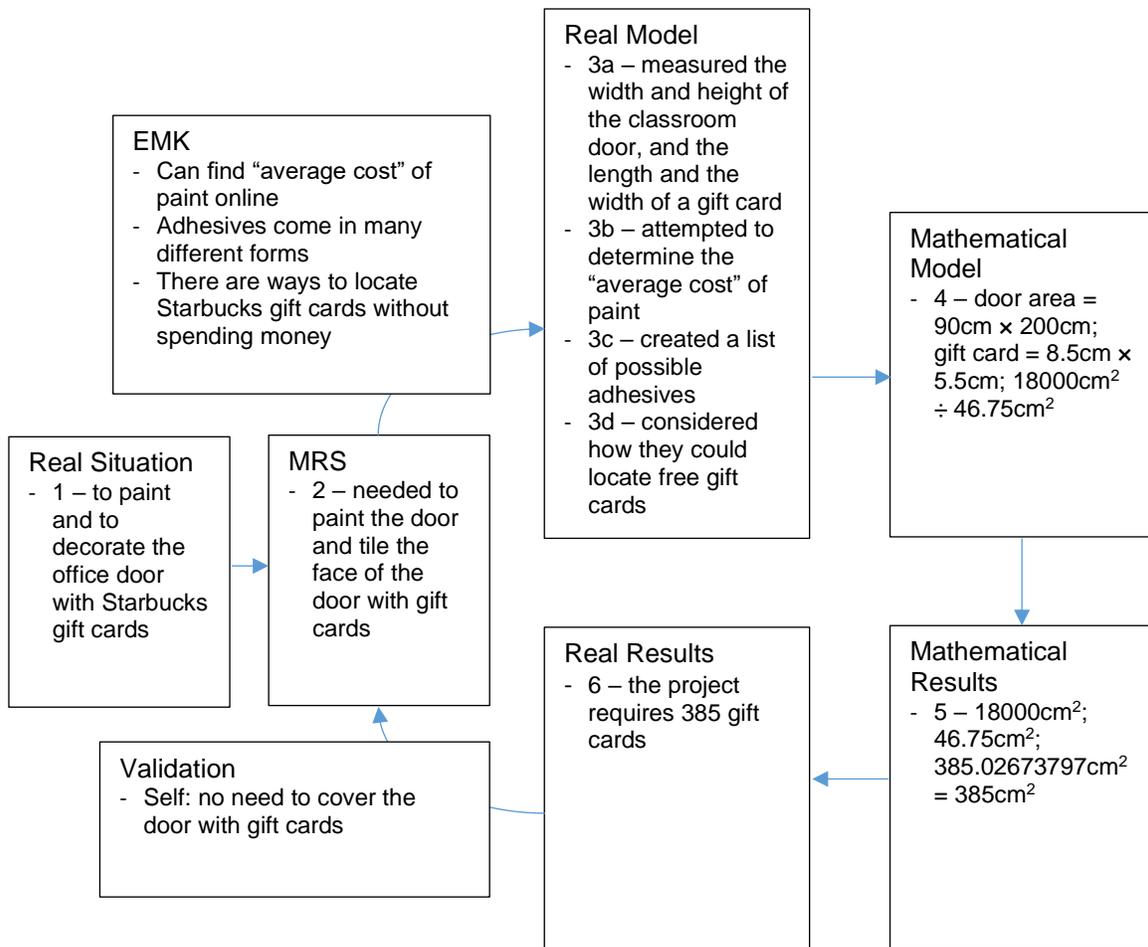


Figure 34. Group C-D’s first modelling cycle (gift cards)

Their new designs (front and back) require only 1 gift card and 4 different colours of paint (RM8). Aiming to keep the cost of the project low, Carmen and Carol first recognized and argued for the possibility to get paint from their art class, and later on were convinced that they needed to purchase paint. They made assumptions about the amount of paint they needed and used the Home Depot website as their source to determine the cost for each can of paint they needed and totaled the cost (MM9, MR10). They did not clearly indicate the volume of paint required, but seemed to understand the general relationship between the volume of paint and the cost of paint. They interpreted their solution as the cost for the *Door Project* (RR11).

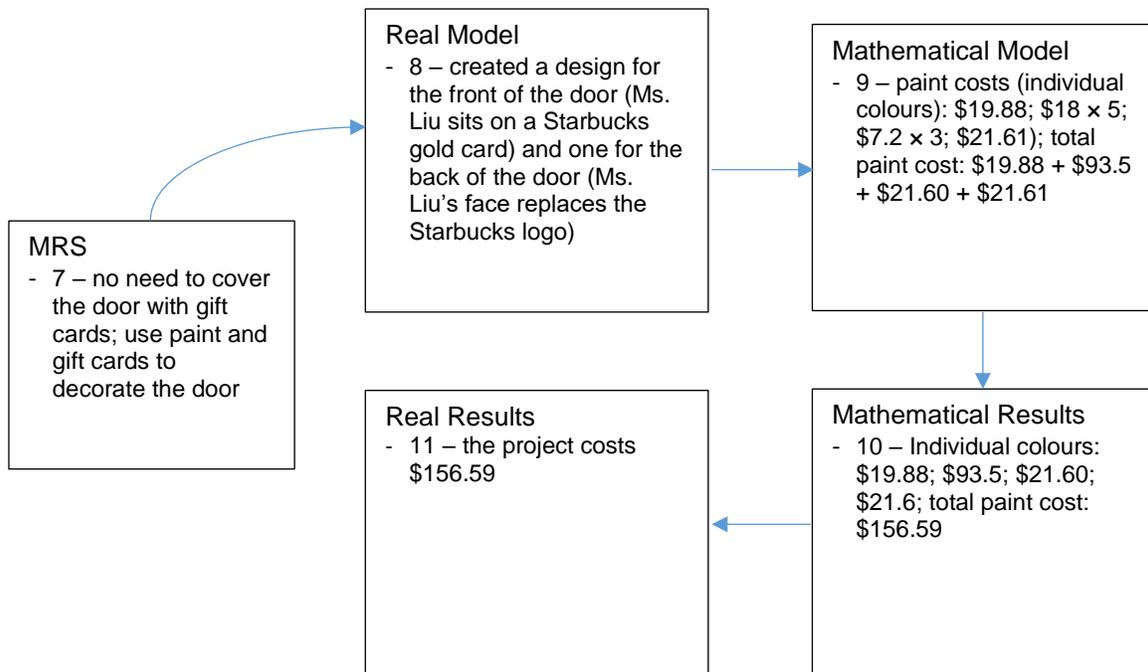


Figure 35. Group C-D’s second modelling cycle (gift cards and paint)

The following is a diagrammatic representation of Carmen and Carol’s modelling process. The red arrows (MC1, steps 1-6) represent their work on tiling the door with gift cards; the blue arrows (MC2, steps 7-11) represent their work on creating a design.

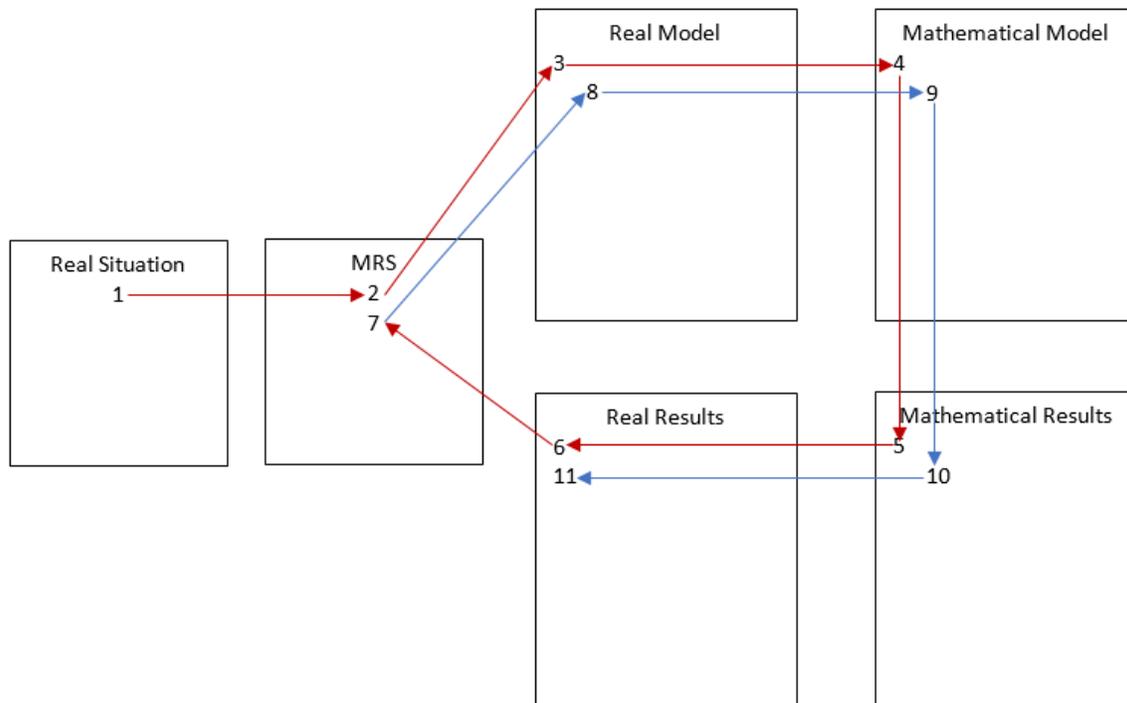


Figure 36. A diagrammatic representation of group C-D's modelling process.

Similar to Group B-D, Carmen and Carol's submitted solution neglected painting tools, adhesives, and sales tax, and did not state the specific volume of paint required. However, Carmen and Carol's decorative solution included a mural on the front and the back of the office door. As their solution requires only one gift card, they might possibly have assumed they could get a gift card from their friends or walk into a Starbucks store and get a free gift card, and therefore did not include the cost of the gift card in their solution. It is also possible that they forgot about their gift card in the end along with the different types of adhesives they considered at the beginning of their modelling process. Also, the pair recognized they needed adhesives to secure gift cards on the door but not painting tools for painting purposes. It is unclear why they recognized the need of one type of tools but not the other. It is almost as if they applied or activated their EMK at the beginning of their modelling process to consider adhesives and ways to locate free gift cards, but were carried away as they considered the rest of the problem and neglected what they took into considerations at the beginning of the modelling process.

Carmen and Carol were the only ones who created an elaborate design for the *Door Project* and incorporated a gift card into a mural. Their work focuses heavily on decorations, or the visual aspect of the task. Their work further forced them to focus on

the number of different colours of paint they needed. Their solution seems to have the correct proportion of paint – assuming each can of paint reported has a similar volume, they included more olive green paint (background of the door) than gold paint (for “everything and anything”), white paint (for the Starbucks cup), and brown paint (for hair and the latte).

8.4. Jasmine, Jessica, John, and Joseph (Group J-D)

As they read the problem, Jasmine, Jessica, John, and Joseph recognized that they needed both paint and gift cards for the project, and immediately asked me for the possibility to get free paint from the art teacher.

- Jasmine: Are you repainting it?
- T: Yes.
- Jessica: You gonna buy paint?
- T: Probably. What do you have in mind?
- Jessica: Get paint from art?
- T: Get paint from art class for my home use? I think [the art teacher] won't be too happy.

When their idea was rejected, the group moved on to discuss the cost of gift cards. They aimed to keep the cost of the gift cards low and believed that the gift cards were free if they contained no money. They brainstormed creative ways to locate these free cards. These plans include: “walk to a Starbucks store”, “grab a bunch of cards” and “run out of there”; or go to the store and “just ask them for it”.

- Joseph: There's no money.
- John: Yea, there's no money on this.
- Jessica: We walk to the store.
- John: Grab a bunch of cards. Yea, and then run out of there, and then we go... And it's the cheapest...
- Jasmine: Can we just ask them for it?
- John: Oh yea, yea, yea, they probably have them in store or something.

Jasmine: Yea! They probably give it to us.

After hearing their plans, I confronted the group about not paying for the cards and rejected these ideas. Instead of making changes, the four students laughed it off and ignored my comments.

Without resolving any of the tensions regarding free gift cards, the four students moved on and discussed with each other the surfaces of the door they needed to paint and asked me additional questions to painting. They questioned whether they needed to paint all six surfaces of the door, or only the front and back sides of the door. Instead of answering their questions, I walked away²³.

Jasmine: And then we'll cover it with it, and we are trying to find out paint.

T: How are you covering it?

Joseph: Do we need to cover all of it?

Jessica: Do we need to paint it?

John: Both sides?

Jessica: Well how do you wanna cover it?

Jasmine: We are gonna paint the [front and back] sides, are we gonna paint the bottom?

Jessica: Paint the door, do we need to paint the whole thing?

T: Interesting questions... (walks away)

The four students asked many clarification questions about painting the office door. As they further discussed the *Door Project* with each other, they also considered the possible assumptions they could make about the office door and asked additional questions, such as whether the office door is a sliding door, and whether the door is a solid door or a door with a window just like the classroom door. At this point, they

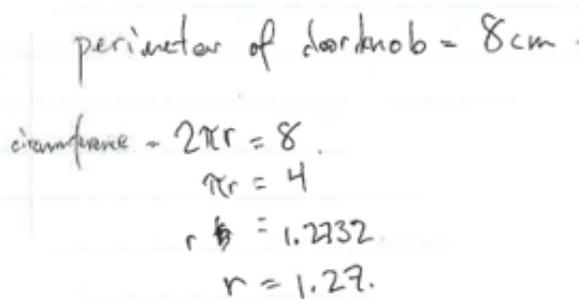
²³ It is very customary for me to walk away from students instead of providing them with a direct answer as they work on these tasks. Students understand that rather than refusing to answer their questions, I want them to try to answer their own questions by reflecting on the questions they asked, doing some online research, and making reasonable assumptions based on their reflections and/or their research.

focused on exploring all possibilities but did not explain how these assumptions might play a role in their approach or their solution.

- Jasmine: Can we have like, one of those sliding door?
- T: I don't have a sliding door! I have an actual door, with a door knob.
- John: Is there a window?
- Jasmine: There's a window on the door! So there is a window!
- T: No, no window. Just a standard door.

So far, Jasmine, Jessica, John, and Joseph spent time to explore the assumptions they could make about the office door. After they asked questions and gathered information, the four students assumed that the office door has the same dimensions as the classroom door, and measured the dimensions of each side of the door and determined the areas, including the face of the door (width \times height; which they refer to as the front and the back of the door), the top edge (width \times thickness) and the side edge (length \times thickness, the side with the locking mechanism, not the one with the hinges) of the door.

Jasmine, Jessica, John, and Joseph also assumed the door knob on the office door is different from that of the classroom (see Figure 9). Their drawing shows that the door knob does not have a metal plate. When asked, Jessica claimed that "nobody has that door knob at home. School rooms are different than home rooms". The group interpreted the door knob as a two dimensional circular object and determined its area by first measuring the circumference (they referred to it as the perimeter). Using this measurement, they calculated the radius of the door knob:



perimeter of doorknob = 8cm.

$$\text{circumference} = 2\pi r = 8$$
$$\pi r = 4$$
$$r = 1.2732$$
$$r = 1.27$$

Figure 37. Group J-D's work on the radius of the door knob.

Afterwards, they calculated the area of the door knob using the calculated radius (10.13, no units reported) and the area of the “side slof of door” (66cm):

The image shows two pieces of handwritten work on lined paper. The left piece is a calculation for the area of a circle: $area = \pi r^2 = \pi (1.29)^2 = \pi (1.6129) = 5.067 \approx 5.07 \times 2 = 10.13$. The right piece is a calculation for the area of a rectangle: $side\ slof\ of\ door = 2 \times 33 = 66\ cm$.

Figure 38. Group J-D’s work on area of the door knob and the locking mechanism.

They then totaled the areas of these four sides (the front, the back, the top edge, and the side edge with the lock) and subtracted the area of two door knobs (10.13) and the lock on the side edge (60, which they refer to as the “side slof of door”) from the sum, and determined the surface area that required painting to be 37603.87 (see Figure 39). Throughout this process, the four students made a few minor mistakes: some of the units were wrong or missing (such as the area of the “side slof of door”, the area of the door knob, the total area of the door, etc.), and they changed the area of the “side slof of door” from “66cm” to “60”. These mistakes did not contribute to their final solution.

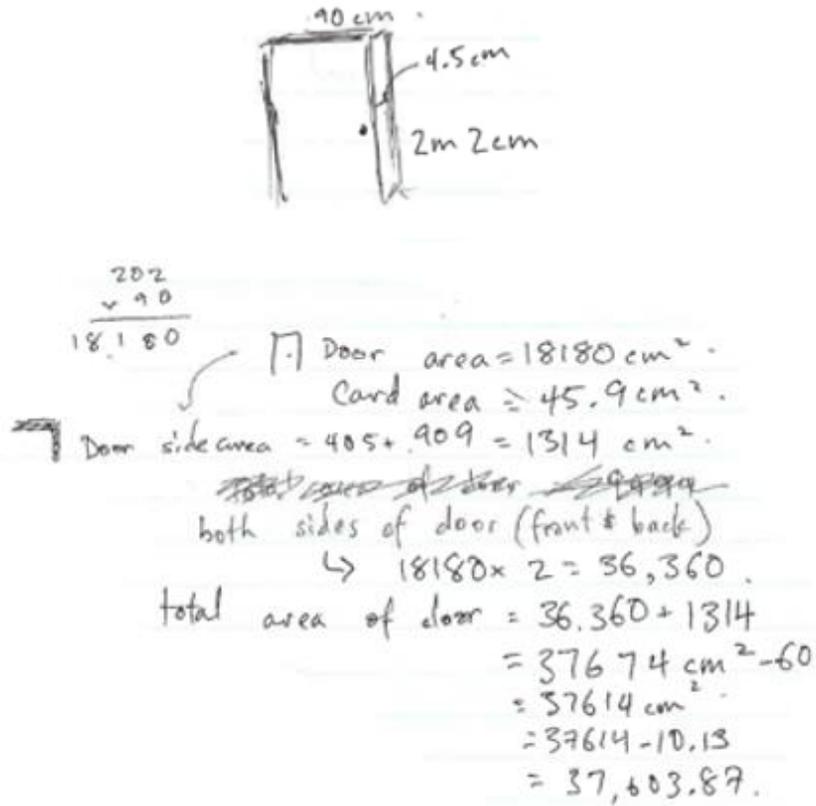


Figure 39. Group J-D's calculations on the area of the door to be painted.

After they determined the area to be painted, they went on the Home Depot website and decided that they needed 3.78L of white paint, which costed \$36.97 before tax, and \$41.41 after tax. The group did not provide any details on how they came to this decision.

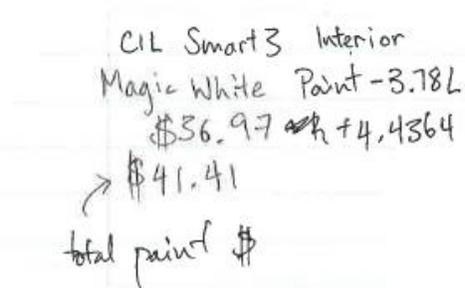


Figure 40. Group J-D's work on paint.

When they settled on paint, they moved on and discussed with each other how they could decorate the office door using gift cards. Eventually, the group decided to put one gift card on the front and one on the back side of the door (see Figure 41).

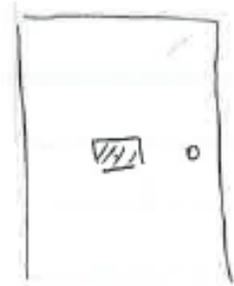


Figure 41. Group J-D's first design

After settling on their first design, Jasmine, Jessica, John, and Joseph calculated the cost for the project based on the paint they chose.

Jasmine: It costs that much.

T: Forty-one forty-one, is that with tax?

Jasmine: Yea.

T: Forty-one forty-one, and then, magic, magic white?

Jessica: Uhuh.

Jasmine: Yeah.

T: Nice. And then what? Door area, card area, and then... what about the cards?

Jessica: It's free.

Jasmine: It's free. So we are gonna just put one, on one side, and on the other side, we are just gonna put like

Jessica: Doesn't matter how many there are.

In this conversation, the group argued that it didn't matter how many gift cards they planned to use because the cards were *free*. The group was stuck on using *free* gift cards for the project no matter how much I protested. Jasmine, Jessica, John, and

Joseph spent most of the time during the conversation arguing with me over the *free* gift cards and refused to make changes to their approach and the cost of the project.

After a short but intense discussion, the four students became dissatisfied with their 1-card design. They modified their design on the back side of the door to include more cards (see Figure 42). This new design includes two rows of 10 cards at the top and the bottom of the door, and one card in the centre. As for the cost, Jasmine, Jessica, John, and Joseph insisted that the cards were free and the project would only cost them forty-one dollars and forty-one cents, which is the cost of paint.

- T: So you have ten cards going across?
- Jasmine: Yes.
- T: And then... one in the middle?
- Jasmine: And ten cards at the bottom.
- T: And ten cards at the bottom.
- Jasmine: And on the front there's one card.
- T: And on the front, hey, that's cool. And then... so, twenty-two cards?
- Jasmine: And paint costs forty-one forty-one.
- T: Forty-one forty-one? So how much does the whole thing cost me?
- Jasmine: Forty-one forty-one.
- Jessica: Coz the cards are free.
- T: No? How are they free?
- Joseph: They are free?
- Jessica: They are free?
- Jasmine: If you don't like, put money in them, they are free.
- Jessica: Yeah.
- T: So what, just grab them?
- Joseph: Yeah.

Jessica: Grab a lot.
 Jasmine: Yea. You can ask them.
 T: Ask them for it?
 Jasmine: Yeah.

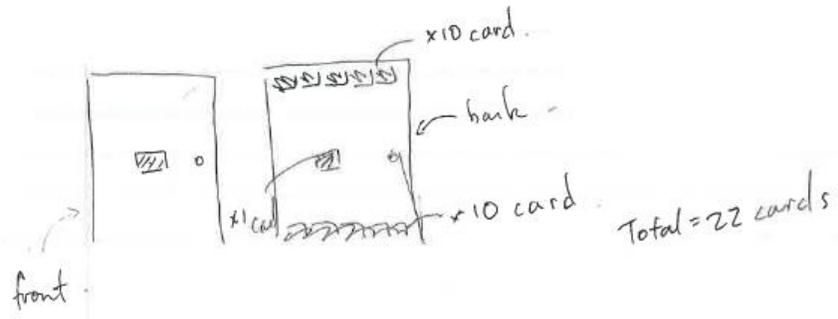


Figure 42. Group J-D's modified design

The group was adamant about *free* gift cards. As the conversation unfolded, Jasmine, Jessica, John, and Joseph also revealed during their conversation that they recognized they needed painting tools, but assumed that I already owned paint brushes. When told otherwise, the group went around the problem by lending me their paint brushes. When I later on asked about adhesives, Jessica was taken by surprise that I did not have glue at home and that they needed to include glue in their budget. Instead of purchasing glue, the four of them decided that I could borrow a glue gun from Joseph. They also added a foam roller to the list, but quickly switched back to paint brushes after. It is possible that they switched from a foam roller to paint brushes because the group members did not own any foam rollers. They insisted that I did not need to spend money on any of these tools as I could have Jessica's paint brushes and borrow Joseph's glue gun. These students were very focused on finding the cheapest way possible to complete the project.

T: Okay? And then paint? Do I need anything else?
 Jasmine: No. You have paint brushes.
 T: I don't have paint brushes.
 Jessica: Okay, never mind. We need to buy paint brushes.
 T: Okay, and how am I gonna get the gift cards on?

Joseph: You glue them on.

Jessica: You have glue!

T: No, I don't.

Jessica: You have a glue gun?

T: No I don't.

Jasmine: You can borrow mine!

T: I can borrow yours!?

Jessica: So it's free!

Jasmine: And borrow paint brushes, like the, the, the roly stuff.

Jessica: Borrow her roly stuff.

T: The roly stuff.

Jessica: So it's free.

T: Okay, I can do that?

Jessica: Yeah!

T: Okay, can you give me a list, of all the things?

Jessica: Okay... [start writing and creating a list]

T: Do I have to return them (the paint brushes)?

Jessica: You don't have to return them.

T: That means I can have your paint brushes?

Jessica: You can have my brushes.

T: Okay.

Jessica: Free paint brushes.

By now, the group believed that they had more or less finished the problem. As a final attempt to satisfy my complaint about the *free* gift cards and to justify their act of taking the gift cards from the counter, the group suggested for me to take a gift card when I purchase a one dollar drink from Starbucks. They later on discarded the idea and maintained that gift cards were free.

- T: Why are the cards one dollar each?
- Group: [giggle]
- T: No seriously, why are they one dollar each?
- Jessica: When you buy a drink, just grab them.
- T: Huh?
- Jessica: If you want a drink, just grab one on the door.

In the end, the group submitted a proposal for the door that costs \$41.41 which includes a can of paint (\$36.97 plus tax), free paint brushes from Jessica, a free hot glue gun from Joseph, 22 free gift cards from Starbucks, and a design of the door (Figure 43).

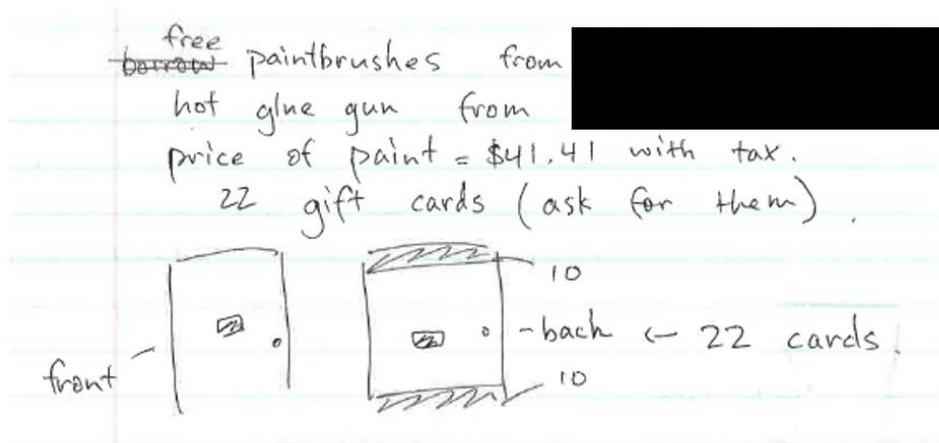


Figure 43. Group J-D's final solution

8.4.1. A discussion on group J-D's modelling process

After reading the instructions (RS1), Jasmine, Jessica, John, and Joseph quickly interpreted the cost of the project as a priority and were extremely focused on keeping the cost of the project as low as possible. They understood the goal of the question as to decorate rather than to cover the door using gift cards, and never aimed to cover any sides of the door with gift cards. They also understood the primary tools for the project included paint and gift cards. These ideas allowed their first MRS to take shape (MRS2). Based on their understanding and their interpretation of the real situation, it could be said that these students activated their EMK very early on.

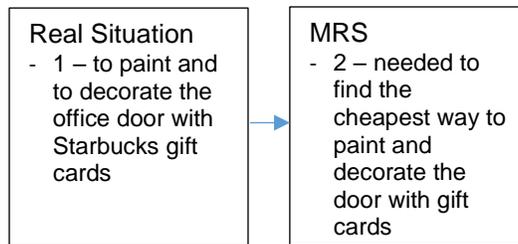


Figure 44. Group J-D’s initial understanding of the door problem

Using their MRS as their guide, Jasmine, Jessica, John, and Joseph began to build their real model (RM3). They first looked into the possibility of getting free paint and free gift cards.

Their plans may be based on two motives: to find an easy way out or to keep the cost low. If Jasmine, Jessica, John, and Joseph could find a way to argue for free paint (and free gift cards, and later on painting tools and adhesives), then it didn’t matter how much paint they needed for the project. In other words, they could claim that the project didn’t cost anything, as all the things they needed were free, and they could be done with the problem quickly. The assumption that everything they needed was free could serve as an easy way out for these students. However, based on their overall work, including the designs they created and changes they made to their designs, it is more likely that they were looking for the cheapest solution rather than the easiest solution at this point.

Their aim to lower the cost as much as possible led the group to focus on locating free gift cards instead of creating a design and figuring out the number of gift cards required for the project. Their work suggests that they have a fair understanding about the gift cards at Starbucks, including where these cards are located. They applied this understanding, or their EMK, to devise an approach and to find a solution for the *Door Project*. They also seemed to understand that their plans to get gift cards might not be appropriate or ethical: John said he would have to “run out of there” after they “[grabbed] a bunch of cards”; Jasmine suggested for the group to “ask [the baristas] for it” instead to justify for these free gift cards; and the group tried to laugh it off when confronted and failed to justify their actions. Unfortunately, their priority to keep the cost down was higher than their ethics. They might also believe that since the problem is one that is given in class, it is not necessarily a real life situation, and therefore they were not responsible for their actions. At this point, I believe they have shifted from keeping the

cost low to avoiding to reflect deeply on the situation, as they might have realized that putting loaded gift cards on the door is not necessarily sensible and avoiding to pay for these cards by assuming they could score free cards was their easiest way out of the sticky situation.

Without resolving any of these tensions, the four students looked into paint, but decided to take a closer look at the office door as door designs may vary (MRS4). They asked many clarification questions and gathered additional information about the office door prior to making assumptions about the door and deciding on an approach. Eventually, they decided to paint the front, the back, the top side, and the long side with the lock, and to leave the bottom side and the long side attached to the hinges untouched. When asked about their decision, they claimed that if “you can’t see them then it doesn’t matter”. This claim sheds light on their assumption about the door: other than the two faces of the door, they could see the long edge close to the door knob but not the edge attached to the hinges; and that they could also see the top edge of the door but not the bottom edge. These assumptions seem unrealistic – one should be able to see both long edges (one side with the lock and the side with the hinges), but probably not the top side (since a door is at least 2m tall) and the bottom side. It is unclear on what basis their assumptions were made.

Other than the sides to be painted, the group also assumed that the office door has the exact same dimension as the classroom door. Similar to all the other students, the dimensions of the office door acted as a hurdle and this particular assumption allowed all students to move forward. The assumption could have roots in students’ insufficient EMK about interior doors, or the availability of the classroom door for measurement purposes. All these ideas and assumptions – free gift cards, paint, and the surfaces to be painted – added to their real model (RM5). Based on their decision about the surfaces of the door to be painted and their assumption about the dimensions, they built a mathematical model (MM6) and determined the total area to be painted (MR7). Afterwards, they chose a can of paint from the Home Depot website and mathematically determined the cost of the paint including sales tax (MM8 and MR9).

Their work here suggests that their EMK fell short: their choice of paint does not fit the project very well. First, the paint they chose, a can of CIL Smart 3 Interior Paint, is listed as a ceiling paint. While it is entirely possible to use this to paint a door, their decision

seemed to suggest that they did not look closely at their choices, that they made hasty decisions, or that they were not very knowledgeable about paint.

CIL Smart3 Interior Magic White Ceiling Paint - 3.78L

★★★★☆ (11) | Write a Review | Questions & Answers (0)



Figure 45. Group J-D's choice of paint

Also, they calculated the area to be painted in detail, but did not apply this information to choose a reasonable amount of paint. Their submitted work does not suggest any relation between the calculated area of the door and the volume of paint they chose, as the determined volume far exceeds the required volume for the job.

The amount of paint suggested by this group is sufficient to cover approximately an area of 350 to 400 square feet (single coat)²⁴. Given the area to be painted as 37603.87cm² (see Figure 39, based on the group's detailed calculations), or roughly 40ft², they over-estimated the amount of paint required. With the group's priority as keeping the cost of everything they needed in the project as low as possible, their choice of paint seemed to go against this priority, as a smaller can of paint would suffice and it would cost them a lot less than their submitted solution. Therefore, it is likely that Jasmine, Jessica, John, and Joseph lacked experiences in painting and did not acquire the appropriate EMK to help them make well-informed decisions.

Jasmine, Jessica, John, and Joseph interpreted their mathematical result (MR9) as the cost for the project (RR10), but not the final solution, as they understood they had not yet created a design. Instead of validating their work, they went straight ahead to

²⁴ The area of coverage is determined via: <http://www.cil.ca/Tools/PaintCalculator> on July 7, 2016.

decorate the door with gift cards. Eventually they settled on putting one gift card on the front and one on the back side of the door (RM11). During their work, they never interpreted the problem as to tile any sides of the door with gift cards. Afterwards, they included 2 gift cards in their solution (RR12), and validated their work with me, which resulted in a discussion on the cost of gift cards. Jasmine, Jessica, John, Joseph, and I were not able to resolve our tensions and our differences.

Soon after, these four students added 20 cards to the back side of the door (RM13), and concluded that the project costs \$41.41 (RR14), which includes a can of paint (\$41.41) and 22 gift cards (free).

During the validation process, Jasmine and Jessica revealed their assumption about my access to painting tools and adhesives. As their assumption got rejected (MC2, MRS15), the group made modifications to their real model (RM16). They added paint brushes and a glue gun, but decided that I could borrow these tools from them so that they could avoid any additional cost to the project. It is unclear whether the group members actually own all these tools or whether they made these up. Their assumptions and decisions led to no changes in the mathematical model and mathematical result. They determined that the project costs \$41.41 (RR17), and this concluded their modelling process.

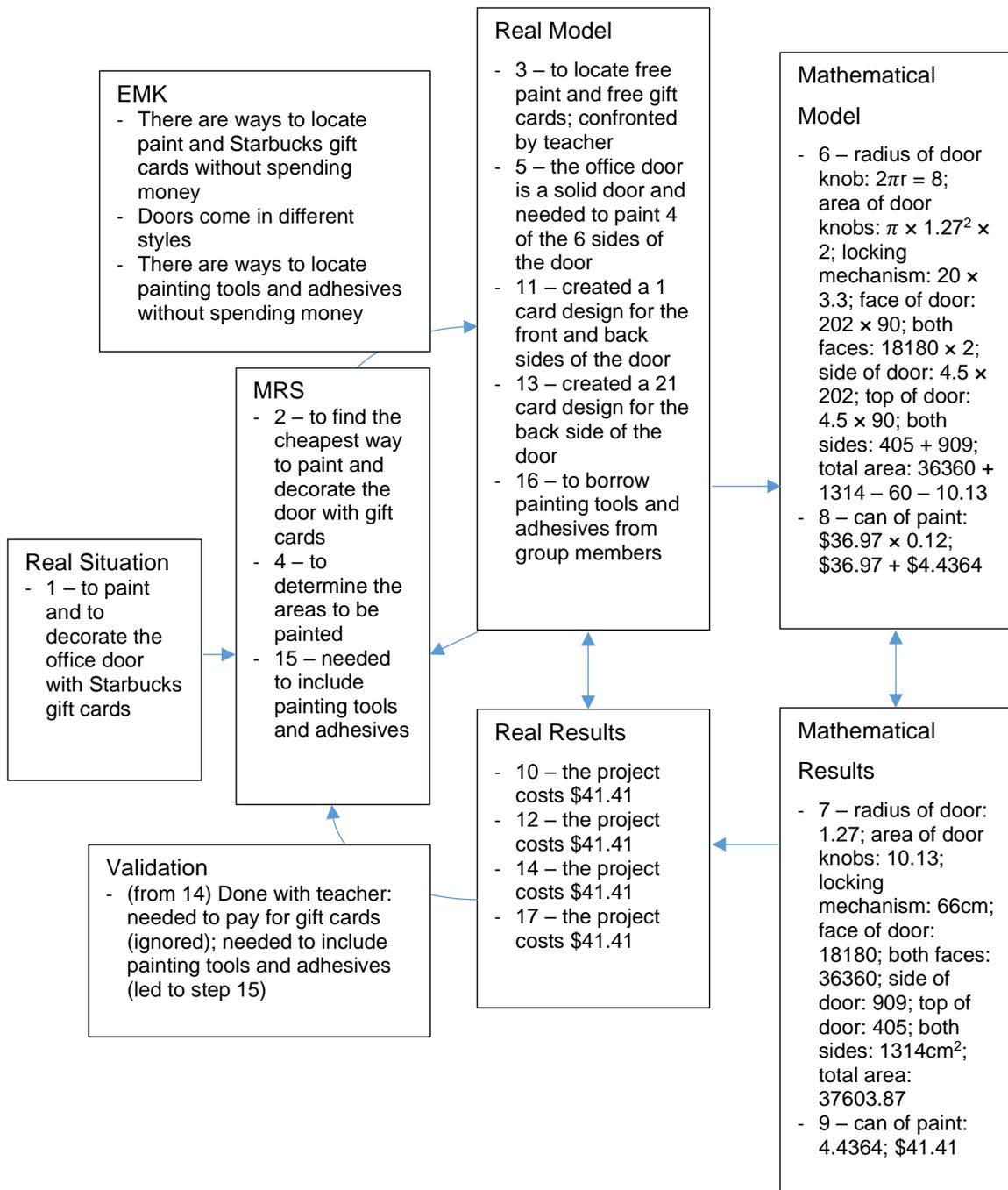


Figure 46. Group J-D’s modelling process

There is something peculiar about the approach these four students took. Their decision to keep the cost down could be interpreted as a realistic approach as cost is most often a realistic concern in projects. However, they also made unrealistic assumptions along the way to achieve this goal, including their assumptions about gift cards, their initial

assumption about paint, and possibly their assumptions about painting tools and adhesives.

It is unclear why students refused to pay for the two gift cards in their initial design, which might cost them as little as \$10. It is possible that they were so focused on finding the cheapest possible solution they were willing forgo their ethics and bend rules. It is also possible that they recognized the problem was given in a mathematics class, and therefore they were not responsible for the real life choices they made during the process of creating a solution for the problem.

Overall, Jessica, Jasmine, John, and Joseph engaged with the problem at a somewhat realistic perspective since the beginning of their modelling process. While their priority of keeping the cost low led them to look for creative ways to locate free tools such as paint brushes and hot glue gun, it also led them to forgo their ethics and to bend rules. The following is a diagrammatic representation of Jessica, Jasmine, John, and Joseph's overall modelling process.

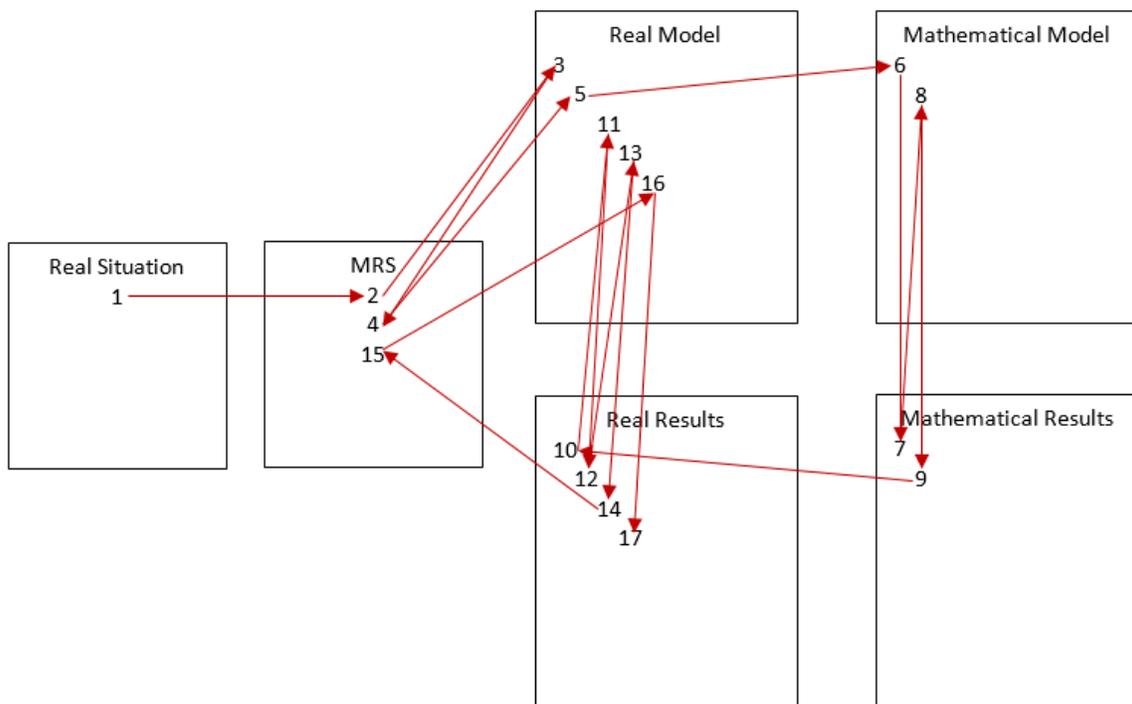


Figure 47. A diagrammatic representation of group J-D's modelling process.

Chapter 9.

An analysis of the results of the door project

In the previous chapter, I provide the data and a discussion on four groups of students' modelling processes on the *Door Project*. These students applied different EMK and took different actions to combat for their insufficient EMK during the *Door Project*.

EMK is required throughout students' modelling processes in the *Door Project*. This knowledge influenced students' assumptions regarding the office door, their decorative designs, and the inclusion of additional tools required for the *Door Project*. In this chapter, I provide readers with a discussion and an analysis on students' EMK activation with a focus on their decorative designs, on students' insufficient EMK with regards to paint and on the office door, and on students' refusal to apply EMK during their modelling processes.

9.1. Students' decorative designs and the activation of EMK

Three of the four groups of students (A-D, B-D, and C-D) interpreted the *Door Project* as a mathematical word problem when they worked on a decorative design²⁵. As discussed, when dealing with word problems, students choose a basic arithmetic operation based on the key words in the problem, extract numerical values from the question, and apply the arithmetic operation to these numerical values to determine a solution (Greer, 1997). These students picked up phrases such as "office door", "Starbucks gift cards", and "how much will this project cost me". The phrases they picked up directed or led them to believe that they had to tile at least one side of the office door with gift cards and to determine the cost of the gift cards. As such, they measured the classroom door and a gift card (to extract numerical values), picked an arithmetic operation (to divide the two areas, to divide the width and height of the door by an edge of the gift card, and/or to multiply the number of cards by the cost per card),

²⁵ Group C-D recognized additional tools such as adhesives at the very beginning of their modelling process (RM3c, see Figure 34). However, they did not apply these realistic considerations to their decorative design until after they became dissatisfied with their real results. Their aim to tile the door with gift cards shows that they were interpreting the problem from a mathematical perspective.

and determined a solution based on their chosen operations. At this stage, students considered the *Door Project* with minimal realistic perspective. Their mathematical approaches resulted in some unrealistic and incomplete solutions that requires hundreds of gift cards and cost thousands of dollars. Most of these students were confident about their solutions and that they were interpreting the question correctly. When confronted, for example, Heidi and Helen²⁶ were shocked to find that their answer was not necessarily the best solution and had room for improvement. They argued that their work would be considered correct if the question was to be interpreted in a mathematical manner.

- Helen: You get, ten dollars gift cards right? That's three thousand six hundred and eighty dollars.
- T: Three thousand six hundred and eighty dollars!? ... That, my dear, is too expensive.
- Heidi: Wah!?
- Helen: If you view this as math, you'll be like, oh, you passed math this term.

Students' reactions in this conversation were likely the result of their interpretation and expectation of the *Door Project*. As they interpreted the *Door Project* from a mathematical perspective and solved the problem accordingly, their solutions were mathematically founded. As such, they expected to have produced a *correct* solution as long as they picked the right arithmetic operation and carried out their calculations correctly. However, the *Door Project* was more than a mathematical word problem. This is not to say that tiling one side of the door with gift cards is not a possible solution. However, their mathematical result had room for improvement – to include a realistic perspective in their approach and to grapple with reason. For example, if they were to tile the door with gift cards from edge to edge, could they close the door completely thereafter?

Students' mathematical interpretation of the *Door Project* might have been the result of the problem being given in a mathematics class. Students seemed to be confined by socio-mathematical norms which resulted in specific expectations and behaviors (Cobb and Yackel, 1998), including how problems should be solved in a mathematics class

²⁶ Heidi and Helen's modelling process is not discussed in this thesis.

(Leder, Pehkonen, and Törner, 2003). These beliefs might have led students to treat the *Door Project* as a mathematical word problem or from a mathematical perspective. It is possible that students might have approached the *Door Project* completely differently if the problem was given in, for example, an art class.

Two of these three groups of students (A-D and C-D) eventually recognized the realistic aspects of the *Door Project* as they validated their decorative designs. These students also activated and applied their EMK and progressed from tiling one side of the door with gift cards to creating a decorative design. Students' activation of EMK is different from that suggested by literature. Instead of activating their EMK towards the beginning of their modelling cycles, these students activated their EMK as they validated their real result. As for group B-D, they refused to modify their approach even after they recognize the possibility of a decorative design. I provide a discussion on their refusal to apply EMK in Chapter 12.

Group J-D was the only group who never tiled any side of the door with gift cards in their modelling process. It could be said that the group likely situated the *Door Project* in reality and activated their EMK early on in their modelling process. However, they still measured and calculated the area of the door to be painted and the area of a gift card. Group J-D was possibly affected by peer pressure – they felt the need to take these measurements since the rest of the class was doing it. The following table (Table 2) summarizes students' designs, including the number of cards and the cost of these cards in each design, and students' EMK activation.

Table 2. The number and the cost of cards in students' designs

Groups	Design #1; # of cards involved; cost per card; total cost of cards	Design #2; # of cards involved; cost per card; total cost of cards	EMK activation
A-D	EMK not activated: tile the front side of the door with cards; 408 cards; \$5 per card; \$2040	EMK activated: Letter "C"; 28 cards; \$5 per card; \$140	EMK activated during the validation process of design #1
B-D	EMK not activated: tile both the front and the back sides of the door with cards; 1090 cards; \$5 per card; \$5450		EMK activated when a classmate explicitly explained to the group what it means to decorate with gift cards after design #1, group B-D refused to make changes to their design
C-D	EMK not activated: tile the front side of the door with cards; 385 cards; \$0 per card (ask friends for used cards, take blank cards from the counter); \$0	EMK activated: Front: incorporated a gift card into their mural; Back: no gift card required in their mural; 1 card in total; \$0 (gift cards were not mentioned in the final solution, likely to ask friends for a used card or take a blank card from the counter at Starbucks)	EMK activated at the very beginning of their modelling process (adhesives) and then during the validation process of design #1
J-D	EMK activated: Front: 1 card in the centre; Back: 1 card in the centre; 2 cards total; \$0 per card; (take blank cards from the counter or ask baristas for blank cards); \$0	EMK activated: Front: 1 card in the centre; Back: 10 cards across the top and the bottom and 1 card at the centre; 22 cards total; \$0 per card (take blank cards from the counter or ask baristas for blank cards); \$0	EMK activated early on in their modelling process

A similar situation where validation led to the activation of additional EMK is found in some students' inclusion of painting tools and adhesives. When groups A-D and J-D validated their work, they were reminded of painting tools and adhesives. In these scenarios, students activated additional EMK for the *Door Project* and these EMK contributed to an improvement of their real result.

As shown in these scenarios surrounding decorative designs and additional tools, there is a difference in the quality of the real result before and after EMK was activated. Prior to EMK activation, students approached the RMCTs mostly from a mathematical perspective and their solutions contained minimal realistic considerations. Therefore,

while their solutions might be mathematically correct, these solutions were not realistic. It was not until students recognized the realistic aspects of the RMCTs then they activated and applied their EMK to produce a realistic solution.

9.2. Students' insufficient EMK and their assumptions

All students have some experiences with painting but might not know much about paint. Paint comes in many different colour, types (such as interior, exterior, acrylic, oil-based, etc.), textures (flat, eggshell, high gloss, etc.), forms (such as cans and spray cans), and can be purchased in various volume (0.5L, 1L, 4L, etc.). When dealing with a paint job such as the *Door Project*, one would make sure s/he purchase *enough* paint for the job, where *enough* could mean the exact quantity required, or more than what is required. While it is possible to purchase a lot more than what is required, doing so may not be an economically wise decision, unless one is planning on using the paint for other projects.

In general, students included some information regarding paint in their submitted solution but none of them provided a detailed plan regarding the paint to be used in the *Door Project* (see Table 3). For example, only groups A-D and B-D indicated where they were to purchase paint; group J-D provided some information to the type of paint they chose; groups A-D and J-D included the volume of paint; and groups A-D, C-D, and J-D indicated the colour of paint they were to purchase.

Table 3. A summary of the paint students chose to use for the Door project

Group	Where to purchase	Brand and additional information (texture, form, etc.)	Volume	Colour	Cost
A-D	The Home Depot	----	946mL	Green	\$26
B-D	The Home Depot	----	----	----	\$43
C-D	----	----	1 can of brown, 5 cans of olive green, 3 cans of gold, 1 can of white	1 can of brown , 5 cans of olive green , 3 cans of gold , 1 can of white	\$156.59
J-D	----	CIL smart 3 interior (typically used as a ceiling paint)	3.78L	White	\$36.97

While the *Door Project* does not require students to include specific information about the paint they chose other than the cost, it is likely that students did not include this information because of their insufficient EMK in paint. During their modelling processes, students did not demonstrate a clear understanding of how much and the type of paint required for the project. They made assumptions about the paint they needed based on their limited EMK. Also, students did not demonstrate an understanding of where they could get such information, nor did they demonstrate a motivation to find this information. As such, students' insufficient EMK on paint negatively affected the quality of their submitted solution.

On top of paint, students also demonstrated limited EMK with regards to the office door in the problem. However, students' insufficient EMK on the door did not affect the quality of their submitted solution as much as paint did.

Most students represented the office door with drawings. These drawings, along with the conversations they had during their modelling process, provide information to students' existing EMK, the EMK they applied, and the assumptions they made about the office door in the *Door Project*.

A standard door (a conventional one purchased at home hardware stores) is 80 inches tall, but varies in width (from 28 inches to 36 inches)²⁷. To be fair, although a door is an everyday object, it might be something that students have taken for granted. Unless they needed to determine the width and height of a door to fulfill a purpose, such as moving furniture or big items through the door, replacing a door, etc., it is reasonable for students to not have paid attention to its height and width and therefore they might never have noticed that doors come in different sizes. Therefore, it is understandable for students not to have such EMK and assumed the office door is the same size as the classroom door.

What is interesting here is not students' assumptions about the measurements of the door. Rather, it is how students went about finding these measurements. All students had access to the school Wi-Fi when they worked on the *Door Project*. Also, each student group had access to at least one smart phone. However, none of these

²⁷ Home hardware stores in the city at the time of the study mostly use the imperial system as a unit of measurement.

students looked online for the door's measurements. Given students are often joined at the hips with their phones, it is interesting that their first choice was to physically measure the door rather than to search online for an answer. From the way they acted, it seemed that it never crossed their minds to use their phones on this matter. It is possible that their actions were affected by their insufficient EMK. If they never realized doors come in different sizes, they would have never bothered to investigate the matter and therefore might have used what was readily available as their point of reference. Also, doing a search online requires them to formulate search words. Students might have found it easier to physically measure the door than to search online for these dimensions.

All the classroom doors at the school students attended are 36 inches wide. These doors also share a few common features: they are wooden doors; they have a *window* which allows people on one side of the door to see the other side of the door; and they have a piece of metal around the door knob and the lock. The door in Figure 48 is a representation of a typical classroom door.



Figure 48. A typical classroom door at the school which students attend.

Most students took the door dimension as the only similarity between the office door and the classroom door (groups A-D, B-D, and C-D). Group J-D was the only group who made additional assumptions about the possible similarities between the two doors. The group assumed that the office door in the problem has a window but the door knob does not have a metal plate around it. When asked about their assumptions, Jasmine claimed that their assumption about the window was based on the classroom door:

Jasmine: There's a window on the door! So there is a window!

And Jessica claimed that the door knob on the classroom door is unique:

Jessica: Nobody has that door knob at home. School rooms are different than home rooms.

Group J-D was not able to articulate the reasons behind their different assumptions about the door. Maybe the group did not know much about doors and made assumptions as they went about solving the problem without noticing the contradictions in their assumptions. Maybe the group assumed that having a window on the door could lower the cost of paint. Also, although extremely unlikely, maybe the group noticed the uniqueness about the metal plate on the door knob but not the window on the classroom door?

All other students assumed that the office door they were working on has no window and has a different door knob as compared to the classroom door. When asked about their assumptions, students drew from their EMK and pointed out that the doors they have at home are different from those at school, and they made their assumptions about the office door based on the doors they have at home.

Andy: My door at home has no window. And the office is at home. And who has a window looking into your study. That's weird.

Barry: My door at home doesn't have a window.

Carmen: Some door knob are just the circle ones. And some are the handles. That's just the school one.

Carol: That's the school door knob. It's a special door knob for school. And the door knob doesn't have the area around it. It's just on the school door. My door knob doesn't look like that.

To summarize, students' assumptions about paint and the office door demonstrate the negative effect of insufficient EMK on the real result. When students have insufficient EMK, they made assumptions about the situations. However, these assumptions do not necessarily lead to the production of a realistic solution.

9.3. Students' refusal to apply EMK

Of the four groups of students, group B-S applied minimal amount of EMK during their modelling process and approached the *Door Project* mostly from a mathematical perspective even after their EMK had been activated. This is most evident in their decorative design, where they insisted on tiling the door with gift cards and spent over \$5000 on gift cards. It is possible that group B-D refused to make changes because:

- 1) These students did not see reality's relevance within the problem. As such, they were stuck on interpreting the *Door Project* as a mathematical problem where the major focus of the problem is to determine the number of gift cards required to tile one or more sides of the door; and/or
- 2) These students were unable to reason with the sensibility of their solution. They did not recognize the cost of their designs and their decisions to put loaded gift cards on the door for decoration purposes were rather unreasonable.

Group B-D engaged with the problem mathematically with minimal realistic perspective. Their work shows that after the activation of EMK, students still need to be willing to engage with the problem from a realistic perspective and apply the EMK prior to the production of a realistic solution.

9.4. Summary

Students made various assumptions of the office door, gift cards, and paint as they worked on the *Door Project*. These assumptions allowed them to simplify the real situation and to move forward to produce a real model. All students' approaches contain some degree of realistic perspective where they all applied some EMK to solve the *Door Project*. Because the *Door Project* is designed as a realistic situation that forces students to reflect on realistic aspects of the situation, even the most mathematical approach taken involves a small degree of reality. For example, group B-D, who included no painting tools or adhesives and insisted on tiling the front and the back of the door with loaded gift cards, applied their EMK and aimed to purchase paint from the dollar stores at one point of their modelling process. Other students demonstrated their EMK and took into various realistic considerations during their modelling process and generated different degrees of realistic solutions. For example, group A-D aimed to put

loaded gift cards on the door but included additional tools and a sufficient amount of paint for the project. In other words, while some students interpreted the *Door Project* from mostly a mathematical perspective, others approached the *Door Project* somewhere in the mathematical-realistic perspective continuum.

In the next chapter, I present to readers the second of the two RMCTs administered to students and students' modelling processes. This is followed by a discussion of the themes that arose from the data analysis process.

Chapter 10.

Design a new school

In what follows I present the work of three groups of students. What is presented here are representations of students' modelling process on the *Design a New School* task. Similar to the *Door Project*, I chose these cases because they were the most complete and comprehensive of all the cases, and I experienced data exhaustion after analyzing three groups of students' work. Pseudonyms are used to protect students' identities. In the reported transcripts, I use *T* to indicate lines spoken by me, and various pseudonyms to indicate lines spoken by the students.

The *Design a New School* task involves a number of elements: a school building, a soccer field, two tennis courts, a 30 car parking lot, and greenspace. The school which students attended at the time of the study has similar features mentioned in the *Design a New School* task. There are two things about the students' background that are worth mentioning. Firstly, none of the participating students had reached the legal driving age at the time of the study, but all participating students have been driven around town by family members and others. As such, it could be assumed that they had no driving experiences, but had experiences as a passenger inside a vehicle. Also, the *Design a New School* task was administered towards the end of the school year. As such, the grade 8 and 9 student participants would have spent close to one year and two years at the secondary school they attended, and it could be assumed that the students know the layout of the school, including where they could find additional information about the parking lot, the tennis courts, the soccer field, etc. A summary of the *Design a New School* task could be found in Chapter 7.

10.1. Amy and Angela (Group A-S)

Amy and Angela reacted to the task by first asking questions about the parking lot. Amy believed that one of the key factors of the parking lot is the dimensions of a vehicle, and suggested to go outside to the staff parking lot to take some measurements.

Amy: How big is a car [talks to herself]? Can I go outside for a second [asks teacher]?

When she came back, Amy discussed these measurements with Angela, and suggested that they should increase these measurements to accommodate for large vehicles.

Amy: The car I measured was 2.5 metres by 1.5 metres, but it was a slightly smaller car so probably make it a bit bigger? 'Cause there are bigger cars in the parking lot?

Angela: Like Chevres.

Amy: What's that?

Angela: It's a truck.

After this conversation, Amy and Angela decided to put the parking lot on hold and investigated the possible locations and orientation of some of the building structures on the grid. First, they re-read the instructions provided, and paid attention to the areas on the grid which they were allowed to put buildings and the actual length each square represents. They divided 12.5 (distance between the border and all buildings) by 10 (each square represents 10m) and got "one and one-fourth", and outlined a rectangle one and a quarter squares inside the border of the grid to represent the space they could put the buildings (Figure 49).

Angela: So, all fields, courts, buildings, and parking lots must be no closer than 12.5 metres to any of the property lines. So one and one-fourth.

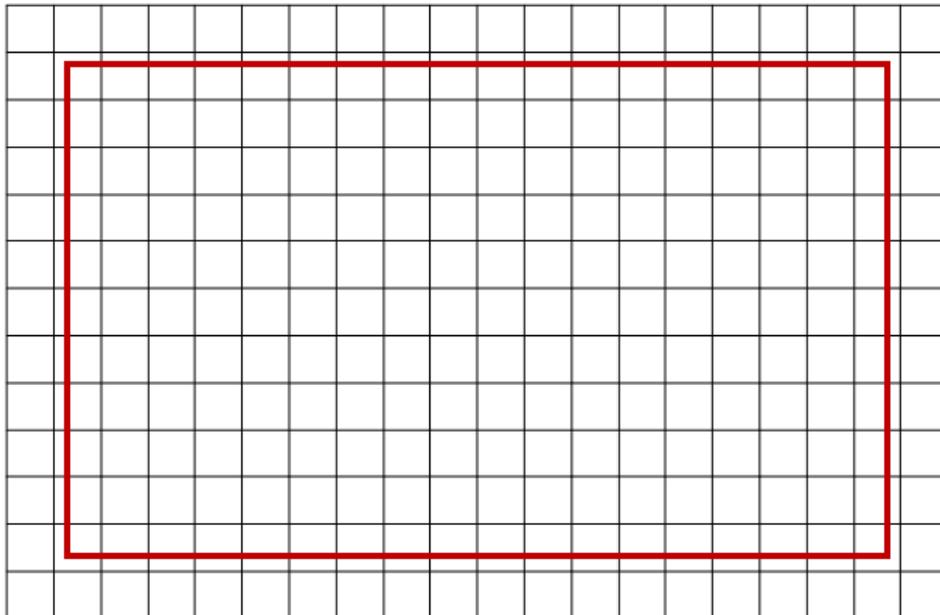


Figure 49. An illustration of group A-S's border

Amy and Angela then used the measurements given in the instruction to work on the soccer field. They divided the length and width of the soccer field and those of the tennis courts by 10, and drew a rectangle that is 10 by 7.5 squares to represent the soccer field (Figure 50).



Figure 50. An illustration of group A-S's soccer field (green)

After penciling in the soccer field, Amy and Angela realized they could not fit the tennis courts beside the short edge of the soccer field, and rotated the soccer field 90° to allow the tennis courts (two rectangles that are 2.75 by 1.5 squares long) to fit beside the soccer field (Figure 51).

- T: Okay what are you doing right now?
- Amy: We are doing a soccer field. ... There we go, that's a soccer field.
- Angela: And it's... it's ...
- T: So you are doing the... looks like you are doing the border... which is... is that your school? Or is that your soccer field?
- Both: Soccer field.
- Amy: Now it's just the tennis fields.

Angela: Aw. Man.

Amy: We can flip it.

T: Ah, like, rotate it. So rotate this sideways, instead of placing it lengthwise. So you can fit the tennis courts

Amy: Besides it.

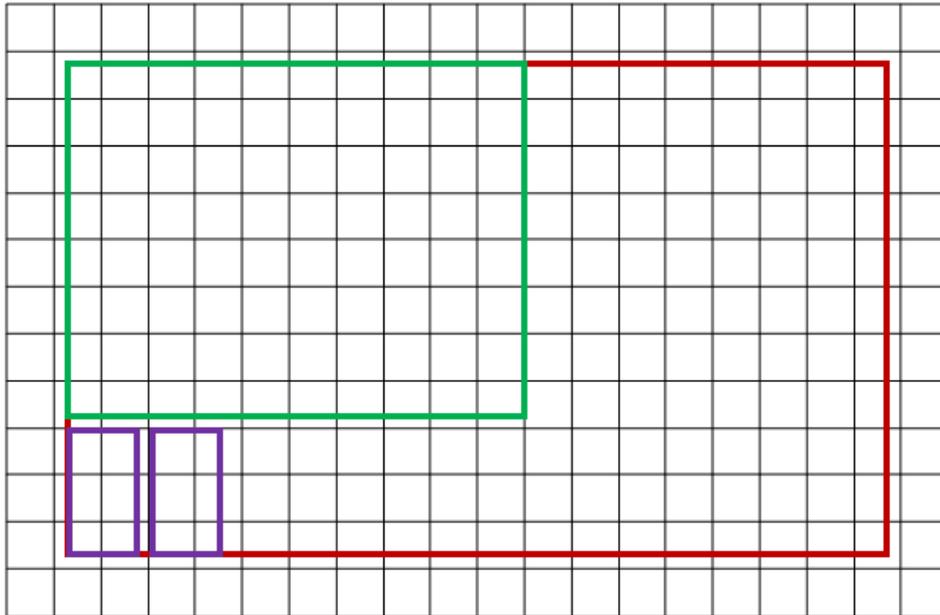


Figure 51. A illustration of group A-S's tennis courts (purple)

After drawing four rectangles to represent the usable space, the soccer field, and two tennis courts on the grid, the girls went back to the instructions and read the information given on the parking lot. They drew a quick sketch of a few parking spots and tried to visualize what the parking lot might look like based on their drawing.



Figure 52. Left: Group A-S drew 5 parking spots within a square; Right: An illustration of group A-S's work.

However, they had difficulties visualizing the parking lot and the things they needed to consider other than the areas taken up by parked vehicles. The girls expressed these difficulties to me. A brief discussion with me led Amy and Angela to realize that there was more to consider than just the area each parked vehicle takes up.

Amy: The car space... 4 by 2. Because, there are bigger cars.

T: But that's the size of a car?

Amy: That's the car I measured²⁸.

T: Okay. So that's the size of a car. But once I parked the car...

Amy: You can't get out.

T: Ahuh. I need to get out. So... would I need... what does that mean?

Angela: You need some extra space!

T: You need some extra space! So, how much is that extra space?

Amy: Like... 0.5 metres?

Angela: A car can't fit!

Amy: Like, 0.5 metres between each car.

Angela: Oh okay.

Amy: You need to be able to go behind a car

T: You need to go behind the cars and to...

Amy/Angela: Drive.

In this conversation, Amy pointed out that cars in parking lots do not park right next to each other. Rather, there is a gap between cars which allows drivers and passengers to enter and to exit their vehicles. Other than a gap between the vehicles, Amy and Angela also pointed out the parking lot needs a driveway, and implied that the driveway is for the vehicles to drive into and out of the parking spaces. After further discussing their work with each other, the girls created an outline of the parking lot. It is rectangular in

²⁸ The vehicle Amy measured was 1.5m by 2.5m. She has already increased the dimension to accommodate for vehicles larger than the one she measured.

shape, 60m long, and 5m wide. All 30 parking spaces are lined up along the long edge of the parking lot, and each parking space is 2m wide and 4m long. The driveway, which runs along the long edge of the parking lot, is 1m wide and 60m long. It is a single direction driveway, where drivers enter the parking lot through an entrance on one end of the parking lot and leave the parking lot using the exit on the opposite end. They did not specify how they arrived at a 1m wide driveway. They divided these measurements by 10, and drew a rectangle that is 6 squares by 0.5 squares on the grid (Figure 53).

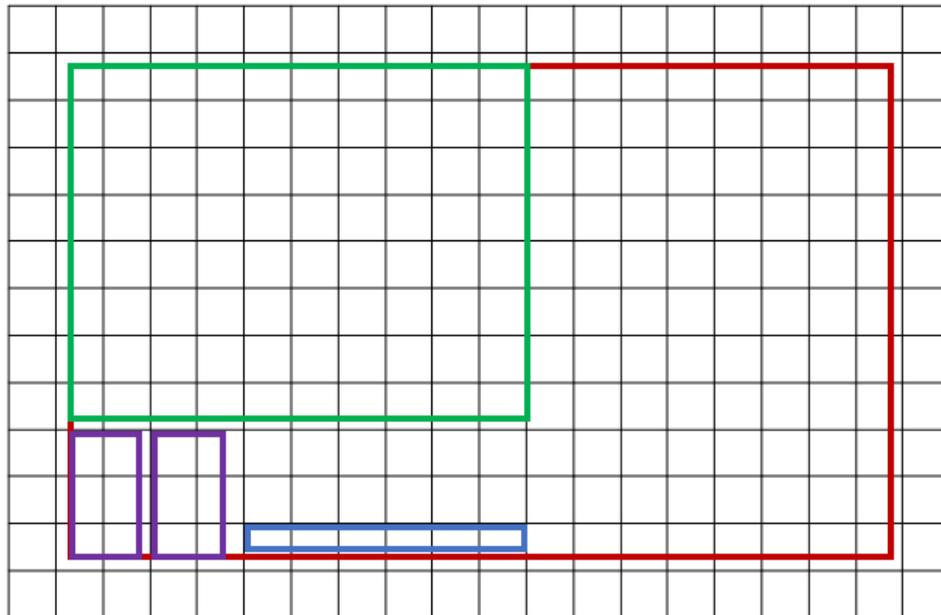


Figure 53. An illustration of group A-S's parking lot (blue)

Afterwards, Amy and Angela moved on to the final requirement, the school building. Based on the instructions, the girls first decided that the length of the school building was 110m and the width was 100m to accommodate for an area of 11000m². Similar to what they previously did, Amy and Angela divided 110m and 100m by 10 and determined that they could represent the school building on the grid using a rectangle that is 11 squares long and 10 squares wide. Very soon, they were stuck. Amy and Angela could not fit an 11 × 10 rectangle on the grid. They then recognized that each square is 100m² and 11000m² means 110 squares, and looked for 110 squares on the grid for the school building. They were stuck again. Angela complained that she couldn't fit the school building on the grid because there was not enough space to do so. Amy described Angela's frustration as a "mental breakdown", and called me over for help. During the conversation, Amy had an "AHA!" moment (Liljedahl, 2005) and realized that

she could "stack" the school building because in reality, it is possible to have buildings taller than one floor.

T: What's wrong?

Amy: [Angela] is having a mental breakdown.

T: Yea... looks that way. Why is that?

Angela: I can't fit it in!

T: You can't fit it in!? Oh my gosh! Excellent!

Amy: Wohoo!!!

Angela: Excellent?

T: Okay. Explain to me what you mean by you can't fit it in.

Angela: I have no more space left!

T: You have no more space left to fit what?

Angela: Um... the rest of the school.

T: The rest of the school. So, tell me, tell me what you have so far. So on the left, you have your soccer field, okay, and next to your soccer field you have your two tennis courts.

Angela: We could properly just move this over?

T: Where is the parking lot?

Angela: I can't fit it in.

T: Oh you can't fit it in? Is the school big enough? The school is big enough. Okay...

Angela: No.

T: The school is not big enough and you don't have any more space. Oh my... Oh no... Oh no... so we need more space.

Angela: But, if we squish it in, it would be ugly.

T: So don't squish it in.

Angela: But there is no space!

T: Oh there is always space.

Amy: Stack them!
T: What do you mean stack them?
Amy: Two floors!

Amy's "AHA!" (Liljedahl, 2005) moment happened about 30 minutes into their modelling process. While they realized that they could have a school building that is taller than one floor, they have not quite grasped what the building might look like and how much space they wanted for each floor. They also joked about creating a "110 floors" building, to which I took the opportunity to discuss with them building shapes and floor area.

Angela: Oh! Can we do those lab rooms thingy? ... Oh windows! [draws an "S" shape school] ... Oh Oh Oh, how about this [points at Amy's drawing]?
Amy: There. It's a circle [draws 2 rectangles, one inside the other]!
Angela: Can we do three floors?
T: Yea you can do three floors. [Our school] has three floors. The basement the main floor and our floor.
Amy: [Our neighborhood school] has four though.
Angela: Oh, you know what, in New York, my school has five floors, I eat breakfast at school, on the fifth floor. [laughs]
T: Yes, imagine all the walking right?
Angela: I did that for one year!

As Amy and Angela discussed the shape of the school building and the number of floors they wanted, they also explored the idea of having a two floor tennis court building, and the idea of incorporating the two tennis courts into the school building by putting them on the roof.

Angela: Wait so tennis court can be two floors.
T: Um, you can, but I don't know if that's cool, though.
Angela: On the roof, the tennis courts!

After discussing the possibility to include the tennis courts as a part of the school building, Amy and Angela decided to not make changes to their tennis courts design and

explored the idea of a school building with three equal-area floors that totals to 11000m², and decided that each floor would take up 3670m², or 36.7 squares²⁹.

Angela: We got this!

T: What's that? What's that 70? Two floors?

Angela: Yea.

Amy: Three floors!

Angela: Wait.

T: Three floors!

Angela: Wait. Oops.

Amy: Why?

Angela: 1-1-0 divided by

T: Why is it 1-1-0 divided by- oooh I get it. You are doing the 110 little squares, that's how much space you'll need. And you divide that by 3, to do 3 floors. So what do you get? 36 or something?

Angela: 36.66667 so I round this to 7.

T: Round this up

Angela: 7.

Amy: 36.7?

Upon further discussion, Amy and Angela interpreted 36.7 as a rectangle with an area of 36.7 squares on the grid. When they discussed the possible length and width of a rectangle made up of 36.7 squares, they quickly dismissed their solution, and further chatted with each other the school building they were in and came to the realization that the school building does not need to have equal floor areas. They eventually settled on a building with two floors where the main floor (100m by 70m) is larger than the second floor (80m by 50m). They then drew 2 rectangles, one inside the other, to represent the school building, and assigned the remaining space on the grid as green space (Figure 54).

²⁹ Three floors of 3670m² per total up to 11010m², which is slightly larger than the required 11000m².

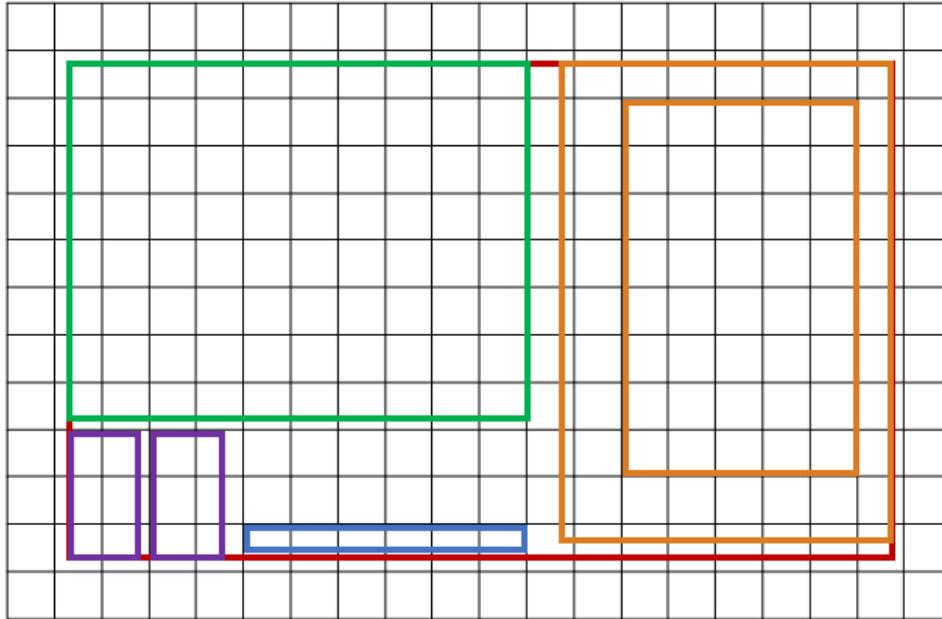


Figure 54. An illustration of group A-S's design

Finally, Amy and Angela read over the instructions again to verify that they have satisfied all the requirements. However, Amy and Angela were not satisfied. They began to make modifications to their design to improve students' school life (Figure 55). First, they redesigned their green space by creating a garden next to the parking lot. The garden is 60m long and 20m wide, surrounded by trees and hedges. It has a gate (approximately 3m wide) on one side of the garden and a bench inside the garden. They then added two doors/gates to the tennis courts and changed their soccer field to a soccer stadium. Amy and Angela did not include any explanations in their submitted work about these changes.

Next, they added a front and a back entrance to the school building. These entrances are approximately 20m wide. The width of these doors was out of proportion, as a 20m wide door is approximately the height of 10 interior doors. Also, these entrances were placed on the second floor of the school building rather than the first. They also added a path that is more than 10m wide leading to the front entrance on the second floor. This is likely a mistake as they overlooked where the outline of the first floor of the school building was, although it is also possible that the girls decided in the last minute to make what was originally the ground floor into a basement and the second floor into the ground floor. After making all these modifications to their designs and verifying that they have satisfied all requirements once more, Amy and Angela submitted their work.

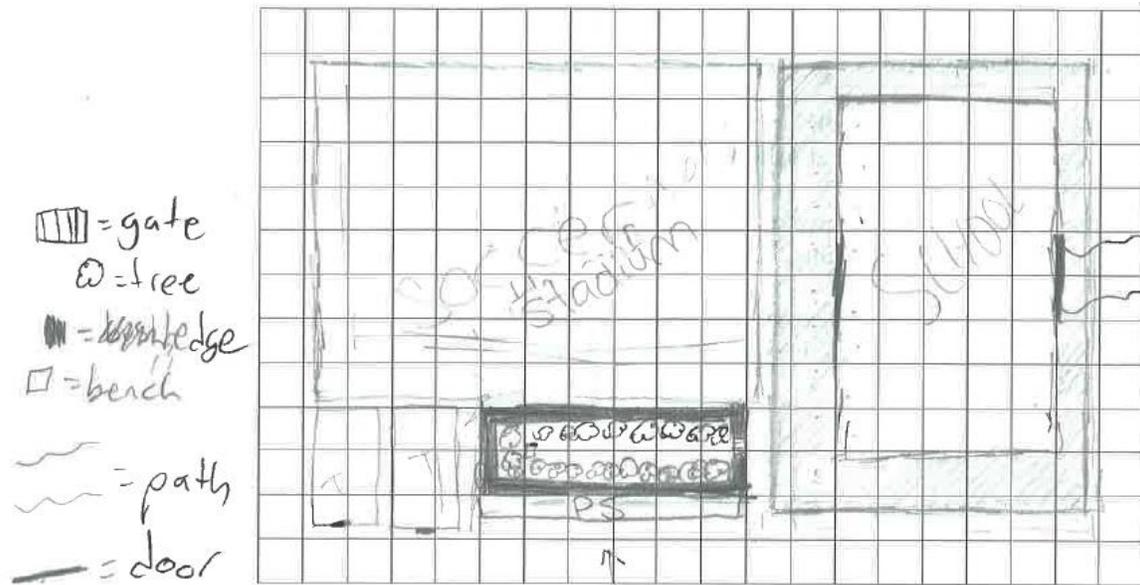


Figure 55. Group A-S's submitted solution

10.1.1. A Discussion on group A-S's modelling process

Amy and Angela's modelling process begins as they identified the parking lot as something ambiguous, as no information was provided other than that it needs to fit 30 vehicles (RS1). Drawing from her EMK, Amy recognized the length and width of vehicles as important factors in the designs (MRS2) but were not certain what to make of these measurements within their design. Their questions and actions helped reduce some ambiguity, but were not sufficient to help them move forward. Instead of spending more time on the parking lot, Amy and Angela put it on hold and worked on other aspects of the problem.

As Amy and Angela re-read the instructions (RS3), they decided to focus on the distance between all buildings and the property line. Amy and Angela interpreted this as a restriction to the "usable space" on the grid (MRS4) and made plans to create an outline on the grid to represent the space they could use (RM5). They drew from the instructions the information they needed and applied their mathematical skills to convert the distance between the property line and the buildings into number of squares on the grid (MM6, MR7). They interpreted their solution as an outline on the grid and used it to represent the space they could use (RR8). This is Amy and Angela's first complete modelling cycle (MC1, steps 3-8). The outline helped them to get organized and to determine real results for the rest of the real situation. Since Amy and Angela decided to

focus on one aspect of the problem at a time, and this forced them to repeat the modelling cycle a few times.

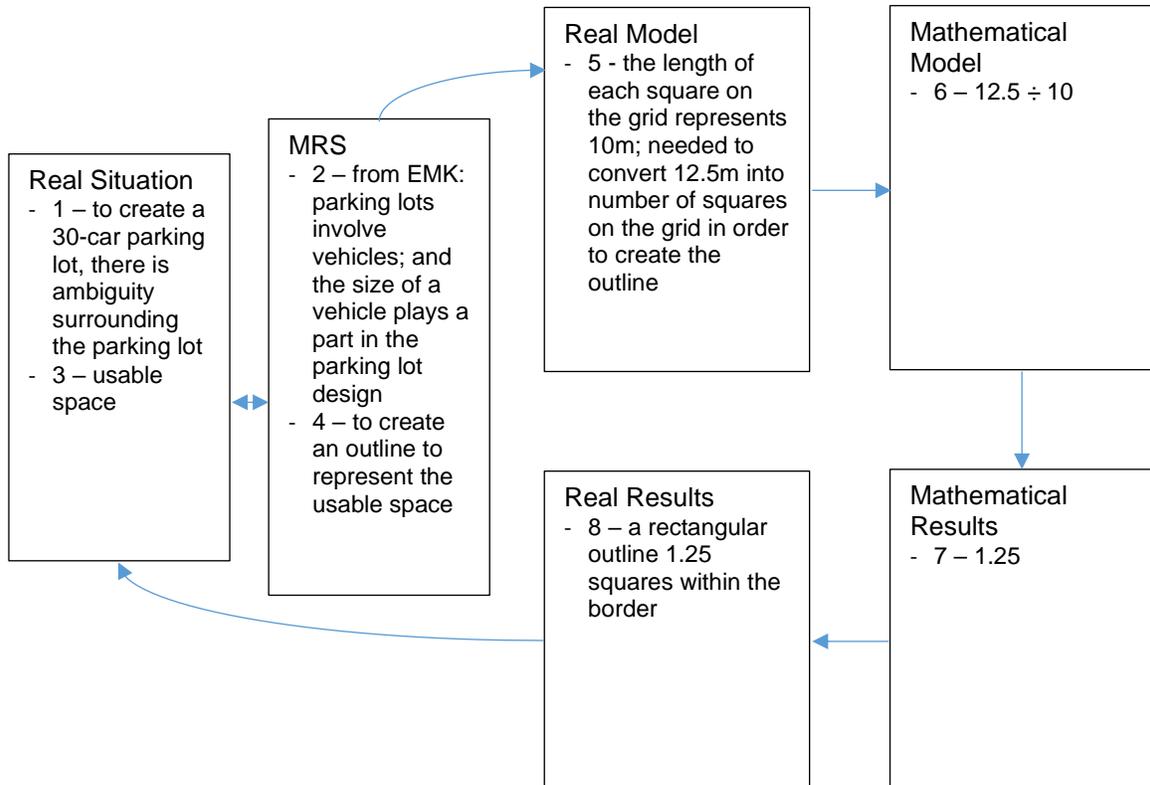


Figure 56. Group A-S’s initial focus (1-2, parking lot) and their first modelling cycle (usable space)

After creating the outline, Amy and Angela focused on the soccer field (MC2, steps 9-14) and the two tennis courts (MC3, steps 15-20). They took a similar approach to the usable space. They first identified the soccer field as their focus (RS9), and understood that they needed to create outlines of these buildings on the grid provided (MRS10). Combining this with the understanding that the length of each square represents 10m, Amy and Angela planned to convert the measurements provided into number of squares and to create rectangles on the grid to represent the soccer field (RM11). They then divided these measurements by 10 to obtain the length and width of the rectangle they needed to create on the grid (MM12 and MR13), interpreted this as the length and width of the rectangle they needed to draw (interpretation), and drew a rectangle on the grid to represent the soccer field (RR14). Afterwards, they worked on the two tennis courts and repeated the process (MC3: RS15, MRS16, RM17, MM18, MR19, and RR20). One thing that is worth mentioning here is Amy and Angela’s rotation of the soccer field after

they generated a real result for the tennis courts, as they recognized the overall relationships between the locations and placements of the buildings, and that the rotation of the soccer field by 90° allows them to have a better use of space.

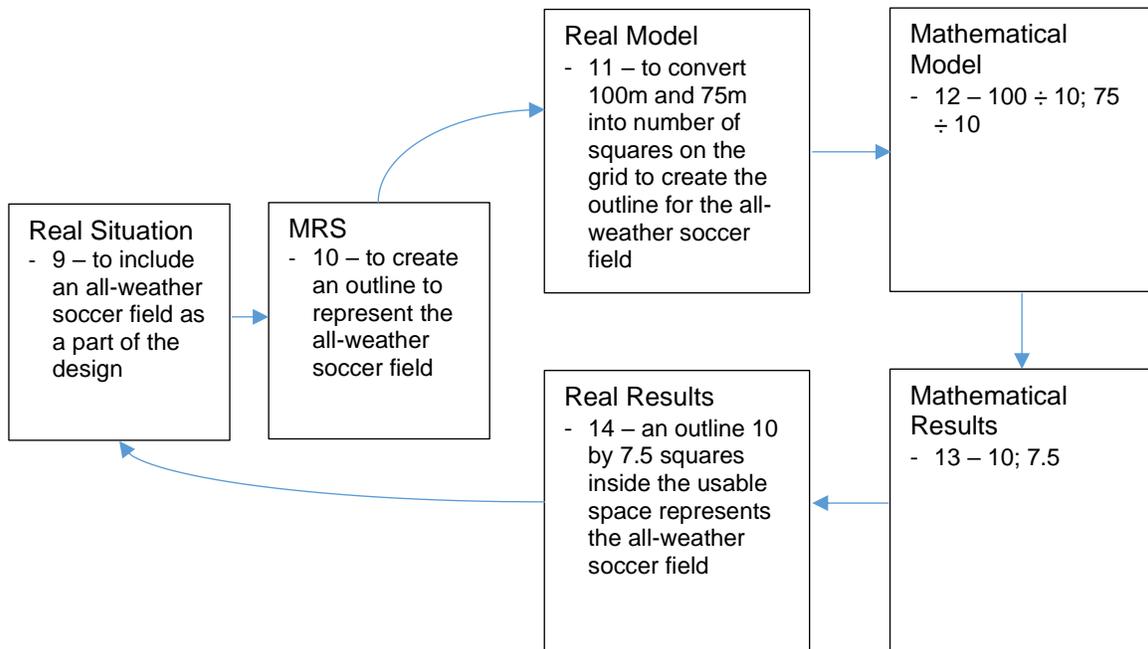


Figure 57. Group A-S's second modelling cycle (soccer field)

Moving on to the parking lot (MC4, steps 21-26), the girls re-read the instructions (RS21), and focused on creating an outline of the parking lot (MRS22). They drew some sketches of parking spaces based on the measurements they took, but experienced difficulties in visualizing the relationship between the vehicles and the parking spaces and the relationship between the parking space and the parking lot. It seemed that they were not able to visualize what the parking lot might look like and their EMK failed to help them with the problem. Although they have both visited various parking lots as passengers, neither had reached the legal driving age at the time of the study, and had never experienced parking lots from a driver's perspective. Their lack of EMK became a hinderance to their modelling process. A brief discussion with me led them to deepen their understanding of parking lots and to generate their real model (23): 30 parking spaces on one side with enough space for drivers and passengers to enter and exit the vehicles, a driveway which allows vehicles to enter the parking lot, to reach all driving spaces, and to exit the parking lot. Based on their decisions on the length (4m) and width (2m) of the parking spaces and the amount of space vehicles need to back out of

the parking spaces (1m), they created a mathematical model (24) where the length of the parking lot (in terms of number of square on the grid) is the width of each parking space multiplied by 30 and then divided by 10, and the width of the parking lot is the length of each parking space plus 1 and then divided by 10. They then generated their mathematical results (25), 6 and 0.5, based on their mathematical model. They interpreted 6 and 0.5 as the length and width of the parking lot, where the length of the parking lot is 6 squares long and the width of the parking lot is 0.5 squares long (interpretation). Afterwards, they drew a rectangle that is 6 by 0.5 squares on the grid to represent the parking lot (RR26).

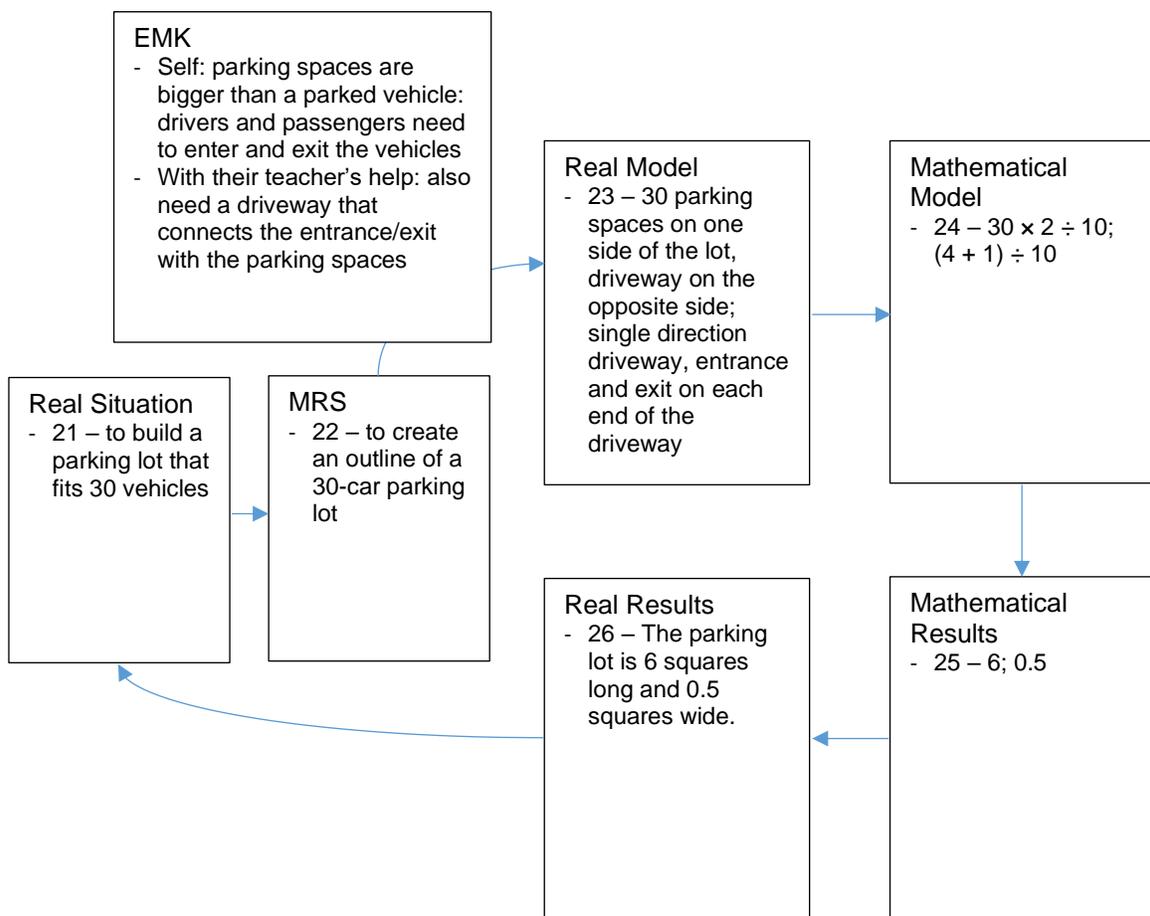


Figure 58. Group A-S's fourth modelling cycle (parking lot)

Unfortunately, despite all their hard work, Amy and Angela did not create a realistic parking lot design: there was no indication of an entrance or an exit to the parking lot in this design; they did not include any driveways that connect the edge of the school property with the parking lot; and their driveway inside the parking lot is far too narrow

for most if not all vehicles. It seems that their discussion with me and with each other was insufficient to expand their EMK for the purpose to generate a realistic parking lot design.

Finally, Amy and Angela proceeded to work on the school building (MC5, steps 27-32). They re-read the instructions again (RS27) and aimed to determine the amount of floor space the school building needed on the grid (MRS28). They proceeded to make decisions about the shape of the school building (RM29), built a mathematical model (30) by looking for factors of 11000, converted these into number of squares (MR31), and interpreted 11 and 10 as the length and the width of the school in terms of squares (RR32). When they were unable to find the necessary rectangular space on the grid for the school building, they questioned if they could fit the school building on the grid. They validated this by dividing 11000 by 100, and looked for 110 squares on the grid. This is when they got stuck, as they confirmed that there were less than 110 usable squares on the grid and they could not fit their school building on the grid.

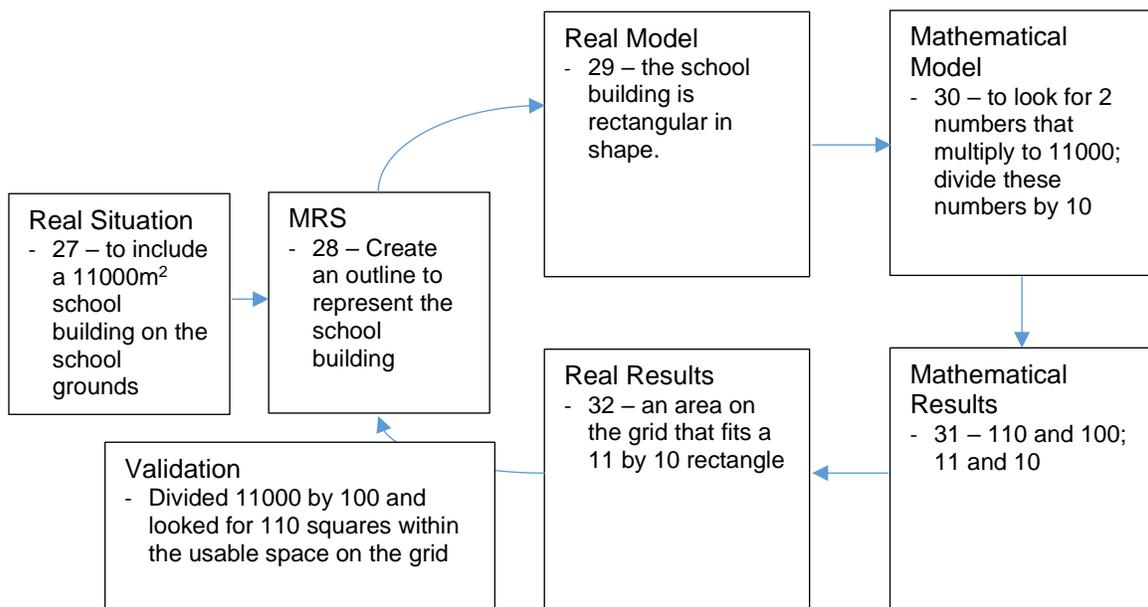


Figure 59. Group A-S's fifth modelling cycle (school building)

Amy's "AHA!" moment (Liljedahl, 2005) during their discussion with me helped them to recognize the possibility to extend the vertical height of the school building in order to satisfy the floor area and to decrease the construction area required (MC6, steps 33-41). This realization contributes to their new understanding (MRS33) and allowed the girls to

draw from their past experiences, whether it was visiting various neighborhood schools or staying aboard schools from overseas, to think about the various shapes and the possible height of the school building, and the possibility to make changes to the designs of the tennis courts. In other words, Amy and Angela’s recognition to incorporate vertical space into their solution demonstrated their deepened understanding and offered them the opportunity to reorganize their approach and solutions to the problem.

As they moved forward, they toyed with the idea of a three-floored school building where each floor has an equal floor area (RM34), built a mathematical model (MM35) to determine the number of squares they required on the grid by dividing 11000 by 100, and then by 3. Afterwards, they carried out the calculations and generated a mathematical result ($36.\bar{6}$), which Angela rounded to 1 decimal place (MR36). Amy and Angela interpreted this as a rectangle with an area of 36.7 squares (RR37). Their interpretation of school buildings restricted them to design rectangular buildings. But as they drew from the EMK in attempts to make sense of their solution, they realized that their school building did not need to have an equal floor areas (MRS38), and they re-entered the modelling cycle one last time.

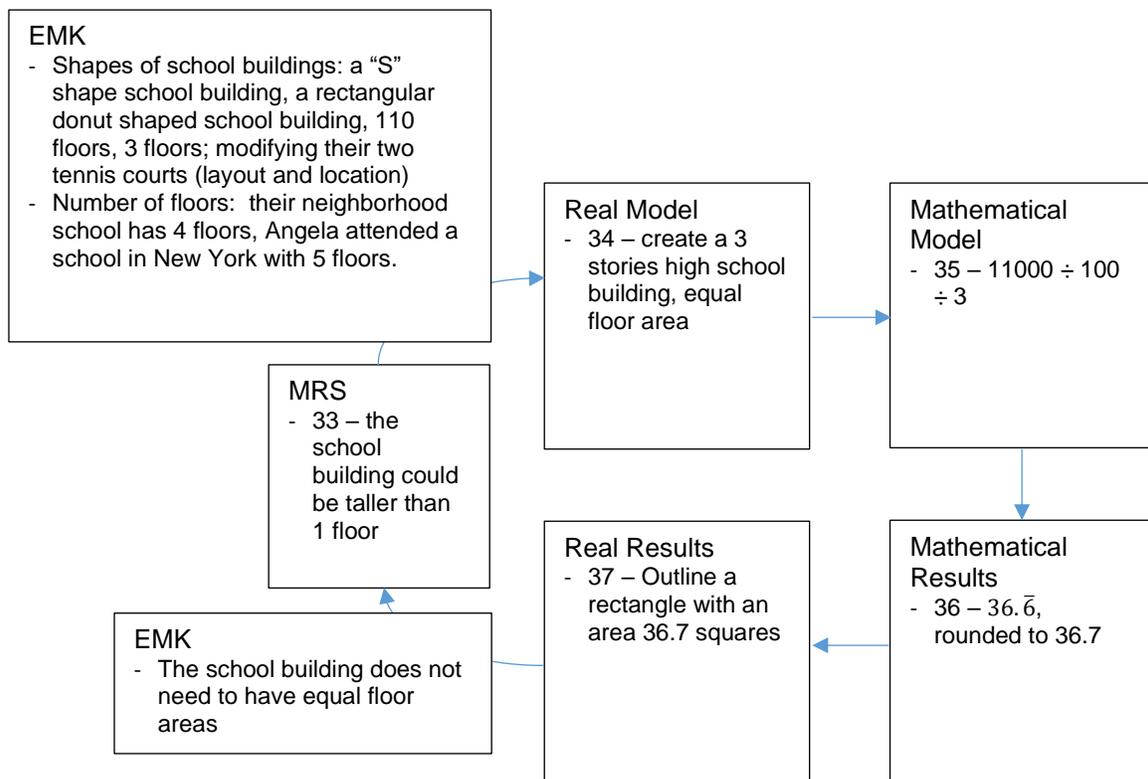


Figure 60. Group A-S’s sixth modelling cycle (school building)

They eventually settled on a building with two floors where the main floor is larger than the second floor (RM39). They also decided that the main floor would be 100m by 70m, and the second floor would be 80m by 50m. Amy and Angela did not explain how they arrived at these measurements. I speculate that they first reduced 11000m^2 to 110 squares. They then picked 10 and 7 squares as the length and width of the main floor to generate a floor area of 70 squares, realized they needed 40 more squares to satisfy the requirement, and chose 8 and 5 squares as the length and width of the second floor to make up for 40 squares (M40, MR41).

Afterwards, they outlined two rectangles on the grid, one within the other. The bigger one is 10 squares by 7 squares and represents the main floor of their school building, and the smaller one is 8 squares by 5 squares and represents the second floor of their school building. These two rectangles represent their real result to the school aspect of the problem situation (RR42).

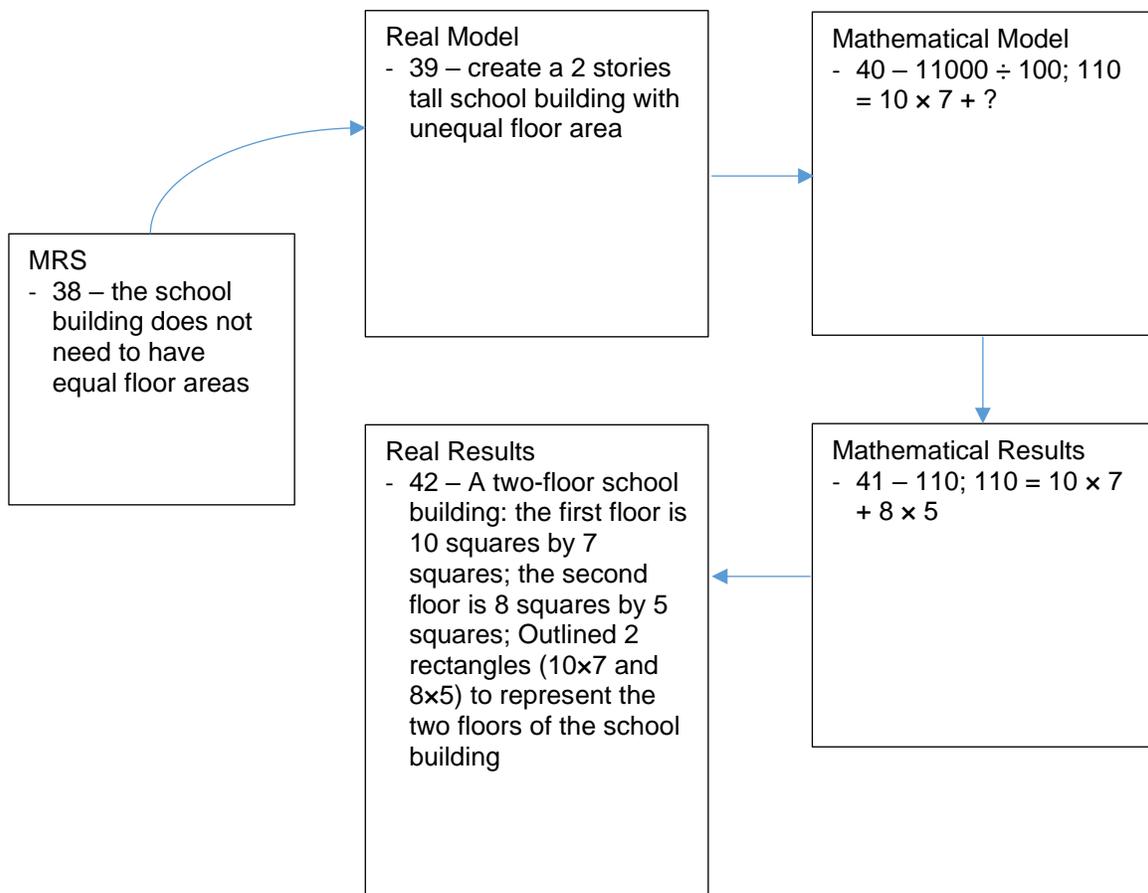


Figure 61. Group A-S's seventh modelling cycle (school building)

Finally, Amy and Angela re-read the instructions one last time to verify that their solution is complete. Instead of concluding their modelling process, however, the girls made additional changes to improve their solution. These changes, including the addition of a garden, a path to the front entrance of the school, and the front and back doors of the school, do not qualify as additional modelling cycles. Mathematical modelling refers to “the process of model building, leading from a real situation to a mathematical model, or the whole applied problem-solving process, or again any manner of connecting the real world with mathematics” (Blum, 1993, p. 5). Amy and Angela’s work does not demonstrate any of the processes indicated here. While their want/need to build a garden, path, and a gate may be interpreted as a real situation, there is no indication of the generation of a mathematical model or mathematical results. They simply outlined a rectangle next to the parking lot to represent a garden, added drawings of trees, shrubs, a bench, and a gate to decorate the garden. There is no indication of consideration of the relative dimensions of the plants, the bench, and the gate in their work. Similarly, they did not seem to have considered the width of the path and the two doors in their drawings in relation to reality. Amy and Angela seemed to have added these features as a finishing touch, and did not account for the measurements in their drawings in relation to reality.

In summary, Amy and Angela went through the modelling cycle 7 times and focused on a different aspect of the problem each time. They first identified the parking lot as something they would work on, but found it too challenging at the time. Instead of carrying on with the parking lot design, the girls moved on to create an outline to represent the usable space (MC1), the soccer field (MC2), two tennis courts (MC3), prior to applying their EMK to create a parking lot design (MC4). Afterwards, they further applied their EMK and created a two-floored school building (MC5-7). Finally, they verified that they have satisfied all requirements and added a garden, a path to the front entrance of the school, and a front and a back door to the school building. The following is a diagrammatic representation of Amy and Angela’s overall modelling process. The red arrows in the diagram (3-8, MC1) represent their work on the usable space; the orange arrows (9-14, MC2) represent their work on the soccer field; the green arrows (15-20, MC3) represent their work on the tennis courts; the blue arrows (1-2, 21-26, MC4) represent their work on the parking lot; and the purple arrows (27-42, MC5-7) represent their work on the school building.

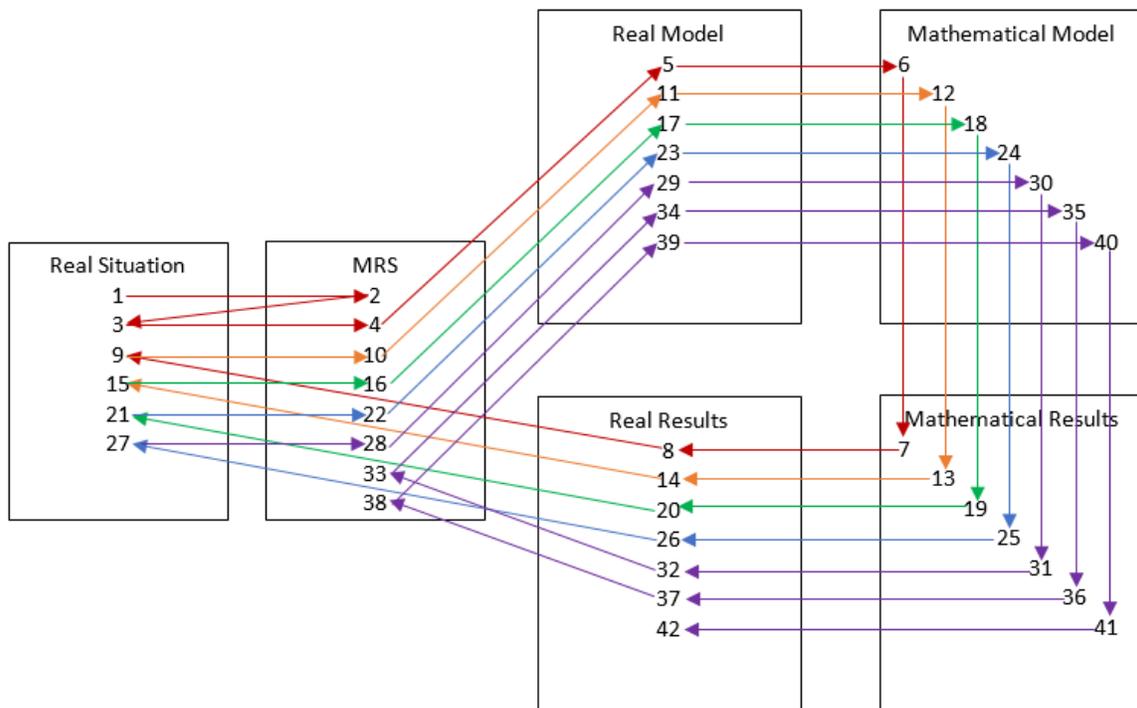


Figure 62. A diagrammatic representation of Group A-S's modelling process

Throughout their modelling process, Amy and Angela demonstrated their application of EMK to the design, but showed little interest in validating their real results against the original real situation. For most of their modelling process they drew from their EMK to satisfy the requirements listed in the instructions and took a much stronger mathematical approach than a realistic one. They attended to the realistic aspects of the problem towards the end of their modelling process after they satisfied all the requirements listed in the original real situation and considered how they could improve students' school lives. While these ideas might seem to originate from a realistic perspective, the girls did not fully consider these ideas from a realistic perspective. For example, they turned their soccer field into a soccer stadium, but did not include additional space for bleachers, change rooms, washrooms, food vendors, etc.; they added two gates to the tennis courts, but did not indicate whether they were putting a fence around the courts and did not include additional space around the courts; they installed two doors to their school building, but did not reflect deeply on the width of these doors; etc. As such, their designs are mathematically accurate, but lack details and do not reflect reality. Their lack of validation against reality could be a result of the pair treating the task as one with roots in mathematics rather than one with roots in reality. This led them to focus on the

mathematics and away from making sense of their solution in terms of the original real situation. For example, the girls attended school on a daily basis, but did not apply their knowledge of the shape of the school building to design a sensible building. Similarly, they used the vehicles in the staff parking lot for parking space dimensions but did not use the parking lot to determine the proper width of the driveway. It is unclear whether their work here is a result of their insufficient EMK, a result of them not accessing or acquiring additional EMK, a result of them making convenient assumptions without deep reflections as they installed these additional features, or their minimal sense of the relationship between their drawing and reality.

10.2. Becky and Bianca (Group B-S)

After reading the instructions, Becky and Bianca rushed into solving the problem and immediately took the measurements provided in the instructions (in metres) and converted them into number of squares by dividing the actual measurements by 10. They first took the length and the width of the soccer field, 100m and 75m, and divided them by 10 to get 10 and 7.5, and drew a rectangle 10 squares long and 7.5 squares wide at the upper left-hand corner on the grid to represent the soccer field.

T: You're gonna park the soccer field right at the corner?

Becky: Yea.

T: Okay.

The girls quickly repeated the process and drew two rectangles to represent the two tennis courts after they penciled in the soccer field. They divided the length (27.5m) and the width (15m) of the tennis courts by 10, and outlined two rectangles that are 2.75 squares long and 1.5 squares wide on the grid next to the soccer field (Figure 63). While Becky and Bianca were quick with their calculations, they did not read the instructions clearly. They neglected the distance between the property line and the building structures and put their soccer field and the tennis courts along the property line (Figure 63).

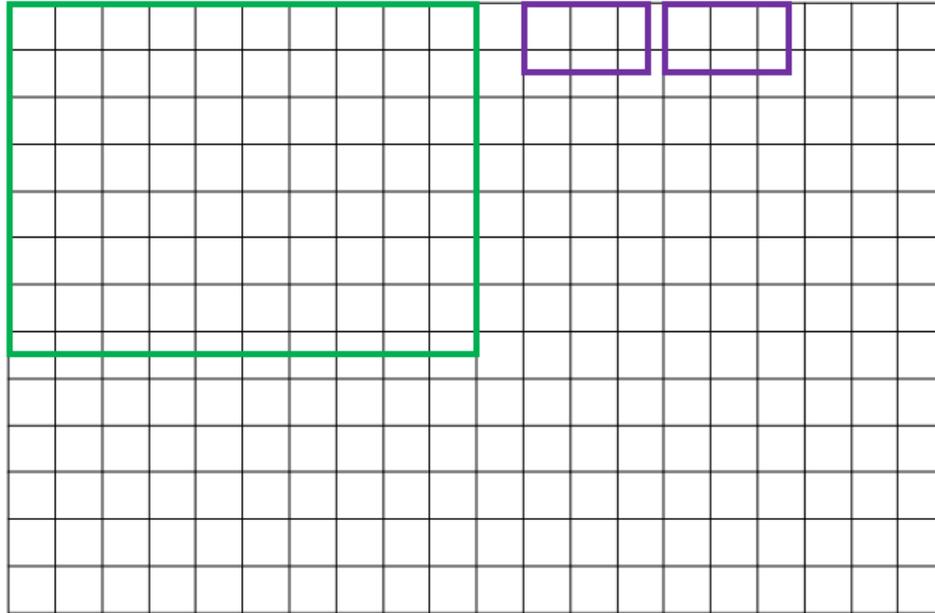


Figure 63. An illustration of group B-S's soccer field (green) and tennis courts (purple)

Afterwards, Becky and Bianca worked on the parking lot. Very soon, they were stuck and asked me for help. The girls were not certain how big the parking lot needed to be, as they were not certain how many cars their parking lot needs to fit. After they re-read the instructions, Becky suggested that the length of a vehicle is roughly 2 metres long, multiplied 2 by 30, and claimed that the product, 60, was the solution. Bianca disagreed with Becky's estimation of 2m, but was not confident with her own estimation either. In order to help them move forward, I suggested for them to either look up some information online, or to go outside to get measurements from the school parking lot. Becky and Bianca chose to use their phones to look up information online.

- T: Well how big do you need?
- Becky: Well not that big, like 2 times 30 equals... to 60...
- T: What do you mean by 2 times 30?
- Becky: I was thinking like, a car would be around 2 metres, right?
- T: Two metres... long? Or two metres wide?
- Becky: Long...
- Bianca: Wouldn't it be like, two and a half metres long?

- T: Well, can I give you some suggestions?
- Becky: Yea.
- T: Do you all have your phones here? There are a couple of things you can do. You can go online, and check out how big a car is...
- Becky: Oooooohhh
- T: Or you can go outside and measure a car.
- Bianca: But it's cold outside. Okay! Phone it is!

After looking up information online, the pair discussed what their parking lot would look like, but could not agree on the size of a parking space and the dimensions of the parking lot. Becky called me over for help again. At this point, about 35 minutes into the modelling process, I took the opportunity to point out that all their buildings needed to be 12.5 metres away from the property line, which Becky and Bianca were not too pleased with. During this discussion, Bianca also pointed out that the amount of space given to build a school, a soccer field, two tennis courts, and a parking lot was rather small.

- Becky: We, cannot agree, on anything.
- T: Why not? So did you move the school in yet? The 12.5 metres, so you should not be putting anything
- Bianca: Oh.... My God.
- T: Like
- Bianca: Are you serious!?! But this is such as small space!

Moving quickly, Becky and Bianca divided 12.5m by 10m and outlined a rectangle 1.25 squares within the property line to represent the usable area, and relocated the soccer field and the two tennis courts to the right side and the lower left-hand corner of the school ground respectively, and rotated the tennis courts 90°. However, the tennis court on the right side at its new location is slightly shorter than the one on the left side (Figure 64). The girls did not explain the reasons behind the change of locations and the rotation, nor did they explain why one tennis court is shorter than the other. Afterwards, the girls worked on the parking lot again, and quickly got stuck. They believed that a vehicle is roughly “3 to 4 metres times... 1 or 2 metres”, but were not certain what they could do with the information to help them build a parking lot. When they called for help,

I guided them through their parking lot design. In the first part of this conversation, Becky pointed out that “the cars need spaces” but did not specify what she meant by “spaces”. Following on Becky’s comment, Bianca interpreted “spaces” as the “spaces between cars”.

- Becky: We are having trouble with the parking lot.
- T: Parking lot? Why?
- Becky: Um.....because..... [pauses for roughly 3 seconds]
- T: Well, how big is a car?
- Becky: Like, 3 to 4 metres times... 1 or 2 metres?
- T: Okay so let’s say 2 metres by 4 metres. Okay? So whatel, so in a parking lot, what do you see?
- Becky: Cars...
- T: What are the components of a parking lot?
- Becky: The cars need spaces?
- Bianca: Spaces between cars?
- T: Spaces between cars...

While Bianca made an excellent point about spaces between vehicles for drivers and passengers to enter and exit their vehicles, Becky clarified that her “spaces” refers to “the space for the cars to drive in”, or a driveway.

- Becky: The cars need spaces?
- Bianca: Spaces between cars?
- T: Spaces between cars...
- Becky: We need the space for the cars to drive in.

At this point, Becky and Bianca seemed to have a fairly good idea about what they wanted for their parking lot. I then asked a few more guiding questions and provided them with suggestions to help them visualize the parking lot. Becky and Bianca also made decisions regarding some of the measurements of their parking lot.

- T: Okay, good. You have the space for the cars to drive in, you have your parking spots, okay? So let’s take that into

account. So if I have a car here, a car here, a car here, a car here, a car here [draws a row of parking spots on a piece of paper], and then what else can I do? Can I do, I probably should need some space behind the cars so they can drive in.

Becky: You should leave the space between the cars. [points to the line separating the parking spots]

T: Okay and between the cars? How much space do you think we'll need between the cars?

Becky: Half a metre?

T: Half a metre between the cars? So the 2 metres per spot should be pretty good right? So for cars to drive in ... how much space, how wide do you think this road should be, for cars to drive in?

Bianca: Two or three metres?

T: Two or three metres would put it this close [points to the width of a parking spot]. Well, each car is four, right? So two or three would be less than one car space. Would that be enough?

Bianca: No.

T: No? How-

Becky: Five.

Bianca: Five, yeah.

T: So it would be more than, longer than a car's length. Okay? So how long do you think? Would five do?

Becky: Five, yea so may be six for the bad drivers.

T: Okay. Why don't we do that? Do you want to play with that? So we can then see how much space we'll need.

After the discussion, I figured Becky and Bianca had a fairly good grasp of the design of their parking lot. They seemed to understand that the driveway should be at least as wide as the length of a parking space for drivers to enter and to back out of the parking spaces, and they made decisions about the length and width of each parking space based on the guidance provided. The girls worked on the parking lot for a few more minutes and used the grid provided to guide them and designed a parking lot that has 3 rows of 10 parking spaces per row, with a driveway between each row of parking spaces

(see Figure 64). The parking lot is 67.5m by 50m. Each parking stall is 5m by 10m, the area of half a square, and the driveway between the rows of parking stalls is 10m wide, which is the length of one square. They also left some space at either ends of the parking lot. However, they did not indicate what these spaces were for.

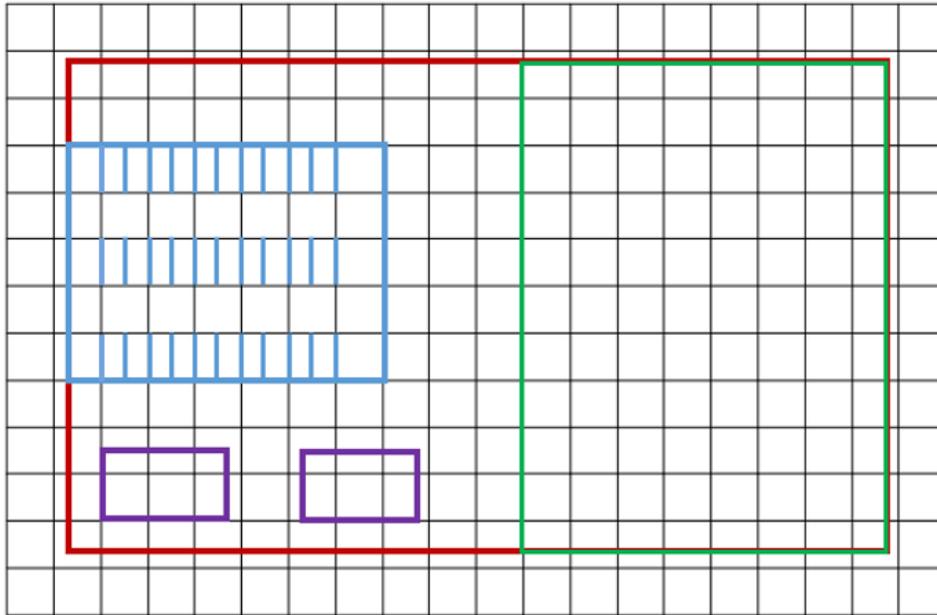


Figure 64. An illustration of group B-S's design so far

Looking at their creation so far, Becky and Bianca realized that the amount of space left was not sufficient for their school building. Instead of pondering on the situation, they quickly called me over for help. In order to guide Becky and Bianca in the right direction without directly giving them the answer, I asked them a question which led them to quickly recognize that their school building could have multiple floors.

Bianca: The parking lot and the soccer field, they are too big to put inside this little space!

T: Okay, very good observation! So we don't have enough space-

Bianca: To make rectangles.

T: To make rectangles. Which floor are you on right now?

Becky: Three.

Bianca: What, wait what???

Becky: You can build oh arghhhh!!!!!! So it doesn't have to all be one floor! You can make it as tall as you want! Skyscrapers.

T: Good job, [Becky].

Bianca: (laughs)

The recognition of a multiple level building opened a lot of possibilities. After working on the problem for another 10 minutes, the girls settled on creating a school building on top of the parking lot at the upper left-hand corner of the school property and turned the parking lot into an underground parking lot. They created a square-shaped school building, and assumed that the school building has the same length as the parking lot, 6.75 squares, or 67.5m. They also decided that the school should be 4 floors. Since each floor is 67.5m by 67.5m, or an area of 4556.25m², a 4 floors tall school building has a total floor area of 18225m², which exceeds the area suggested in the original instructions. After they penciled in the school building, they coloured all leftover space and called this green space, but did not specify what they would use the green space for. Afterwards, Becky and Bianca submitted their work.

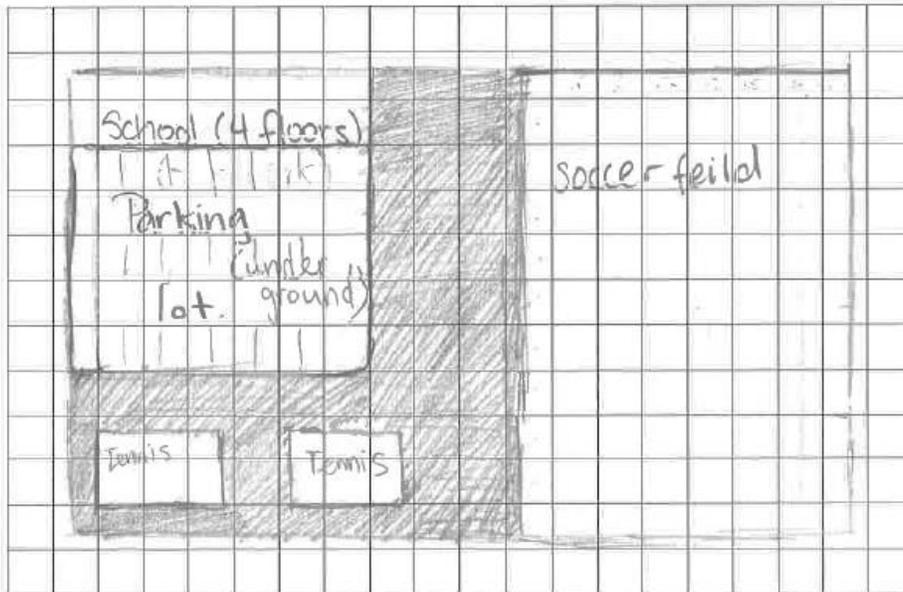


Figure 65. Group B-S's submitted work.

10.2.1. A Discussion on group B-S's modelling process

Similar to group A-S, Becky and Bianca also repeated the modelling cycle multiple times, focusing on a specific aspect of the problem each time. Becky and Bianca's first modelling cycle began as they read the instructions and interpreted the goal of the task as to create a design based on the measurements given in the instructions. Beginning with the soccer field (RS1, Figure 66), they reckoned they needed to create an outline on the grid to represent the soccer field (MRS2) and planned to use the measurements provided in the instructions to do so (RM3). To put their plans in action, the girls took the measurements provided (100m and 75m) and divided them by 10 (MM4), and arrived at 10 and 7.5 (MR5). They then interpreted their mathematical result as a rectangle that is 10 squares long and 7.5 squares wide (interpretation). Finally, they outlined a rectangle at the upper left-hand corner on the grid to represent the soccer field (RR6). This completes their first modelling cycle (MC1). They then repeated the process and outline two rectangles on the grid next to their soccer field to represent the two tennis courts (MC2: RS7, MRS8, RM9, MM10, MR11, RR12).

Becky and Bianca's treatment of the problem thus far suggest that they did not reflect deeply on the situation. With regards to the soccer field and the tennis courts, the girls took the measurements provided in the instructions, applied some basic computational skills, and arrived at their solutions. They approached the problem mostly from a mathematical perspective.

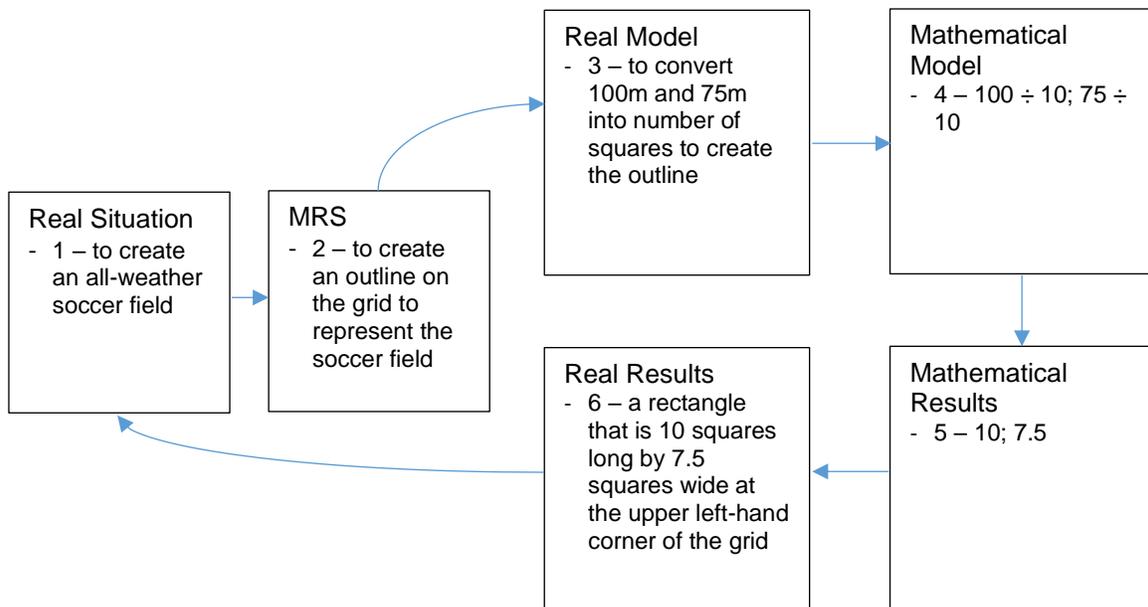


Figure 66. Group B-S's first modelling cycle (soccer field)

After the soccer field and the two tennis courts, Becky and Bianca moved on to the parking lot (RS13), and entered their third modelling cycle (MC3, Figure 67). The pair understood the task had something to do with a 30 car parking lot (MRS14), but were very much limited by their EMK otherwise. They multiplied 2 by 30 (MM15), and believed that the product, 60, is the result (MR16).

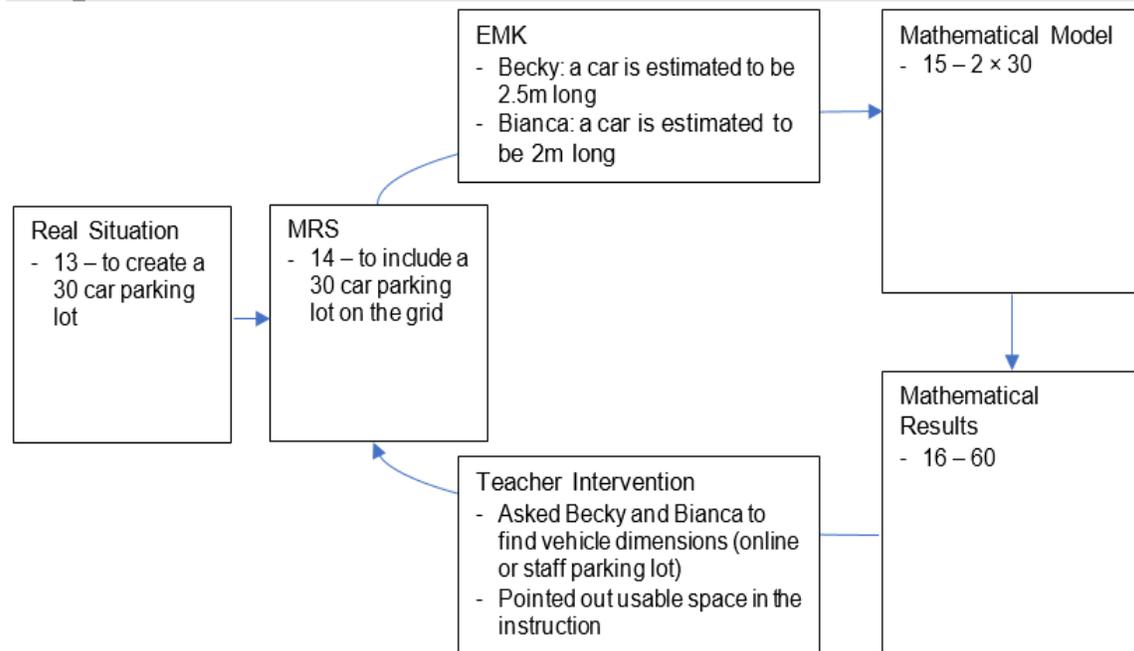


Figure 67. Group B-S's third modelling cycle (parking lot)

Their work here demonstrates that they did not have a fair understanding of the problem, nor did they have a sense of direction where they were going and what they needed to do to arrive at the real results (Maaß, 2006). Similar to group A-S, Becky and Bianca's insufficient EMK hindered their progress. To cope with their insufficient EMK, the girls extracted numbers from the instructions and manipulated these numbers based on their estimation to determine an answer. While they recognized their deficiency and asked for help, the help they received was not sufficient to move them forward. Instead, it redirected their focus to the usable space (MC4, RS17). They created an outline to represent the usable space (MRS18, RM19, MM20, MR21, RR22, see Figure 68), just like they did with the soccer field and the two tennis courts. As they proceeded, they also relocated the soccer field and the two tennis courts within this usable space. The relocation of the soccer field and the tennis courts are not considered standalone modelling cycles as it is merely a re-interpretation of their real results based on the additional work done (RR22). It should be noted that during the process of relocating the soccer field and the two tennis courts, Becky and Bianca made mistakes regarding the measurements of these building structures. The length of the relocated soccer field is slightly longer than that as instructed, and the length of one of the tennis courts (the one on the right, see Figure 64 and Figure 65 for details) is shorter than that as instructed.

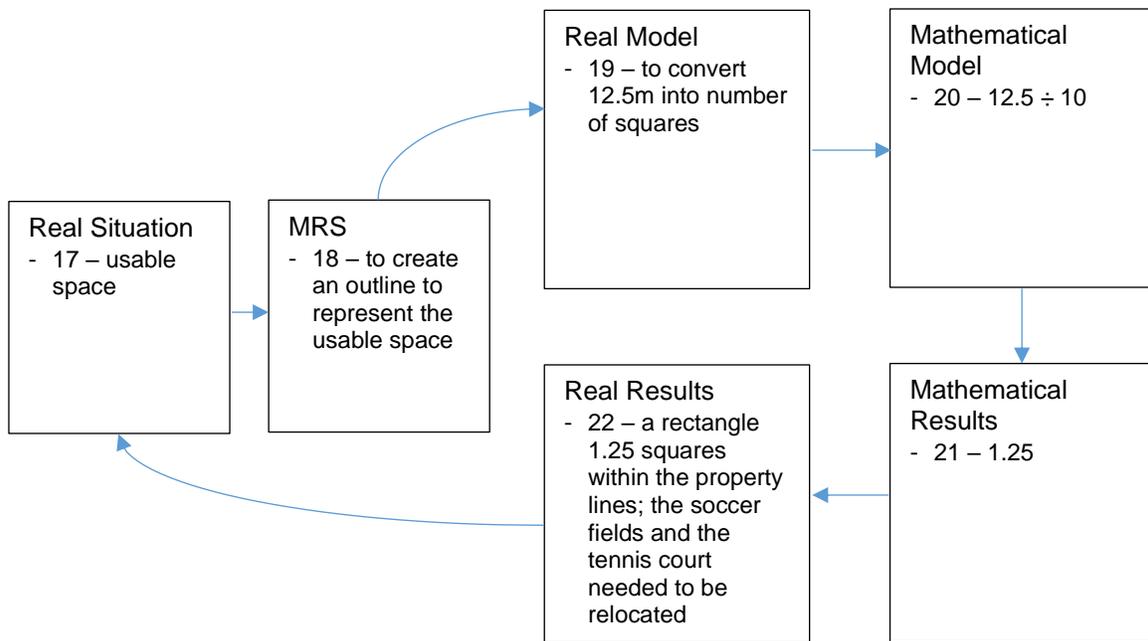


Figure 68. Group B-S's fourth modelling cycle (usable space)

Afterwards, the girls worked on the parking lot again (MC5, RS23), and quickly realized the information they previously found and their EMK were insufficient for them to proceed. As such, they asked for help again. This time, I acted as an authoritative figure and guided them through their parking lot design (MRS24).

Unfortunately, I misjudged Becky and Bianca's understanding of what was being discussed. The girls did not fully understand the content of the discussion and were unable to apply a lot of the things discussed in the conversation to their design. They generated a parking lot with 3 rows of 10 parking spaces, bypassed all mathematical calculations, and drew an outline of the parking lot using the squares on the grid as their guideline (RM/RR25). This resulted in a disproportional parking lot, or a solution that does not fit the situation. Becky and Bianca did not validate their solution, and quickly moved on to the next building structure thereafter.

As I analyzed the data, I find that it is impossible to distinguish between their real model and their real result in this particular modelling cycle. Borromeo Ferri (2006) suggests that modellers idealize and simplify the problem and apply extra-mathematical knowledge as they build a real model, which has strong connections to their interpretations of the problem. Becky and Bianca's work on the parking lot is based heavily on some of my suggestions during the discussions, and shows strong

connections to the original situation. It resembles the characteristics of a real model. On top of these connections to the original situation, a real model should also contain an approach to mathematically solve the original situation (Blum & Niss, 1991). However, what resembles a real model in Becky and Bianca's case does not contain such plan. In terms of the real result, Borromeo Ferri (2006) suggests that it should be in correspondence to the real situation based on the interpretation of the mathematical result. Becky and Bianca's work here is definitely not based on the interpretation of any mathematical work done and does not resemble the characteristics of a real result. However, it IS their real (final) result. As such, I find it impossible to distinguish between their real model and their real result. Rather, I see a collapse of the modelling cycle, where Becky and Bianca were unable to apply what was discussed to generate a real model and then apply mathematics to generate a mathematical and a real result.

Becky and Bianca's work demonstrates the negative impact of students' insufficient EMK on their modelling process. Specifically, Becky and Bianca did not know what information was needed and where to acquire such information. As such, they were unable to proceed independently. Although I attempted to provide the girls with some guidance, their insufficient EMK were too much of a hinderance for them to overcome.

During the modelling cycle, students' extra-mathematical knowledge and their intra-mathematical skills both play an important role to inform students about the problem situation and to help them generate a realistic solution to the real situation. As such, the modelling cycle should tell the story of how students apply their extra-mathematical knowledge and use their intra-mathematical skills to arrive at a solution to the real situation. It should illustrate students' understanding of the situation and their mathematical skills in solving the problem.

Becky and Bianca's work on the parking lot does not illustrate such story. Their work here demonstrates my failed attempt to provide the girls with guidance, including the information and understanding they needed, to create a representation of the parking lot. Their work also demonstrates the process which they coped with their insufficient EMK: they combined the information provided in the question (30 cars) with their understanding of a parking lot (parking spaces and driveways) along with the tool provided in the question (the grid) to generate a solution. Their coping with their deficiency shows that their insufficient EMK did not completely stalled them in generating

a solution, but this deficiency definitely influenced and hindered the way they approached and solved the problem situation.

Maaß (2006) argues that modellers' attitude directly affects their approach to modelling problems, and serves as a trait in modelling competencies. She differentiates between four types of modellers: reality-distant modellers, mathematics-distant modellers, uninterested modellers, and reflecting modellers. Based on Maaß's (2006) descriptions, Becky and Bianca were overall reality-distant modellers. They focused on converting the measurements provided in the question to generate outlines on the grid. This can be seen in their work on the usable space, soccer field, and tennis courts. However, when they worked on the parking lot (MC5), their insufficient EMK prevented them from creating a useful real model, and it did not set them up for mathematization and the application of their intra-mathematical skills to generate a mathematical result. As such, they did not transition from reality into mathematics. They remained in reality and used the grid as a tool to guide them in the production of a solution. In this particular scenario, Becky and Bianca were not only reality-distant modellers, they were also mathematics-distant modellers. However, this does not make them uninterested modellers. Their approach was heavily influenced by their insufficient EMK rather than their attitude towards mathematics. They coped with the situation and generated a realistic solution to their best ability.

Becky and Bianca's work on the parking lot demonstrates the importance of EMK. In this particular example, knowledge regarding the parking lot and the abilities to acquire additional knowledge, including the dimensions of parking spaces, the minimum width of a driveway, the arrangement of parking spaces, the need of an entrance/exit, etc., prevented the girls from the formulation of a real model, which in turn prevented the girls to create a mathematical model and a mathematical result. These deficiencies led to the collapse of their modelling cycle. The following is a diagrammatic representation of Becky and Bianca's work on the parking lot:

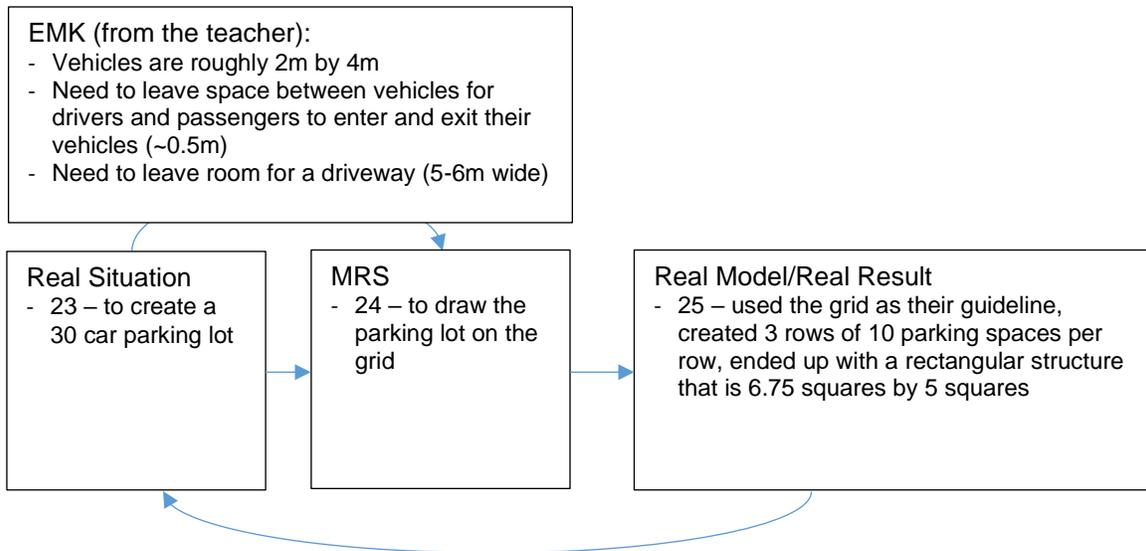


Figure 69. Group B-S's fifth (collapsed) modelling cycle (parking lot)

After the parking lot, Becky and Bianca were ready to work on the school building (MC6, RS26). Similar to what they had done, they aimed to produce an outline of the school building (MRS27). However, before they could proceed, they recognized they did not have enough room for the school on the grid and asked for help. It is unclear how Becky and Bianca arrived at their conclusion. I suspect they made similar assumptions as group A-S, where they assumed the school to be a 110m by 100m rectangular building (RM28), which led them to an outline that is 11 squares by 10 squares (MM29, MR30, RR31); or they converted the floor area into number of squares and realized they needed 110 squares. Since most of the space on the grid is already taken up by the soccer field, the two tennis courts, and the parking lot, they concluded that they did not have enough space for the school building and asked for help.

To help Becky and Bianca out, I asked a guiding question and redirected their approach to use vertical space to increase the floor space without increasing the construction area. This contributed to their new MRS (32).

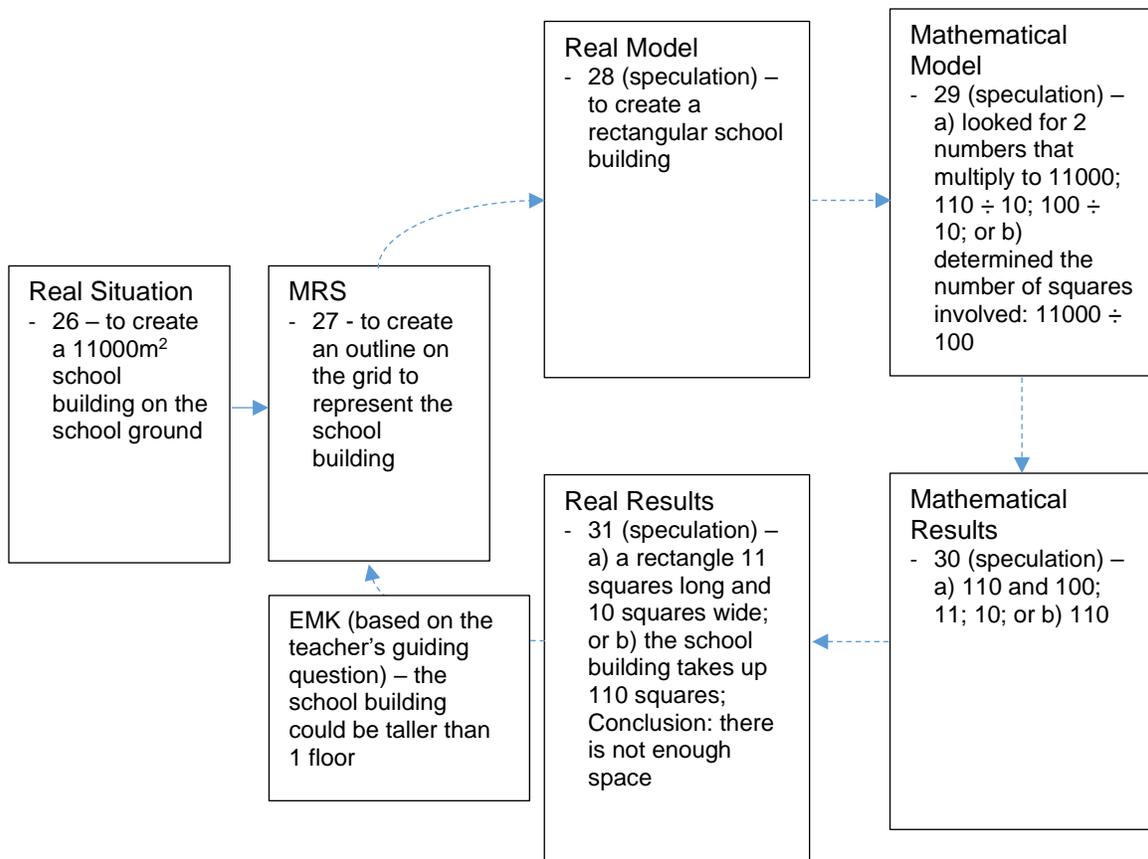


Figure 70. Group B-S's sixth modelling cycle (school building)

Their experience with the school building is different from that of the parking lot. Here, it is clear that Becky and Bianca had the EMK to cope with insufficient land area. As I helped the girls activate their EMK, they immediately recognized a possible solution to the problem they faced, and they demonstrated their understanding in their work afterwards. With this new MRS in mind, they turned their parking lot into an underground parking lot and built their school on top of it. They made assumptions about the school building (4 floors square shaped building) but did not provide any reasons behind their assumptions and their designs. Similar to the parking lot, their plan does not illustrate any strategies to mathematically generate an outline of the building on the grid. As such, I conclude this is another collapsed modelling cycle (RM/RR34). With all this information in mind, Becky and Bianca created an outline on the grid to represent the school building. They decided to build a 4-floor school building without taking into consideration the total floor area required.

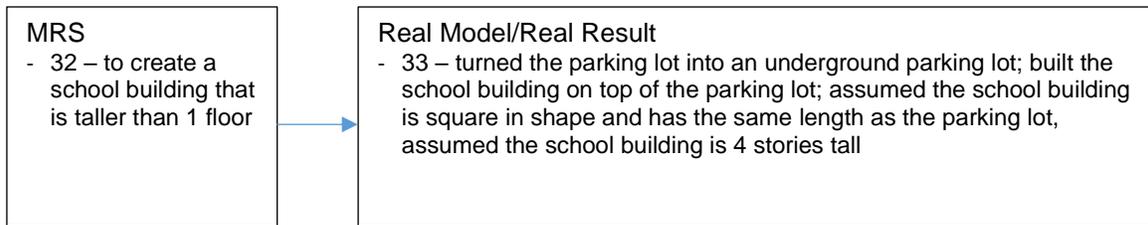


Figure 71. Group B-S's seventh (collapsed) modelling cycle (school building)

Becky and Bianca's seventh collapsed modelling cycle as illustrated in Figure 71 (steps 32 to 33) demonstrates their use of their EMK to solve the problem of insufficient ground area. Their decision to turn the parking lot into an underground parking lot and to build the school building on top of the parking lot likely have roots in their lived-experiences. There are many buildings in the city such as apartments and shopping malls where the main building is built on top of an underground parking lot. However, their work demonstrates that they rushed to finish the problem and did not pay close attention to the instructions given (to build a 11000m² school), and ended up with a solution that does not fit well with the original real situation.

Becky and Bianca's work here on the school building is different from that of the parking lot. The girls had little EMK of the parking lot. Combined with their refusal to use the staff parking lot as a resource, the result was a collapsed modelling cycle. As for the school building, the girls had the EMK (they have spent a lot of time inside the school building) and resources (such as a school map or floor plan) to help them build an appropriate real model, but they only applied their EMK of vertical space to solve the problem. They did not reflect deeply into the problem and likely rushed to finish the problem. This resulted in another collapsed modelling cycle.

I find Becky and Bianca's refusal to use the staff parking lot here interesting. The girls refused to obtain measurements from the staff parking lot, but opted to search for measurements online and relied on the help given to construct a parking lot design. This is a completely differently approach than the one students took in the *Door Project*, where students physically measured the classroom rather than searched for these measurements online. The difference in students' action may be attributed to the efforts required to acquire the EMK.

While both the door and the staff parking lot are physically accessible, students could physically see the classroom door but not the parking lot – students need to exit the school building before they arrive at the parking lot. Students might believe that it is easier for them to look up measurements for cars and the parking lot online than to physically go outside to determine these measurements, whereas they found it easier to physically measure the classroom door than to look up these measurements online.

Finally, Becky and Bianca re-read the instructions and verified they included all the required buildings on the grid, and made some final modifications to the design of the school grounds to satisfy the rest of the requirements. They repurposed the area that is not occupied by the buildings as green space, but did not specify what the green space would look like. Becky and Bianca only validated their work at the very end, and this validation is restricted to making sure they completed the problem rather than making sense of their real result and making sure it fits the original real situation. Becky and Bianca's validation is in line with Galbraith and Stillman's (2006) findings, where novice modellers only validate their work at the end of their modelling cycle and the form of validation is restricted to making sure their work is complete rather than to validate their real result against the original real situation.

In summary, Becky and Bianca experienced a lot of difficulties throughout their modelling process. This is possibly the results of them not paying close attention to details, which could be seen all over their work. In the beginning, they overlooked the distance between the property line and the building structures, and it resulted in the girls redoing parts of their work. They also overlooked the dimensions of one of their tennis courts and the soccer field: the length of one of the tennis courts (approximately 25m) is shorter than the length given in the instructions (27.5m); and the length of their soccer field (105m) is longer than the length given in the instructions (100m). Also, their insufficient EMK became a huge stumbling block and prevented them from generating a realistic parking lot design.

Becky and Bianca approached the *Design a new school* task from a unique perspective. They mostly applied their intra-mathematical skills in the first half of their modelling process (usable space, soccer field, and tennis courts) and approached the problem from a mathematical perspective, where they focused on converting the measurements

given in the question into squares and satisfying the instructions provided in the question.

Their focus on mathematics is also evident during one of their discussions with me as the pair realized they had very little space left for the school building. The pair assumed the focus of the problem was to create a scaled outline of the school grounds and complained about the lack of space “to make rectangles”. While this is not wrong, the task runs much deeper than creating scaled outlines on the school grounds. Their approach and perspective changed as they work on the parking lot and the school building, where they experienced the most difficulties. To cope with these difficulties, they stopped applying their intra-mathematical skills and did not draw from their EMK or acquire additional EMK. Rather, they relied on the tools or resources readily available to them to finish the task. This led to a collapse of their modelling cycles and resulted in the generation of a less than satisfactory solution. The following is a diagrammatic representation of Becky and Bianca’s overall modelling process. The red arrows (steps 1-6) represent their first modelling cycle and their work on the soccer field. The orange arrows (steps 7-12) represent their second modelling cycle and their work on the two tennis courts. The green arrows (steps 13-16) represent their third modelling cycle and their work on the parking lot. The blue arrows (steps 17-22) represent their fourth modelling cycle and their work on the usable space and the relocation of the soccer field and the two tennis courts. The purple arrows (steps 23-25) represent their fifth (collapsed) modelling cycle and their work on the parking lot. The black arrows (steps 26-33) represent their sixth and seventh modelling cycle and their work on the school building. Some of the arrows (between steps 27 and 32) are represented with dash lines as it was unclear the steps Becky and Bianca took to arrive at a conclusion that they did not have enough space to place the school building on the school grounds.

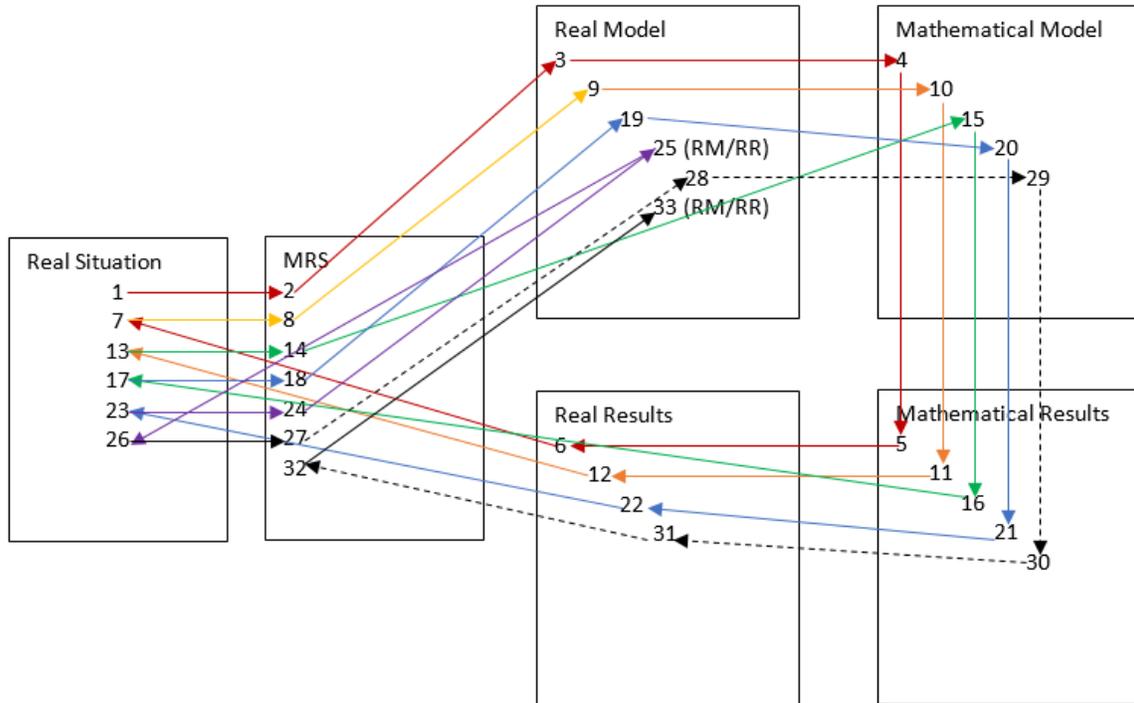


Figure 72. A diagrammatic representation of group B-S's modelling process.

10.3. Gabby, Gloria, Gwen (Group G-S)

Unlike groups A-S and B-S who struggled with vertical space, Gabby, Gloria, and Gwen quickly realized that their school building can have more than one floor after reading the instructions.

Gabby: Okay we measure like houses and apartments.

T: Yeah?

Gabby: But it doesn't matter how, like, the floors are added to the school? Does it have to be completely flat? We can have more than one floor, right? Here to add to the 11000m², so here are two floors you can divide that by 2 right? We can add, the school is two floors so we divide this in half, and we can do that, right?

T: Yup.

Gabby: Okay. Guys, make it like, 6 floors.

At this point, the students also made sense of the size of each square on the grid and attempted to determine the shape of the school building and the number of floors they

wanted. During the process, Gabby randomly decided that she wanted a 2-, then 4-, and then a 10-storey building. She also decided that there were 2 classrooms on each floor, and then changed her mind to create a 20-storey building with only 1 classroom on each floor. She claimed that in both cases, each classroom would be 55m^2 , but was not able to visualize how big 55m^2 is. Gabby did not provide any reasons behind her assumptions. Later on in the discussion, Gwen pointed out that they “don’t have to make [the school] exactly rectangular shape”.

Gabby: If we divide this by two again, which makes it 4 floors, so then we have more space, that’s, or divide this by 10, um ... or, have it 10 floors.

T: I’m glad you guys got this so early. But a better question is, how big should each floor be? So how big do you think the classrooms are? And how many classrooms do you want on each floor?

Gabby: We have to do the classrooms too?

Gloria: Coz we are designing!

Gabby: We should have 2 classrooms on each floor because it’s a 10-storey building.

Gwen: Oh my goodness, okay.

Gabby: And divide that by 2, each classroom is 55m^2 .

Gwen: You don’t have to make the whole building...

T: Are you sure? How big do you think this room is?

Gabby: Probably bigger than 55m^2 .

T: Okay...

Gabby: Okay, yeah yeah. Or... we can do 20 floors, and one classroom each floor.

Gwen: You know we don’t have to make it exactly rectangular shape?

In this conversation, Gabby miscalculated the area of each floor. Provided it was a 10-storey building, each floor should be 1100m^2 instead of 110m^2 . This mistake affected the area per classroom by ten-fold. Instead of 55m^2 per classroom, it should have been 550m^2 per classroom by calculation. It is unclear how Gabby arrived at 110m^2 per floor. It is likely that she divided 11000 by 100, the area each square represents, instead of by

10 floors. Also, the group so far only considered classrooms in their design, and neglected other facilities found inside a school building.

As their conversation unfolded, Gabby pointed out that instead of building a tall building, they could build one with a basement to increase the usable area of the building while leaving room for other buildings on the school grounds. However, Gwen objected to her idea as she believed that people would not want have class in the basement. To her objection, Gabby drew from her everyday experience and argued that their own school has a basement, to which Gwen did not object any further.

Gabby: It doesn't say that we can't have a basement. So can we like, add more, area, if we try?

Gwen: I don't believe anybody would want to coz the school has, would be

Gabby: We have a basement, people still go to school here.

As this conversation came to an end, Gabby, Gloria, and Gwen left these ideas behind and worked on the parking lot. They quickly decided that their parking lot should be an underground one. They used a local shopping centre as an example to explain their decision of the underground parking lot.

Gwen: [The parking lot]'s going to be in the basement.

Gabby: Just like [the local shopping mall].

Gloria: Let's assume each space is 15m².

Gabby: 15m², okay, and then, um

T: But that's, that's the parking space, right?

Gabby: And then you need more spaces so that the cars could park.

T: Good job. Wait, why don't you draw the parking lot so that you have a better idea how much space it is?

Gabby: Sure. Why not?

T: Okay so maybe use a separate piece of paper to draw that out and see what happens?

Uncertain how to begin, Gabby, Gloria, and Gwen first assumed that each parking space is 15m², but also recognized that vehicles require more than a parking space in a

parking lot. When I suggested for them to make a diagram of their parking lot structure for them to better visualize their design and to understand the amount of space required, the girls decided to put the parking lot on hold and worked on the two tennis courts instead.

Gabby, Gloria, and Gwen decided to put the two tennis courts on top of their school building. However, when they made the decision, they were not yet certain of the location of the school building, the number of floors, and the size of each floor.

Gabby: Tennis courts' gonna be on top of the school.

T: Tennis courts are gonna be

Gabby: Yea.

T: Where's your school? Can you show me your school?

Gabby: Okay. It's around here (points at an area on the grid).

T: Around where?

Gabby: This area.

T: Okay. What shape is your school gonna be?

Gabby: We don't know yet.

So far, Gabby, Gloria, and Gwen discussed various ideas including a multi-floor school building, a parking lot that has more than just parking spaces, and the possibility of installing the tennis courts on top of the school. Thus far, they moved between various components of the problem situation, including the school building, the parking lot, and the two tennis courts, but had not put anything down on the grid yet.

It had been more than 30 minutes into their modelling process. Feeling that time was running out, Gloria took charge and suggested for her group to first outline the space they could use. Responding to Gloria's request, Gabby referred back to the original instructions, to which Gloria modified. Gloria reduced the 12.5m restriction to 10m. Gabby and Gwen did not protest to these changes. Rather, they went along with the changes Gloria made. Gloria then divided 10 (metres, instead of 12.5m) by 10 (metres per square), and Gabby suggested 1 square was the answer they needed. Gloria then quickly outlined a rectangle 1 square inside the border of the grid, and directed her group to work on the soccer field. Gabby referred back to the original instructions once

again, divided these measurements by 10, took the pen from Gloria, and outlined a rectangle on the upper right-hand side of the grid within the usable area to represent the soccer field.

Gloria: Let's do the outline first.

Gabby: 12.5m or something divide by 10

Gloria: I'm gonna make it 10?

Gabby: Or 1 square

Gloria: Ok then (outlines a rectangle 1 square from the edge of the grid). What about the soccer field?

Gwen: We can put it here.

Gabby: So... 100m by 75m

Gwen: That's 10 squares and 7.5 squares

Gabby: Okay so one two three four five six seven point five and then one two three four five six seven eight nine ten. There.

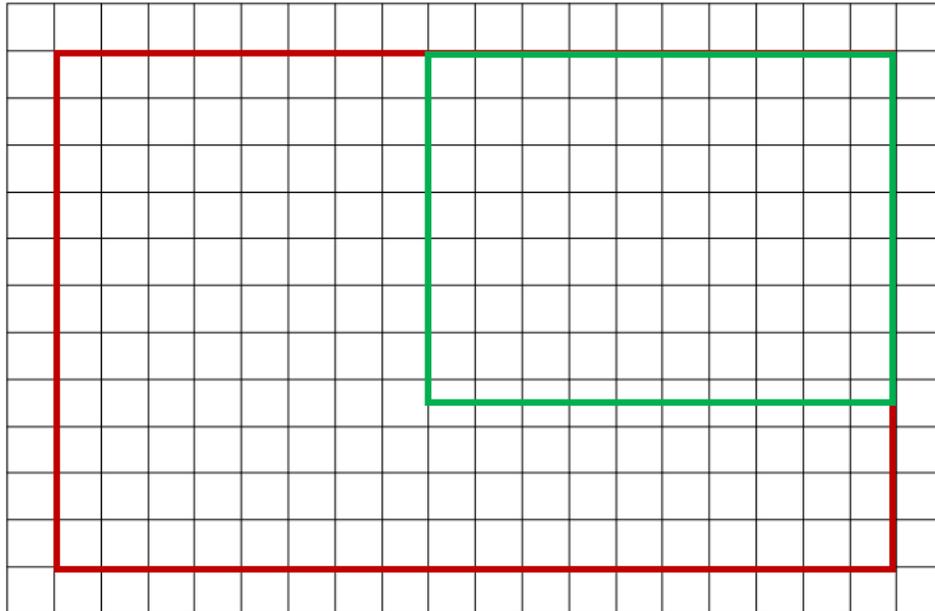


Figure 73. An illustration of group G-S's usable space (red) and soccer field (green)

The girls then outlined two rectangles to represent the two tennis courts. They did not pursue any further the idea of putting the two tennis courts on top of the school building

as they previously discussed, but put the tennis courts at the lower right-hand corner of the usable space. They multiplied 15m (width) by 2 (2 tennis courts), and then divide the answer, 30m, by 10 (metres per square) to get 3; and multiplied 27.5m (length) by 2 (2 tennis courts), and then divide the answer, 55m, by 10 (metres per square) to get 5.5 and rounded it up to 6. They then drew a rectangle that is 3 squares wide and 6 squares long, and then divided it into two smaller rectangles along the middle.

Gloria: And then the two tennis courts and 15m by 27.5m.

Gabby: We can put them here. One two three one two three four five six. Okay.

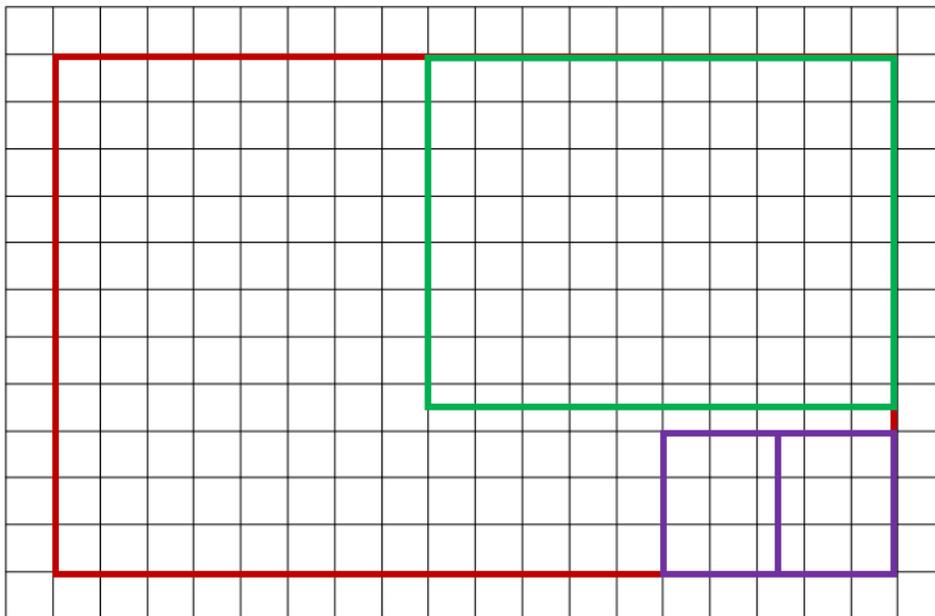


Figure 74. An illustration of group G-S's tennis courts (purple)

By now they were ready to work on the school building. Instead of following up on their initial ideas, the girls decided to design their school building from scratch. They re-read the instructions and understood that the school building needed to be 11000m², and decided that their school building should be 2 floors, where each floor has a floor area of 5500m². They also assumed that the school building is rectangular but did not provide an explanation on their decisions. They then looked for two numbers that multiply to 5500. Gloria suggested “55 times 100”, to which Gabby disliked as it would create a very “long and skinny” school building. Gabby reduced the suggested length to 90m and increased the suggested width to 75m. As Gwen punched these values into her

calculator and determined an answer and suggested rounding the answer up, Gloria recommended for the group to modify the width to 70m. She did not provide an explanation to this modification.

Gabby: Just make it 2 floors.

Gloria: 5500

Gwen: That's 55 times 100.

Gabby: That's so long and skinny. Let's make it 75 times 90.

Gwen: Six thousand and seven hundred something I'm rounding this up.

Gloria: 70 times 90 then.

Gabby: That's 6300.

After Gabby, Gloria, and Gwen finalized their design, they converted these measurements into number of squares by dividing these measurements by 10, and outlined a rectangle on the upper left-hand corner on the grid within the usable space to represent their school building. Finally, they worked on the parking lot again.

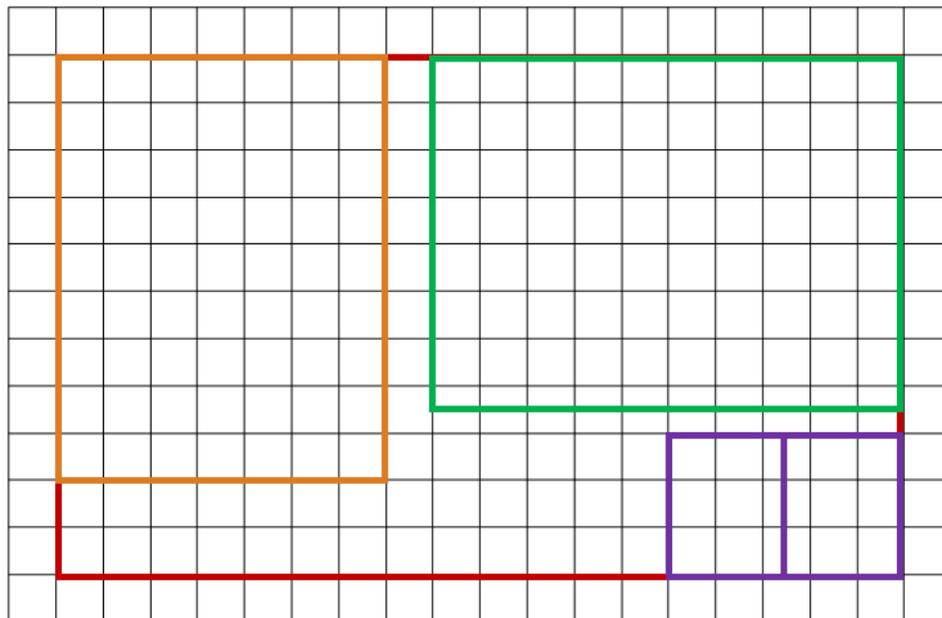


Figure 75. An illustration of group G-S's school building (orange)

Gabby, Gloria, and Gwen understood that they needed to include both parking spaces and a driveway in their design but were not certain how to proceed. Seeing that the girls were struggling, I approached them, asked them guiding questions, and made suggestions.

- T: How are the cars gonna park?
- Gwen: I dunno, just here?
- T: Can you show me the lines on the parking lot?
- Gwen: Lines?
- T: Yea...
- Gabby: Oh, like this (drew one row of parking spaces using the grid as their guideline, labeled it 180m long).
- Gwen: Oh yea.
- Gabby: And then like this (added three more sides to create a rectangle, labeled the width of the rectangle 100m long, added parking spaces to the opposite row of parking spaces).
- T: Okay so you have two rows, um, so you want two rows of parking spaces, and so people are just gonna go through the middle, so
- Gabby: Yea
- T: The middle's gonna be your driveway
- Gabby: Yea
- T: So they either turn right or left into the parking spots. So a hundred metres, a hundred and eighty metres long, how many cars do you plan to fit?
- Gabby: 30.
- T: 30? So you have the left side and the right side.
- Gabby: Yea so 18 cars, so 36 yea, 18 boxes.

The conversation revealed the girls' difficulties and the way they coped with their difficulties. Similar to group B-S, these girls used the grid as their guideline. They created an 18 squares long and 10 squares wide rectangle to represent their parking lot. Their work suggests that each parking space is 10m wide (1 square) and shows that

they understood very little about cars and parking lots. As such, I intervened and wanted to provide the group with some direct guidance. Gabby, Gloria, and Gwen refused my help.

The girls worked on the parking lot some more but were unable to determine the amount of space required to park 30 vehicles. They decided to modify the instructions and to create a space that could fit well more than 30 vehicles.

Gabby: Okay so, our parking lot is under here, underneath the school. This is our school on top, and this school has two floors, our field is going this way, and our tennis courts are right between the field and the, the, the lot boundary. And now we have space on top of the school for lawn chairs.

T: So your school is one two three four five six seven by, nine. And it's two floors, so it's more than 11000, okay? And then you are gonna have your parking lot underneath the school, as a basement.

Gabby: Yes?

T: Okay, so how big is your parking lot?

Gabby: Seven by nine.

Gwen: It's as big as the school.

Gabby: Yes.

T: How many cars do you wanna fit?

Gabby: 30. But there's a lot of space for people who can't park. (laughs)

T: Brilliant! (laughs) Can we, can we make the parking area a little smaller?

Gabby: But why would we, if it's under the school and we would have space.

T: Okay. Coz it looks like you could park like, 300 cars there.

Gwen: What if... we need like, like

Gabby: We are gonna drive donuts there.

Gloria: Yeah.

Gwen: It's one quarter of the, like, one quarter of uh, one third of the school, like basement of the school, is the parking lot.

Gabby, Gloria, and Gwen did not know how much space was needed to park 30 vehicles, but were certain that their design could fit well over 30 vehicles. Their quick chat with me led them to believe they did not break any rules and that they satisfied the instructions provided. They put the parking lot in the basement of the school building, and made it 90m long by 70m wide, the same dimensions as the school building. As I pointed out that they over-estimated the amount of space required, Gabby first tried to argue her way out, and Gwen later on reduced the area occupied by the parking lot based on my suggestion without showing further understanding of what was needed to be done.

To help Gabby, Gloria, and Gwen move forward, I asked them for a design of the parking lot again. It turned out that the girls made modifications to their previous design based on the parking lot of a local shopping centre, but they made assumptions about the length and width of the parking lot without reflecting deeply on the amount of space each parking space and the driveways require. There are 14 parking spaces along the 2 centre rows (7 parking spaces per row), 16 parking spaces along the 2 outer rows (8 parking spaces per row) of the parking lot, and 2 driveways (one on either side between the centre row and the outer row) in this design.

T: Show me some pictures.

Gabby: So this is our parking lot, like this.

T: So like a rectangular shape.

Gabby: Yeah. We got this space, like cut the road in half but stick [two rows of parking spaces] together.

T: So you, so what I'm seeing, so your parking lot has two driveways.

Gabby: Yes.

T: So, um, four rows of parking and two driveways. Okay. So the two driveways, so I can choose to turn right or left when I enter and then I can follow the driveway and park either on the left or right. So there's the U-turn, going around to, so basically it's a donut shape parking lot, where you have

Gabby: Yes.

T: Um, two rows in the middle, and two more rows on either end?

Gabby: Ahuh. And there's a lot of space in between so people can go around

T: So there's a lot of space in the driveway.

Gabby: Yes.

T: Okay

Gabby: Yes.

T: A very good design.

Gabby: Thank you.

T: So how did you come up with it?

Gwen: It's like [the mall]. You drive into the basement, you turn, and then you look for parking.

T: Okay, very good. A few questions. How wide is your parking lot. So that includes um, one two three-

Gwen: 70 metres.

T: 70 metres wide? And how do you distribute that length?

Gabby: Um...

Gwen: This, 15 metres, right? And then

Gabby: 15 metres squared

Gwen: Squared. So... uh.... I dunno.

During the conversation, Gabby began to understand that the length and width of each parking space, along with the width of the two driveways, contribute to the overall length and width of the parking lot, and that it requires much less space than what they had to park 30 vehicles. The girls later on claimed that each parking space is 3m wide and 5.5m long. They did not specify how they arrived at their assumptions. Since each parking space is 5.5m long and they planned to put 4 rows of parking spaces across the parking lot, the sum of the length of parking spaces is 22m. This plus the width of the 2 driveways (10m each) equals to the length across the entire parking lot, 42m, which they

rounded to 40m. Also, since each parking space is 3m wide and they planned to put a row of 8 spaces across the parking lot, the width of the parking lot is 24m.

Towards the end of this conversation, I felt that Gabby, Gloria, and Gwen had a fairly good understanding of what was needed to be done, and left them to consider the width of the driveway.

- Gwen: How long is [each space]?
- Gabby: 5.5m.
- T: 5.5m? So if the length of each parking lot is 5, or parking space, is 5.5m long. So that totals to 22m.
- Gwen: Is it gonna be a lot smaller?
- T: Yup. So how wide is your driveway?
- Gabby: We should give it 10m.
- T: 10m across?
- Gabby: Yeah.
- T: Okay. Very good.
- Gabby: That's 40. We should take these squares off.
- T: So 22 yea, so roughly 40 would do. So how long is your parking lot, how many spaces do you plan to put?
- Gabby: We need space here.
- T: That's right, to make the turn. And probably this here for people to enter as well.
- Gabby: So 7 here.
- T: So 7 spots in the middle (moves a finger along the parking lot), so that's 14 cars (points at the two rows in the middle), so 7
- Gabby: We can have 16 along the sides
- T: So this way, and then 8 along the sides, so 16, so that's good! So that's just 30. So how wide is this?
- Gabby: 3 each so 24
- T: 24, wide? So 24 by 40! There you go!

Gabby: That's, that's it!???

T: Yeah.

Gabby: Two by four!? So small!!!

Gwen: Um.

Gabby: Okay.

T: One suggestion though, since we are on the topic, if you are doing 7 cars [for these two centre rows], that means you are not leaving um, you are leaving very little space here, for the [width of the driveway where people need to] turn around at the front and back.

Gabby: Twenty, five metres.

T: Okay. You can add that up. Good stuff! There you go, you got this! I'm glad you made sense of this. Proud of you!

As the girls further discussed their parking lot with each other, they made modifications to their plans, specifically to the number of parking spaces along each row. They increased the number of parking spaces along the two outer rows from 8 to 10 parking spaces per row (20 in total), and decreased the number of parking spaces along the two centre rows from 7 to 5 per row (10 in total). They also changed the length and the width of each parking space from 3m to 4m and the orientation of the parking spaces in the two centre rows.

Gabby: 10 cars so that's 40.

Gwen: And across?

Gloria: 10m is too wide (refers to the driveway).

Gabby: So 5.5?

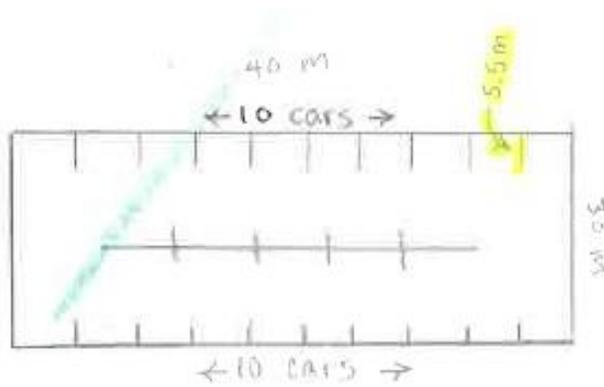
Gloria: Yea...

Gwen: So 5.5 then, that's 30.

Gloria: Okay?

Gabby: (counts and draws on the grid) So one two three four, and that's that. One two three four one two three, okay this, so there's our parking lot.

In this updated design, the width of each parking space is 4m wide. As such, the total length of the parking lot, which is based on the width of 10 parking spaces, is 40m. Other than these two rows of parking spaces, there are also 2 rows of 5 parking spaces along the centre of the parking lot. These parking spaces are rotated 90 degrees compared to the previous design. Since the length of each parking spaces along the two centre rows is 5.5m in length, 5 parking spaces take up 27.5m, which leaves 12.5m for 2 driveways, or 6.25 per driveway. The width of the entire parking lot is based on the length of 2 rows of parking spaces along the edge (5.5m each), the width of 2 rows of parking spaces along the centre (4m each), and the width of 2 driveways (5.5m each) going along the parking lot. In other words, the width of the parking lot is 30m. The following is Gabby, Gloria, and Gwen's drawing of their parking lot:



↑ Parking Lot plan!!

Figure 76. Group G-S's parking lot plan.

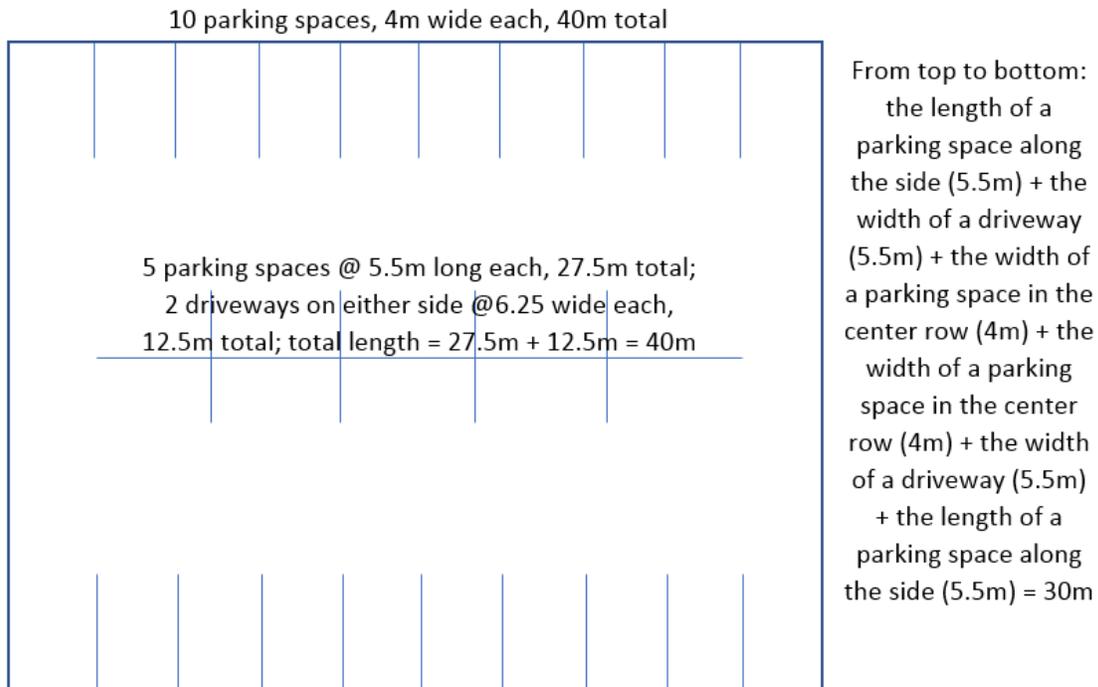


Figure 77. An illustration of group G-S's parking lot.

Afterwards, they installed the parking lot at the lower left-hand corner of their school building as an underground parking lot. Revisiting the instructions once more to make sure they have included all building structures, they realized they needed to include greenspace in their designs. To satisfy the requirement, Gabby, Gloria, and Gwen turned all leftover space into greenspace. Afterwards, they submitted their solution.

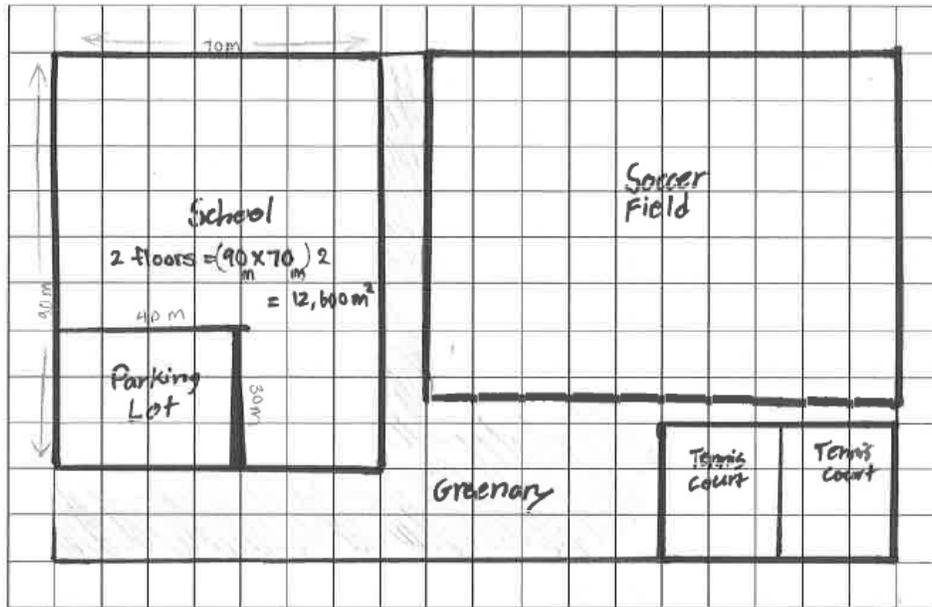


Figure 78. Group G-S’s submitted work

10.3.1. A Discussion on group G-S’s modelling process

After reading the instructions provided, Gabby, Gloria, and Gwen first focused on the school building (MC1, RS1). They applied their EMK and related the school building in the problem to “houses and apartments”, and immediately recognized that they could build a multi-floor school building (MRS2). Because of this, they never ran into the problem of insufficient land space for the building structures. However, the girls were not certain how to approach the school building beyond creating a multi-floor building. Gabby suggested to create a building with 2 floors, then 4 floors, 10 floors, 20 floors, or a building with a basement. The group also briefly discussed the possible number of classroom on each floor and the floor area of each classroom. These ideas contributed to their real model (RM3) and added to their understanding of the problem. However, I consider their real model (RM3) incomplete as these ideas did not represent a complete plan to approach or to solve the problem, and these ideas were not ready to be mathematized into a mathematical model.

The group’s work on the school building also demonstrates the role of EMK at the very beginning of a modelling cycle during the generation of a MRS. While Borromeo Ferri (2006) suggests that EMK is required as students build a real model and a mathematical model and during the process of interpretation after they generate a mathematical result,

the use of EMK is not restricted to these phases during the modelling cycle. In the same paper, she also suggests that students understand and interpret the real situation as they build a MRS. As such, the MRS could be built by associating the situation to the students' own experiences. In other words, students could have activated and applied their EMK as they built a MRS, which seems to be the case here.

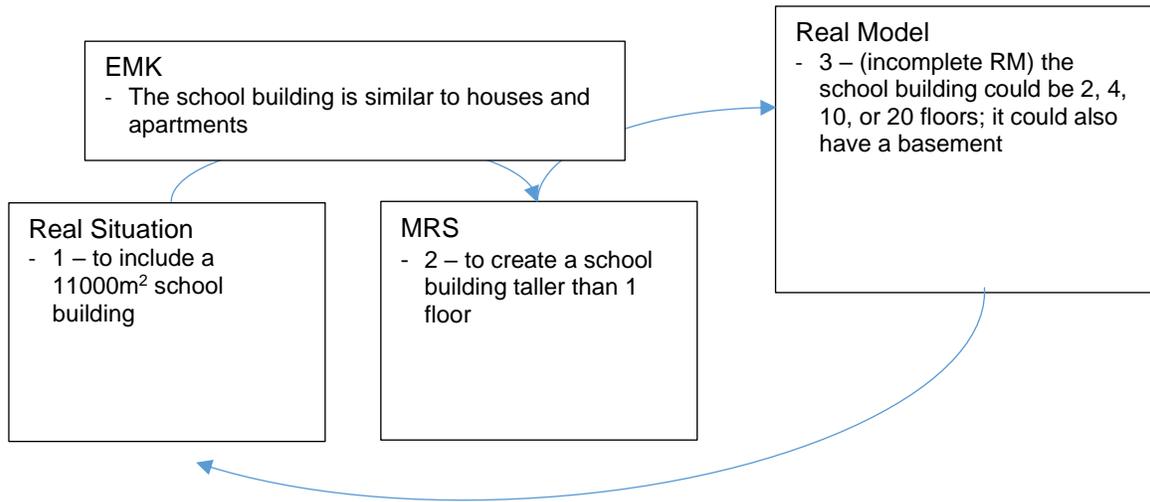


Figure 79. Group G-S's first modelling cycle (school building)

Instead of carrying on with these ideas, Gabby, Gloria, and Gwen put them aside (MC1 incomplete) and worked on the parking lot (MC2, RS4). The girls recognized that they needed to include a parking lot that fits 30 vehicles in the school ground. The group drew from their EMK and planned to create an underground parking lot and recognized that there are more to a parking lot than parking spaces (MRS5). However, they had very limited EMK which they could apply to solve the problem. Unlike group A-S, who went outside to the staff parking lot to measure the length and the width of each parking space, and group B-S, who went online and searched for the possible lengths and widths of a vehicle, Gabby, Gloria, and Gwen assumed that each parking space has an area of 15m^2 but did not provide any details nor explanation to their assumptions. Their assumption served as a way for them to simplify the problem and contributed to their real model (RM6). What the group developed here is considered an incomplete real model, as it does not contain a well-defined plan to mathematically solve the problem (Blum & Niss, 1991). Seeing the group's struggles, I asked for them to create a drawing, or a design, of their parking lot in attempts for them to visualize the various components and therefore the dimensions of the parking lot. However, instead of taking my advice,

Gabby, Gloria, and Gwen put the parking lot aside (MC2 incomplete) and worked on the two tennis courts.

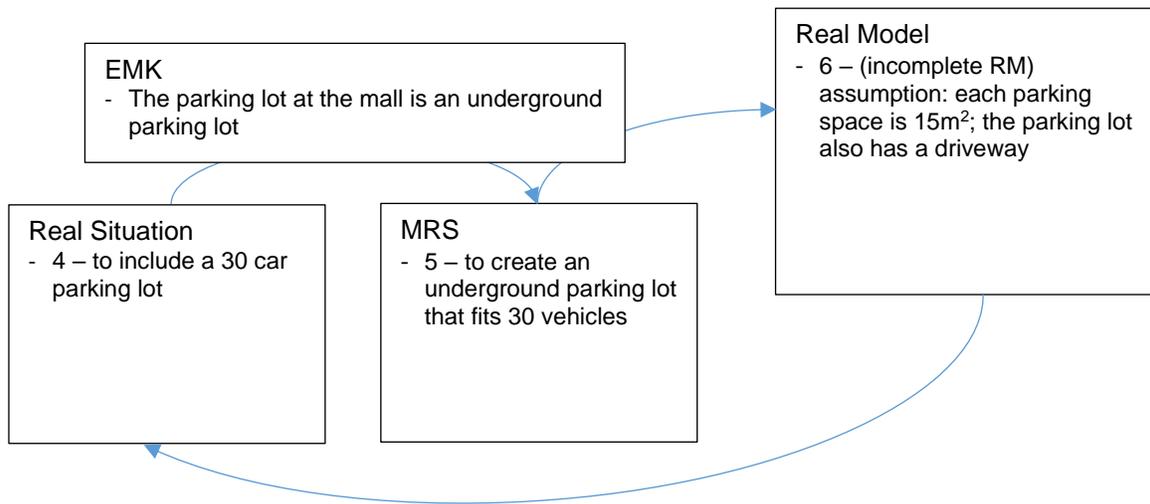


Figure 80. Group G-S's second modelling cycle (parking lot)

First, they reread the instructions (MC3, RS7). Applying their understanding of vertical height, the group recognized the possibility to put the tennis courts on top of the school (MRS8). It is unclear whether they meant to put the tennis courts on the rooftop (outdoor tennis courts) or the top floor of the school building (indoor tennis courts).

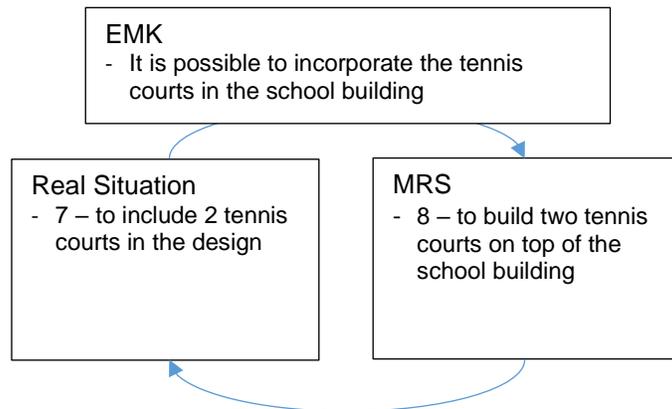


Figure 81. Group G-S's third modelling cycle (tennis courts)

While their idea to incorporate the tennis courts into the school building seems to be a good one, the group did not immediately proceed with the idea because they had not yet figured out the location and the shape of their school building (MC3 incomplete). As Gloria felt that they were pressed for time, she took charge and referred back to the

original instructions (MC4, RS9) and pushed for the group to create an outline to represent the usable space on the grid (MRS10). However, instead of following the instructions closely, Gloria made modifications and reduced the 12.5m restriction to 10m. Gloria did not provide any explanations to her modification and the rest of her group members did not object to these changes either. While the modification of the instructions could be a way for the group to simplify the problem, it is also possible that the group viewed the instructions as suggestions rather than something to which they needed to follow.

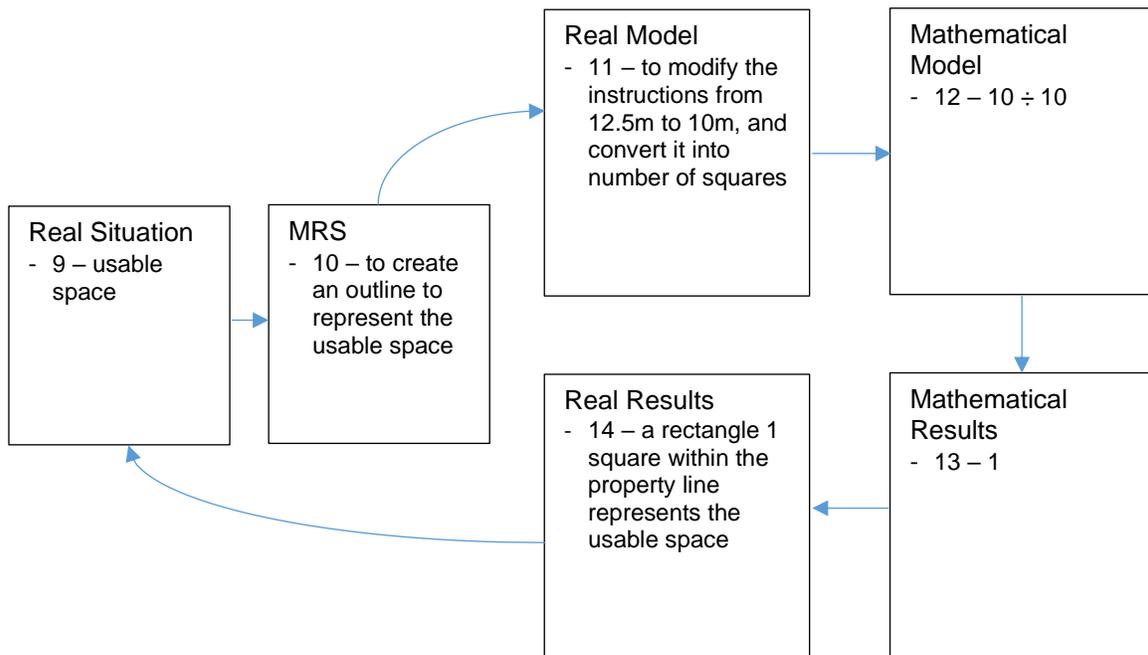


Figure 82. Group G-S’s fourth modelling cycle (usable space)

The group immediately worked on the soccer field thereafter (MC5). They re-read the instructions (RS15), and understood the task as to create an outline on the grid to represent the soccer field (MRS16). They planned to convert the measurements provided in the instructions from metres to number of squares (RM17), divided 75 and 100 by 10 (MM18), got 7.5 and 10 as answers (MR19), and outlined a rectangle that is 7.5 squares by 10 squares on the right-hand side of the grid (RR20). This concludes their fifth modelling cycle.

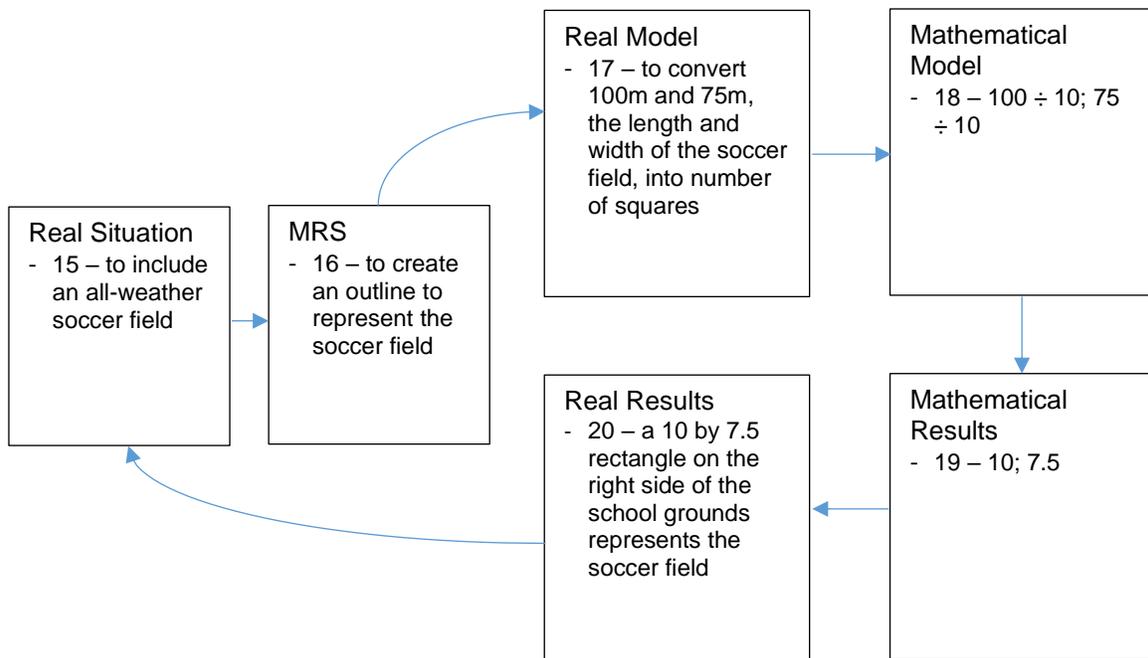


Figure 83. Group G-S's fifth modelling cycle (soccer field)

They then worked on the two tennis courts (MC6). Following the same process as the soccer field, Gabby, Gloria, and Gwen planned to convert the measurements of the two tennis courts into number of squares (RS21, MRS22, RM23). However, they made mistakes during the process and represented the two tennis courts using two 3 by 6 rectangles (MM24, MR25, RR26). Gabby, Gloria, and Gwen's actions here demonstrate their misconceptions regarding area: they multiplied both the length and the width to double the size or the area required. While this is definitely an interesting topic of discussion, it is beyond the scope of this thesis and I choose to stay focused on the analysis of modelling and not discuss this topic in this thesis.

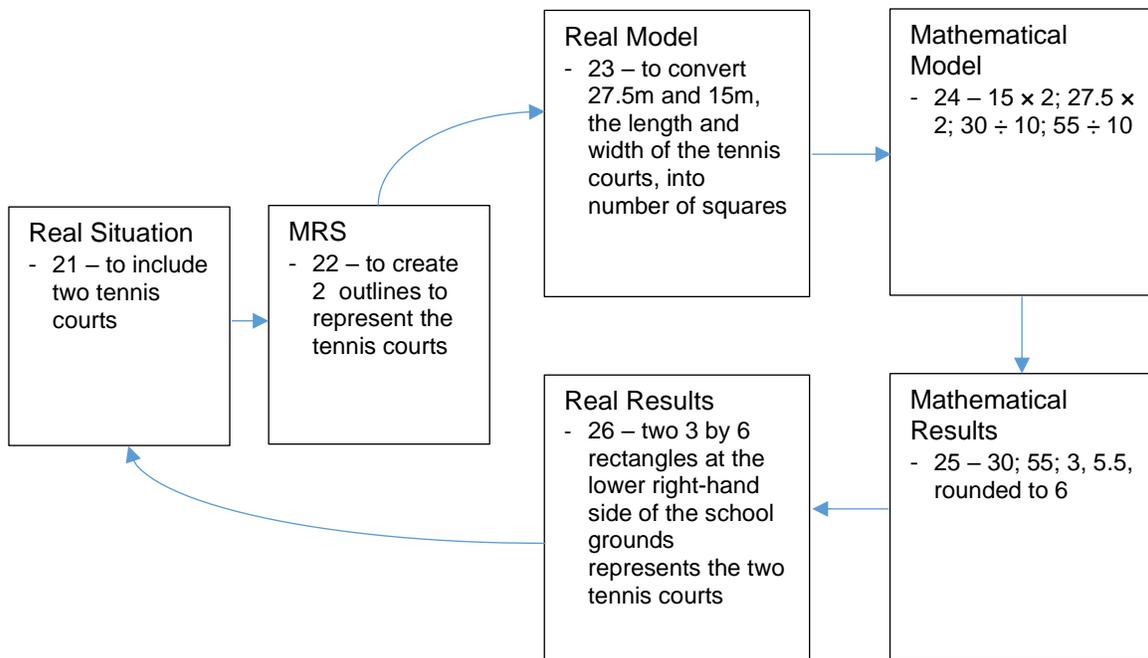


Figure 84. Group G-S's sixth modelling cycle (tennis courts)

The girls quickly moved on to the school building afterwards (MC7, RS27).

Understanding that they were to generate an outline to represent a 11000m^2 school building and that their school building could be taller than 1 floor (MRS28), they settled on a 2-storey school building (RM29). They divided 11000 by 2 (MM30) to determine the floor area, and suggested to create a 55m by 100m building to satisfy the instructions (MR31). Feeling dissatisfied with their real result (32), Gabby changed the dimensions to 75 by 90 (MM33, MR34, RR35) and eventually to 70m by 90m (MM36 and MR37). They then divided 70 and 90 by 10, and outlined a 7 by 9 rectangle on the grid to represent their school building (RR38). The girls had possibly interpreted the instructions to create an 11000m^2 as a minimum requirement rather than an exact requirement. This interpretation is also evident in their work on the parking lot. In other words, they interpreted the instructions as to create a school building that is *at least* 11000m^2 rather than one that is *exactly* 11000m^2 .

While it is clear that the girls were unhappy with their first mathematical result (MR31) as it contributes to a “long and skinny” school building, it is unclear why they were unhappy with their second mathematical result (MR34). It is possible that they felt the need to round up the generated floor area, 6750m^2 (from $75\text{m} \times 90\text{m}$), to the nearest hundreds. This thinking is evident in one of their conversations, where Gwen told the group, “[The

floor area is] six thousand and seven hundred something I'm rounding this up". Instead of going along with Gwen's suggestion, Gabby further changed the length and the width so that no rounding was required. And knowing that they now exceeded the required floor area (5500m^2), Gabby reduced the width from 75m to 70m to reduce the floor area.

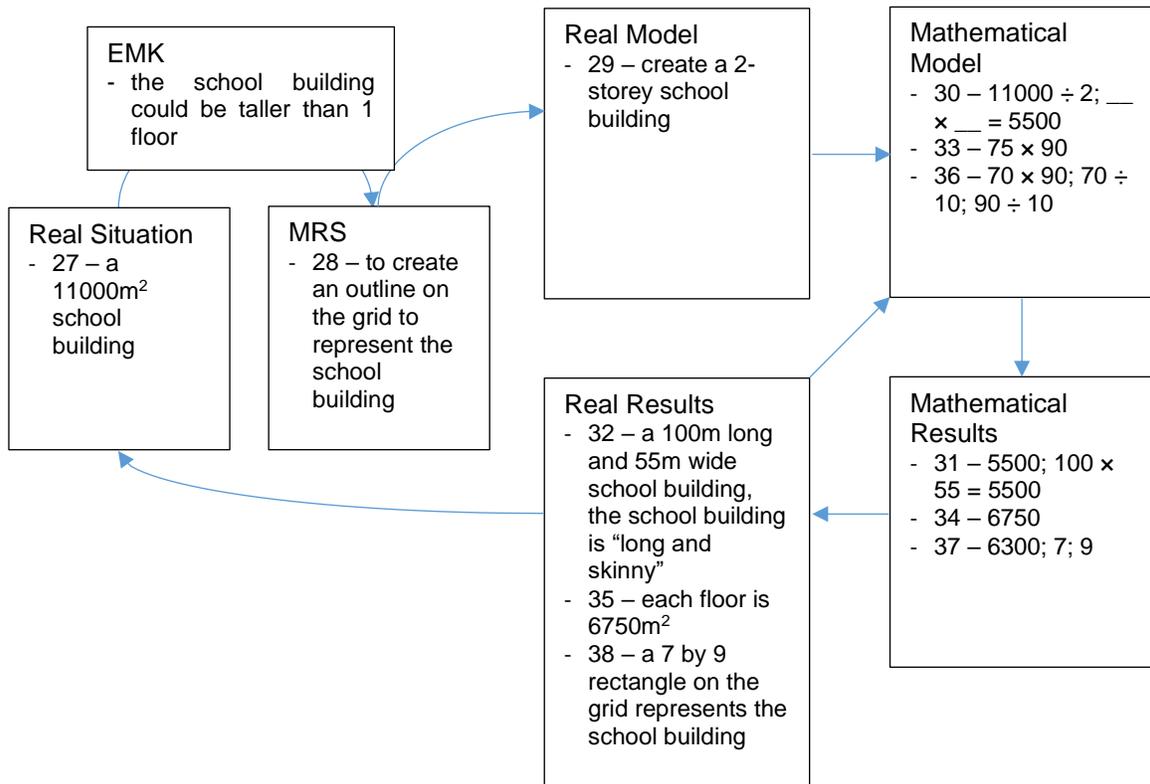


Figure 85. Group G-S's seventh modelling cycle (school building)

Finally, Gabby, Gloria, and Gwen worked on the parking lot, the last building structure of the problem (RS39). They understood they were to design a parking lot to fit 30 vehicles, and that they needed to take into consideration both parking spaces and a driveway that leads to these parking spaces (MRS40). Other than that, Gabby, Gloria, and Gwen had minimal EMK regarding the dimensions of these features found within a parking lot. As such, they used the grid provided as their guideline, just like group B-S did, and created a 180m long and 100m wide parking lot (RM/RR41). While the design was out of proportion, it served as a start. As I pointed out the flaw in their work, the girls refused my help, but decided to modify the instructions to simplify the situation and to justify their solution – they created an underground parking lot with the exact same dimensions as the school building, a space that could park well over 30 vehicles (RM/RR42).

When Gabby, Gloria, and Gwen discussed their work with me, I decided to intervene and took on an instructional role to help the girls to make sense of the parking lot and to generate a more realistic solution than the one they created. I asked the group for a design of their parking lot (RM43) and guided the group to determine a reasonable length and width for each parking space, driveway, and eventually the length and width of the parking lot (MM44, MR45, RR46). After I left the group, Gabby, Gloria, and Gwen made modifications to their designs (RM47), which led to changes in their mathematical model (MM48), mathematical results (MR49), and their real results (RR50). Finally, they read over the instructions one last time to make sure they satisfied all requirements, turned the remaining space into greenery, and submitted their work. This concluded Gabby, Gloria, and Gwen's modelling process.

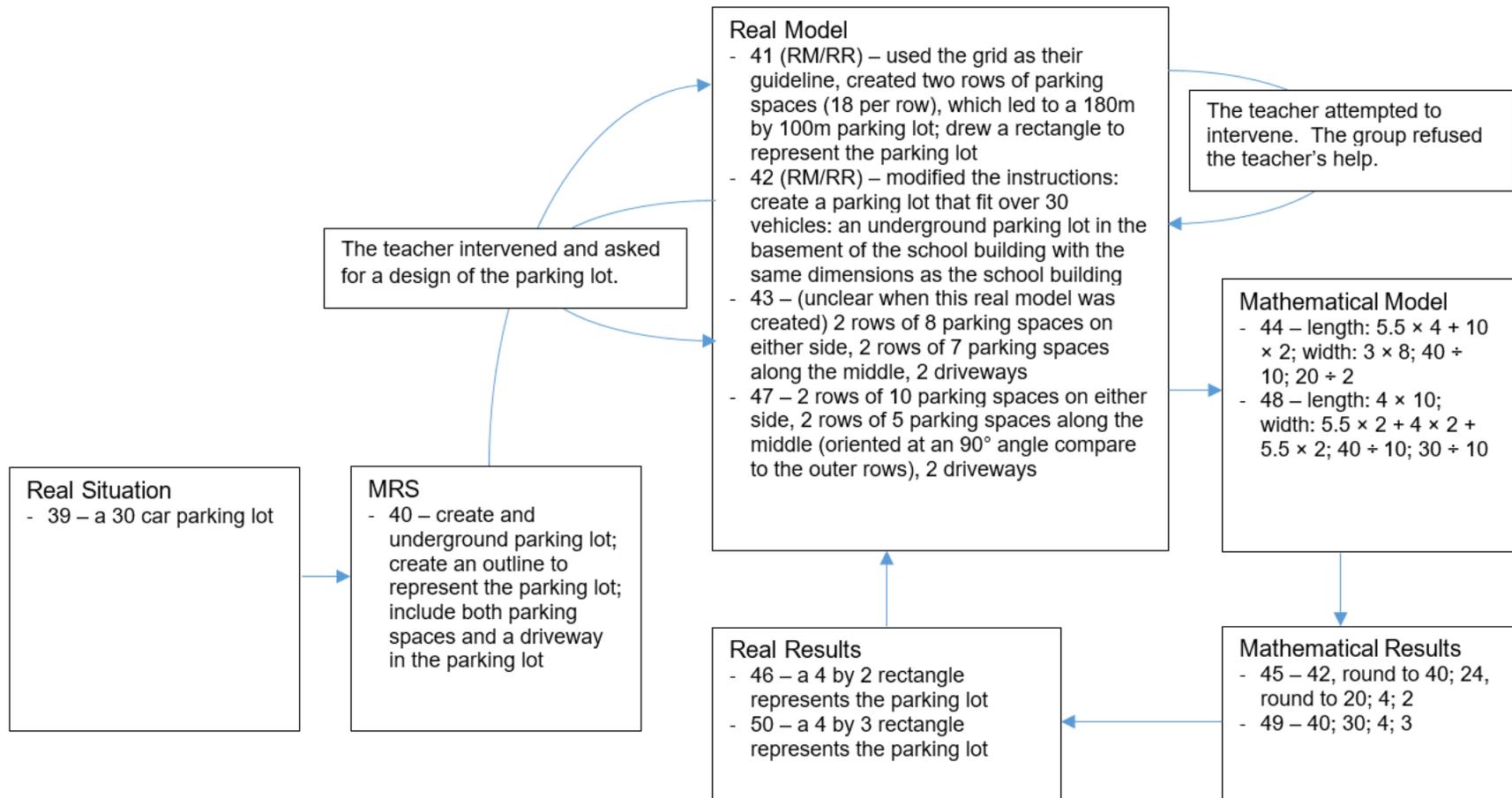


Figure 86. Group G-S's eighth modelling cycle (parking lot)

It is interesting to see the changes Gabby, Gloria, and Gwen made to their parking lot structure during this modelling cycle. They changed the width of each parking space from 3m to 4m, and the width of the driveway from 10m to 5.5m. The girls did not mention the reasons behind these changes. It is possible that they increased the width of each parking space to avoid hitting a car parked in the next parking space when opening the car door. As for the width of the driveway, they might recognize that it could be narrower than what they had previously thought.

In summary, Gabby, Gloria, and Gwen presented one of the most complex modelling process. They drew from their EMK since the very beginning of their modelling process and related the real situation to the world they live in. They attempted to gain a deep understanding of each and every aspect of the problem prior to putting anything down on the grid. However, this proved to be overwhelming and challenging, as they did not have sufficient EMK to approach all aspects of the problem. They generated a lot of ideas during their initial discussion (such as incorporating the tennis courts into the school building), but these ideas were lost as they did not make a record of them.

Throughout their modelling process, Gabby, Gloria, and Gwen treated the instructions provided as suggestions rather than rules they needed to obey. These instructions were created to push students to reflect on the situation and to approach the problem from a realistic perspective. For example, the sizes of the building structures combined with the usable area are in place for students to expand their thinking to use vertical space in their design; the area of the school building pushes students to reflect on the possible dimensions and shapes of the school building; and the 30 car parking lot forces students to reflect on the size and components of a parking lot. However, Gabby, Gloria, and Gwen consciously modified these instructions to simplify the problem or to avoid the complexity of the problem. I provide an in-depth discussion of these actions in Chapter 12.

The following is a diagrammatic representation of Gabby, Gloria, and Gwen's complete modelling process. They generated ideas on the school building (red arrows), followed by the parking lot (orange arrows), and the tennis courts (blue arrows), and then worked on the usable space (green arrows), the soccer field (purple arrows), the tennis courts (black arrows), the school building (pink arrows), and finally the parking lot (grey arrows).

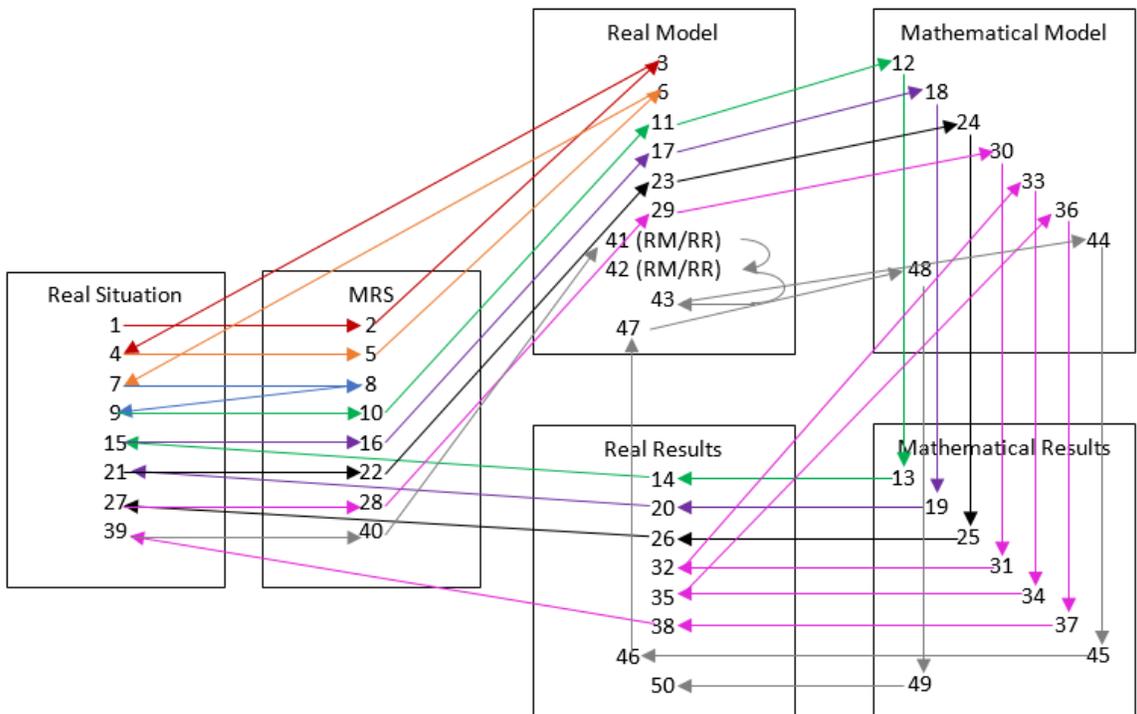


Figure 87. A diagrammatic representation of group G-S's modelling process

Chapter 11.

An analysis of the results of the *Design a New School* task

In this chapter, I provide readers with a discussion and an analysis on the *Design a New School* task, focusing on: students' multiple modelling cycle approach to the problem task; students' collapsed modelling cycles; students' use of vertical space; and EMK.

11.1. Students' multiple modelling cycles

The *Design a New School* task involves multiple building structures: the school building, the parking lot, the soccer field, and two tennis courts. Data demonstrate that participating students naturally focused on one building structure at a time, with an understanding that the task consists of multiple building structures. In other words, students broke down the *Design a New School* task into smaller problems, and their approach required them to repeat the modelling cycle multiple times to generate a complete solution. For example, group B-S first worked on the soccer field (MC1), followed by the tennis courts (MC2), the parking lot (MC3), the border (MC4), the parking lot (MC5), and finally the school building (MC6). These multiple modelling cycles focused on individual buildings are apparent in all of the cases. However, this is not to say that students treated each building structure as isolated problems. While students worked on these building structures individually, they also paid attention to how these building structures fit together to form a complete solution. For example, groups B-S and G-S created an underground parking lot in the basement of the school building; and group A-S rotated their soccer field to accommodate for their two tennis courts.

Students' decision to generate real results for each building structure one at a time is likely the results of the complexity of the problem. While the mathematics skills required to solve the *Design a New School* task is rudimentary, piecing together all the information or real results all at once is rather difficult. Therefore, it becomes logical for students to break down the *Design a New School* task into smaller, more manageable tasks and piece together the solutions to form a final overall solution thereafter than to

approach the *Design a New School* task holistically. This results in the multiple modelling cycles in students' modelling processes.

Having said that, it is not impossible to solve the *Design a New School* task by accounting for all building structures in a single modelling cycle: to create a MRS, a real model, and a mathematical model to represent the overall design, to generate a mathematical result based on the real and mathematical models, to validate the complete solution against the original problem, and to repeat the modelling cycle to improve the overall real result if needed. However, this would also be an extremely demanding process, as students would need to keep track of all the information provided in the instructions along with all their thinking, design, and work at all times.

11.2. Collapsed modelling cycles

As discussed in Chapter 4, modelling cycles begin with a real situation in reality, move between reality and the world of mathematics, and describe the process which modellers use their extra-mathematical knowledge along with their intra-mathematical skills to solve problems situated in reality (Blum & Leiß, 2005, 2007; Borromeo Ferri, 2006). Data from the *Design a New School* task demonstrate a number of scenarios where students' modelling cycles were disrupted, resulting in collapsed modelling cycles, where some of the modelling stages became deeply intertwined and inseparable from each other.

In the collapsed modelling cycles found in this study, students remained in reality and did not venture into mathematics. As such, they generated a solution without applying their intra-mathematical skills. Rather, their solution is based on their MRS, some EMK, and the tools readily available to them. In these scenarios, students generated a real result as they built a real model, making the two stages deeply intertwined.

Some examples of collapsed modelling cycles include: group B-S designed their parking lot using the lines on the grid as their guideline (MC5, see Figure 69); group B-S recognized the affordances of vertical space and created a 4-floor school building without using any mathematics to support their work (MC6, see Figure 70); and group G-S assumed each square on the grid represents one parking space in an early parking lot design (MC8, step 41, see Figure 86).

Students' modelling behaviors which resulted in the collapse of a modelling cycle are not completely comparable to Maaß's (2006) descriptions of mathematics-distant modellers or uninterested modellers. Yes, similar to mathematics-distant and uninterested modellers, these students remained in reality and did not venture into mathematics. However, unlike mathematics-distant and uninterested modellers, these students have all the required mathematical skills to construct a mathematical model and to generate a mathematical result and do not demonstrate negative attitudes towards mathematics or the tasks.

These students' modelling behaviors do not fit well with Borromeo Ferri's (2007, 2010) visual thinkers either. Acknowledging students' thinking style is affected by their EMK, Borromeo Ferri (2007) suggests that visual thinkers associate the problem with their experiences, tend to argue in realistic contexts, and switch between reality and mathematics frequently as they build a mathematical model. While the aforementioned students applied their EMK to produce pictorial representations of the parking lot, these representations were deeply affected by students' limited EMK. Also, these students remained in reality during their collapsed modelling cycles and did not apply their mathematical skills to build a mathematical model or to generate a mathematical result. Rather, their real model represents their real result.

As seen from the data, all collapsed modelling cycles involved scenarios where students had to deal with complex structures such as the parking lot and the school building. In these scenarios, the collapsed modelling cycles are closely related to the difficulties they experienced with the problem:

1) Insufficient EMK:

Students' insufficient EMK led to the collapse of the modelling cycle and greatly influenced the quality of their solutions. All students demonstrated limited EMK on parking lots and experienced difficulties with the parking lot in the question. While some students recognized their deficiencies and accessed the staff parking lot for additional EMK (group A-S), others used tools readily available to generate a solution. Groups B-S and G-S' use of the grid to create a parking lot are examples of the latter.

Group B-S had very limited EMK of parking lots and refused to use the staff parking lot to gather additional EMK during their modelling process. They were not able to find relevant EMK in their online search either. In their second attempt after a lengthy discussion with me, they used the lines on the grid as a tool to create the layout of their parking lot (MC5, steps 24-26, see Figure 69). In this scenario, they remained in reality and did not venture into mathematics, their real model is inseparable from their real result, and their modelling cycle collapsed.

Group G-S's initial work on the parking lot is another example of a collapsed modelling cycle due to insufficient EMK (MC8, see Figure 86). Similar to group B-S' fifth modelling cycle (MC5, Figure 69), group G-S used the grid to help them design a parking lot. They did not generate a real model that allows for a mathematical approach (Blum & Niss, 1991). Similar to group B-S, they did not venture into mathematics, and their real model is inseparable from their real result.

2) Additional work:

A collapsed modelling cycle is also observed in cases when students aimed to avoid additional work that was required to produce a realistic solution. Group B-S's work on their school building towards the end of their modelling process is an example – they built a 4-floor school building above their underground parking lot, but did not bother to verify that their work satisfied the mathematical requirements listed in the instructions (MC7, steps 33-34, see Figure 71). Prior to the school building, group B-S re-did parts of their work as they missed the instructions regarding the border and experienced difficulties with the amount of space available on the grid and with the parking lot. It is likely that by the time the group worked on the school building, the difficulties and frustration they experienced exceeded their perseverance (Liljedahl, 2018). As such, they avoided additional work and aimed to produce a solution quickly so they could be done with the task. As a result, their modelling cycle collapsed.

Overall, students' collapsed modelling cycles are related to the difficulties they encountered during their modelling processes. When faced with various difficulties, students tackled these difficulties to the best of their abilities and moved forward with

solving the task. These strategies, whether it was to create a solution based on their limited EMK or to avoid additional work and aim to finish the problem as quickly as possible, allowed students to get out of these difficult situations. However, not all the applied strategies led to positive results or reasonable solutions.

11.3. The use of vertical space

The *Design a New School* task provides students with an insufficient land area if all building structures are laid out on the school grounds at ground level only. As such, the incorporation of vertical space into the problem is crucial to successfully develop a solution for the task.

The school which students attended at the time of the study has 3 floors: the basement, the main floor, and the second floor. The mathematics classroom is located on the second floor. Students' school experiences, combined with their daily experiences in the three-dimensional world, contribute to their existing EMK.

I did not expect students to have too much trouble with recognizing the affordances of vertical space in creating a solution for the *Design a New School* task. To my surprise, about half of the students in this study had trouble recognizing that the task is situated in a three-dimensional world, where they could design a multi-floor school building and to incorporate the parking lot or athletic facilities into the school building. For example, it was not until Amy and Angela (group A-S) ran out of space then Amy realized that they could build a two-floor school building.

Angela: I can't fit it in!

...

R: Okay. Explain to me what you mean by you can't fit it in.

Angela: I have no more space left!

...

R: Oh there is always space.

Amy: Stack them!

R: What do you mean stack them?

Amy: Two floors!

During this particular episode, Amy experienced an AHA! moment (Liljedahl, 2005) and realized that she could create a school building that is taller than one floor. Prior to her AHA! moment, Amy and Angela approached the problem mostly from a mathematical perspective and attended to the computations of the problem, where they focused on using the numerical values provided in the problem instructions to generate a possible solution to the problem. A possible reason behind Amy and Angela's mathematical focus is their expectation of the problem. It seemed that Amy and Angela expected to apply only their intra-mathematical skills (to convert actual measurements to number of squares on the grid) to solve the problem. However, the problem required them to also draw from their EMK on top of these intra-mathematical skills to create a solution. The limited land space forced the pair to reflect on the situation and to look for an alternative way to solve the problem, which eventually led them to activate their EMK. The way they approached the problem also demonstrated that they were not aware of the potential problems. Rather, they attended to problems as they arose. Similarly, group B-S did not consider vertical space until they ran out of land space.

Conversely, group G-S had no trouble visualizing the problem in a three-dimensional world right from the beginning of their modelling process. They applied their EMK and recognized that they could create a multi-floor school building and an underground parking lot, and that they could incorporate the tennis courts into the school building. This recognition of three-dimensional space freed them from insufficient land space for building structures and allowed them to reflect on how the buildings could fit together on the school grounds.

The *Design a New School* task requires students to make use of vertical space to produce a solution. However, roughly half the students approached the problem from mostly a mathematical perspective where they attended to the measurements of the buildings but did not truly reflect on what these building structures, especially the school building, may look like in reality until they were stuck and were forced to do so.

While it is not wrong for them to focus on mathematics during a mathematics class, it is important for students to recognize the realistic aspects and to activate their EMK while working on RMCTs such as the *Design a New School* task. In what follows, I discuss

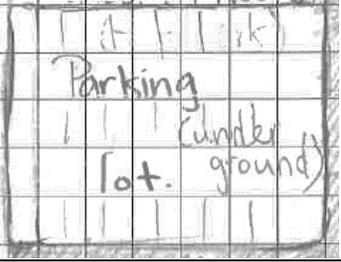
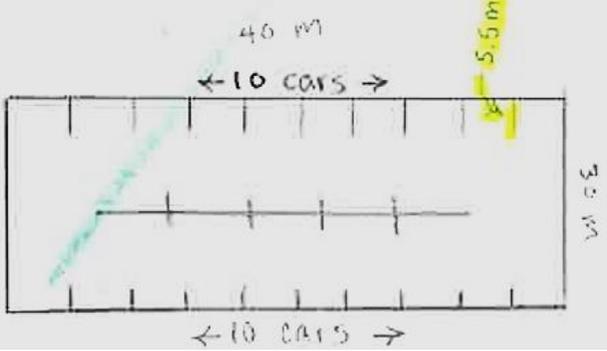
the EMK involved, including students' existing and acquired EMK, in the process of creating the parking lot and the school building, the two most complex structures in the *Design a New School* task.

11.4. Students' EMK and the parking lot

The instructions in the *Design a New School* task asks students to include a 30-car parking lot in their solution. The instructions do not ask for a layout of the parking lot. However, all students developed a layout of the parking lot during their modelling process. Some included this layout in their submitted work (groups B-S and G-S), others included a partial design and an outline (group A-S). To begin the discussion of the EMK involved in these designs, I first remind readers of the processes students took to arrive at their designs (Table 4).

Table 4. Students' submitted parking lot design

Group	Steps taken to create a parking lot and Students' submitted design
Amy Angela	<ol style="list-style-type: none"> 1) Measured the length and width of a vehicle in the staff parking lot. 2) Decided to work with parking spaces instead of vehicles, and that each parking space is 2m by 4m. 3) Recognized that they needed to build a driveway for drivers to access the parking spaces. 4) Decided all 30 parking spaces line up on one side of the parking lot. These parking spaces are connected by a driveway. 5) The parking lot is 60m long (2m × 30 spaces) and 5m (4m + 1m) wide. They also decided that vehicles would enter the parking lot at one end of the driveway and exit at the other end.  <p>Note: This is a representation of Amy and Angela's parking lot design. Amy and Angela verbally discussed their parking lot in detail and drew a few parking spaces on the grid but did not include these details in their written submitted solution.</p>
Becky Bianca	<ol style="list-style-type: none"> 1) Becky and Bianca assumed each vehicle has a measurement of 2m and multiplied 2m by 30. 2) Searched online for dimensions of vehicles. 3) Assumed each vehicle is 2m by 4m. 4) Recognized (with help) they needed to leave space between each vehicle, and they needed to include a driveway. 5) Used the lines on the grid provided to create an underground parking lot that is 65m by 50m: 3 rows of parking spaces (each 5m by 10m, 10 spaces per row), and two driveways that are each 10m wide. They also left some space (5m and 10m) at the end of each row of parking spaces but did not designate a use for these spaces.

	
<p>Gabby Gloria Gwen</p>	<ol style="list-style-type: none"> 1) Assumed an underground parking lot, where each parking space is 15m². Also recognized they needed a driveway to connect the parking spaces. 2) Used the lines on the grid provided to create a 180m by 100m parking lot (two rows of 18 spaces per row connected by a driveway that runs along the centre). 3) Made modifications to their designs based on their EMK. The new design has 4 rows of parking spaces, 2 outer rows (8 spaces per row) and 2 centre rows (7 spaces per row), and 2 driveways between the outer rows and the centre rows. 4) Assumed the parking lot is located at the basement of the school building, outlined a space of 90m by 70m. They were not certain how to distribute the space. 5) Determined (with help) the length (5.5m) and width (3m) of each parking space, the width of the driveway (10m), and the overall dimensions of the parking lot (40m by 24m). 6) Made modifications to their design, changed the dimensions of each parking space to 4m by 5.5m, and the width of each driveway to 5.5m. The overall dimension of the parking lot is 40m by 30m. 

As previously mentioned, it was assumed that students had some EMK (they have been to parking lots) and access to additional EMK (the staff parking lot on site) to construct a reasonable parking lot design. As such, I did not expect students to experience a lot of difficulties with the parking lot design. However, some students encountered tremendous difficulties during the construction of the parking lot, including the key components of a parking lot, a possible layout, the dimensions of the parking spaces and the driveway, the entrance and the exit, traffic patterns, etc.

Students' parking lot designs reflect their understanding and their EMK with parking lots. In comparing the three groups of students, group B-S had the least understanding of parking lots. Their unfamiliarity with parking lots led them to reduce the parking lot into a

mathematical word problem (initial approach). Although they received help from me later on, their limited EMK was too big a hurdle for them to successfully overcome. As such, while the design layout of their parking lot may seem reasonable (and demonstrates the application of EMK and their understanding), the dimensions of their design were not (and demonstrates their insufficient EMK and the insufficient use of resources such as the staff parking lot).

Group G-S related the parking lot in the problem to underground parking lots commonly found at the shopping malls in town (a form of their EMK), and made various attempts to simplify the task but got held back as they were not certain of the relevant EMK to complete their design. As the group and I engaged in a discussion about their parking lot design, I acted as an authoritative figure and guided them to develop a more reasonable solution than they had, just like I did with group B-S. In comparison to group B-S who benefited very little from our discussion, group G-S developed a somewhat realistic solution based on the discussion.

In comparison to groups B-S and G-S, group A-S approached the problem from a relatively realistic perspective. They recognized their deficiencies and sought additional EMK on their own. They applied their EMK and recognized the dimensions of a parked vehicle, therefore a parking space, played a crucial role in their design, but did not recognize the importance of the width of the driveway as a crucial factor. It is unlikely that they ignored the width of the driveway because they did not reflect on the problem situation. Rather, they had likely done so because they had limited EMK and did not recognize the importance of the driveway's width in their design.

11.4.1. Students' EMK related competencies

Students' success in producing a realistic real result lies beyond their intra-mathematical skills and their existing EMK. Rather, their success is also dependent on various EMK related competencies, including: the **activation** of EMK, the **identification** of relevant EMK, and the **acquisition** of additional EMK when needed. The following table (Table 5) summarizes students' EMK related competencies in their parking lot designs.

Table 5. Students' EMK related competencies

	Group B-S	Group G-S	Group A-S
Activation of EMK	N ³⁰	Y	Y
Identification of relevant EMK	N	Y	N ³¹
Acquisition of additional EMK	N	N	Y

While I have only analyzed these EMK related competencies based on students' work on the parking lot, these competencies could be generalized and applied to students' work on the rest of the problem and on other RMCTs. The activation of EMK is essential to students' modelling processes. Without this activation, students would see very little of the realistic aspects but approach the problem from a mathematical perspective (for example, group B-S' initial work on the parking lot).

The second EMK related competency is the identification of the relevant EMK to solve the task. The successful production of a real result is also founded on identifying the relevant EMK for the task, including the identification of students' existing EMK that is relevant to the task and acknowledging that they have insufficient EMK related to the task. For example, groups A-S and B-S identified vehicles as a measurement unit for the parking lot. This identification is based on their existing EMK (parking lot contains vehicles). Both groups also later on recognized the importance of the space between the vehicles and therefore adjusted their measurements to accommodate for their design.

The identification of relevant EMK is closely related to students' insufficient EMK. Students often do not process all EMK they needed to solve a task. As such, it is also crucial for them to acquire additional EMK as they see fit. The acquisition of EMK is a different skillset than the identification of relevant EMK. On top of recognizing the relevant EMK, students also need to identify the tools that would help them acquire the EMK and to possess the skills that would help them with the acquisition. For example, group A-S first identified the staff parking lot as a possible resource and a metre stick as a possible tool to determine the dimension of a parked vehicle. In this scenario, the group needed to know how to properly read the metre stick to make accurate

³⁰ Becky and Bianca eventually activated some EMK to help them with the design of the parking lot, but their EMK were limited and insufficient for them to produce a realistic parking lot.

³¹ Amy and Angela recognized they needed to measure the dimensions of a parked vehicle and extended their measurements to the dimensions of a parking space. They did not recognize they also needed to measure the width of a driveway.

measurements. However, some students identified the tools but did not successfully acquire the additional EMK required. For example, group B-S refused to measure the vehicles in the staff parking lot and searched online for the size of a vehicle, but their online search was not fruitful either.

Students' EMK related competencies (Table 5) affect the quality of their submitted solution: students who demonstrated these EMK related competencies (groups A-S and G-S) produced more realistic solutions than those who demonstrated minimal EMK related competencies (group B-S).

Table 5 also points to a common phenomenon: students often do not have all these EMK related competencies. For example, both groups B-S and G-S did not independently identify and acquire the relevant EMK to solve the problem. To deal with these deficiencies, group B-S used the lines on the grid (a readily available resource) to generate a layout of the parking lot (MC5, Figure 69) and group G-S assumed that their parking lot has the same dimensions as their school building (MC8, Figure 86). While these strategies allowed students to proceed with the modelling processes and approach the situation with some degree of realistic perspective, not all strategies led to reasonable designs. I will discuss these strategies in further details in the next chapter (Chapter 12).

Conversely, group A-S actively sought additional information that could help them with their layout: they measured the dimensions of a parked vehicle and later on identified a parking space as an important factor in their designs. As RMCTs focus heavily on the situation and demand students to reflect on the situation and to solve these tasks from a realistic perspective, the quality of students' solutions hinges predominately on students' EMK and their EMK related competencies (Table 5). As we see from students' parking lot designs, all three groups experienced difficulties as they all had limited EMK with regards to parking lots. To push forward, the groups activated and applied different degrees of EMK related competencies in their work, which resulted in different degrees of realistic parking lot designs. I provide readers with a detailed discussion of the role of EMK in students' modelling processes in Chapter 13.

11.5. Students' EMK and the School Building

The *Design a New School* task also asks students to include a 11000m² school building in their designs. Given that the floor area of the school building is the only constraint, students were given a lot of freedom to design a school building of their choice. As such, they could apply their existing EMK to their designs, or create the school building of their dreams, or based their design on various movies or fantasy, etc.

As we saw in the data, students did not make use of this freedom. Instead, they focused on the size of the building, 11000m². For some, the size of the building acted as a hurdle before they realized they could extend the building vertically (groups A-S and B-S). For others, the difficulties lie in designing a building that satisfies the constraint of 11000m² (group G-S).

11.5.1. The Shapes of students' school building designs

I expected all students to have some understanding of the general layout and the shape of a school building as they have spent a large portion of their daily lives inside a school building at the time of the study. Because of this, I also expected students to apply this understanding to their school building design. To my surprise, students applied very little school building related EMK to their work.

In general, school buildings are made of composite shapes. Given that students had classes all over the school building and had to travel from one end of the building to the other on a daily basis, I expected students to draw from these daily experiences and create a school building with composite shapes. However, all of the students produced square or rectangular shaped school buildings. When asked about their designs and their use of rectangles in their designs, Amy claimed that:

I see this as a big house. And it's rectangular. I understand our school is *L* shape. Although our school is an *L* shape, I see it as rectangular. But it's not an *L* shape though, there are many hallways. But it's a basic shape though, I'm not gonna draw it with all the different hallways. Coz I don't think of it outside, I just think of it as the hallways inside.

While Amy's observations and claims might not be entirely accurate (the school is closer to an *F* than an *L* shape), these claims and beliefs were built on her lived-experiences. First, Amy related the school building to a big house. While there are some non-

rectangular shaped houses in the city, many houses in the city are generally rectangular in shape. Also, most classrooms at the school are rectangular in shape. These classrooms are grouped by subject area and they form subject hallways. These subject hallways are also rectangular and are attached to other subject hallways at different angles. In other words, while all classrooms and subject hallways are rectangular in shape, the overall shape of the school building, which is composed of various subject hallways, is not rectangular.

What is interesting here is that while Amy demonstrated a fair understanding of both the interior and the overall shape of the school building in this conversation and discussed some of these possible designs with Angela during their modelling process, she did not apply these EMK to her designs. As she noted, she did not think of the school building as a building structure from the outside but focused on the school building's interior, which is rectangular in shape. Her focus was likely the result of her spending more of her school time indoor than outdoor, and perhaps a result of what she deemed important at school (attending classes, hanging out and socializing with her friends in the hallways, etc.). This further demonstrates the delicate relationship between students' recognition and application of relevant EMK during their modelling process and the quality of their produced solution.

11.5.2. Focusing on but not satisfying the 11000m² constraint

Students mostly approached the school building as an area problem, because the area of the school building was the only piece of information provided. In hindsight, I suspect if the problem asked students to create a school building with a specific number of classrooms or a school building to accommodate for a specific number of students (or for students' specific needs, such as a dance room, a robotics room, etc.), then it may have pushed students to consider facilities commonly found within a school building, such as lockers, washrooms, stairs, entrances and exits, an auditorium, a library, a gymnasium, etc. along with a possible floorplan of their design, because the focus of the question shifted from the area of the building to the functionality of the building. However, as the question instructed students to construct a 11000m² school building, that was what students focused on. As a result, group B-S applied little EMK to the school building other than vertical space. The other two groups (A-S and G-S) activated

and applied EMK in their discussions of the school building design. Table 6 summarizes scenarios where students applied EMK other than vertical space.

Table 6. Students applied additional EMK to the school building

Groups	Instances where students applied EMK
A-S	<ul style="list-style-type: none"> - Discussed the possibility of having a S shape or a rectangular donut shaped school - Final design: rectangular school building with unequal floor areas
G-S	<ul style="list-style-type: none"> - Pointed out that the building does not need to be rectangular in shape - Final design: rectangular school building with equal floor areas

Students' submitted school building designs point to the importance of the instructions. When the instructions provide students with a specific focus, these instructions direct students to a specific way of thinking and approach and might limit them from utilizing the freedom found in the problem and relating the problem to their EMK.

These results, of course, are not a surprise, because students will focus on what they are instructed to do. What is surprising here is that while students spent time reflecting on how to satisfy the 11000m² constraint, 2 out of the 3 groups of students' (B-S and G-S) submitted designs that did not satisfy the constraint: Group B-S decided they were to build a 4-floor square building above their underground parking lot and did not look closely into the 11000m² constraint; group G-S initially aimed to satisfy the constraint, but later on adjusted the dimensions without validating these changes against the area constraint. Students' decisions to let go of the 11000m² constraint may stem from their assumptions about buildings having equal floor areas. While these assumptions are not wrong, they put another constraint to the situation. And when students had trouble satisfying both constraints (11000m² and equal floor areas), they held on to their interpretation (equal floor areas) and let go of the 11000m² constraint so they could lower the challenge. Also, the difficulties these students experienced during their modelling processes might have led them to feel frustrated and therefore they aimed to finish as quickly as possible. Therefore, they overlooked the area constraint while generating a solution for the problem.

Overall, students' work on the school building show that they understood how a multi-floor school building would help them with their limited amount of ground space. Their submitted work also demonstrate how instructions could affect students' approach and the EMK they bring to their modelling process.

11.6. Summary

The *Design the New School* task sheds light on students' treatment of a multi-faceted task. While the task demands from the students rudimentary mathematical skills, the task is complex as it contains a lot of information and demands the application of EMK to generate a solution.

During the 75 minutes class time which students solved the *Design a New School* task, students treated the building structures in the task as individual problems. This is evident through students' multiple modelling cycles, where they focused on individual building structures in these cycles. They approached the buildings from different degrees of realistic perspectives and applied different amount of EMK in generating an outline for these buildings. This is most apparent in students' treatment of the parking lot in comparison to the other building structures. When dealing with the parking lot, all students reflected on what they know about the situation, recognized their limited EMK, and acquired additional EMK in some way: independently (group A-S actively sought additional EMK) and dependently (groups B-S and G-S relied on the help given). Their treatment of the parking lot with regards to EMK demonstrates that they solved the parking lot with the most realistic perspective amongst all buildings. Students solved the rest of the problem from more of a mathematical perspective than a realistic perspective. Students' different degrees of realistic approaches are most likely the results of the instructions. As the instructions for the parking lot focused on the number of cars and the rest focused on specific areas, students did what the instructions instructed them to do and took different approaches to solve the problem.

For the rest of this thesis, I provide an overall analysis on the strategies students applied to simplify the real situations and to validate their solutions, a discussion on the role of EMK during students' modelling process, the characteristics of EMK, and students' modelling competencies.

Chapter 12.

Reducing reality and reducing complexity

First, the problem situation has to be understood by the problem solver ... Then the situation has to be simplified, structured and made more precise, leading to a real model of the situation. (Blum & Borromeo Ferri, 2009, p. 46)

Throughout students' modelling processes, I observed a number of strategies which students employed to simplify the situation and to make the situation accessible to them. Their actions to simplify the problem are in line with modelling literature (for example, see Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2006, 2007; Mousoulides et al., 2008), and these simplification strategies led to the production of a solution. However, modelling literature only describes these strategies in general and under ideal circumstances, and do not provide details, including students' intentions behind these strategies and the results of applying these strategies. During my analysis, I differentiate between two categories of simplification strategies based on students' intentions. The first type of simplification strategies allow students to retain the messiness and complexity of the original problem. These strategies are similar to those described in modelling literature. I refer to these strategies as *reducing reality*. The second type of strategies allow students to change the nature of the problem and is different from what modelling literature describe students would do. I refer to these strategies as *reducing complexity*.

Students' application of these strategies, whether it was reducing reality or reducing complexity, rest on students' intention – to simplify the situation enough for them to approach the problem from a real world perspective, or to change the nature or the goals of the problem so they could arrive at a possible solution.

Prior to discussing these two strategies in details, it is important to also take a look at the goals of the two problems presented in this thesis and whether they communicated these goals explicitly to students.

12.1. What was being communicated?

I developed the *Door Project* with a few goals in mind. The *Door Project* aims for students to consider what it means to decorate an office door, to recognize and to consider the tools required for the job and the cost of these tools, to recognize their existing EMK and the need for additional EMK and the possibility to find additional information online or through readily available resources, to effectively acquire these information, to apply these EMK while creating a solution, and to consider and argue for the reasonableness of their solution.

Although I did not create the *Design a New School* task, I believe the task aims for students to recognize the benefits of three-dimensional space in the designs, to recognize their existing EMK and the need for additional EMK, to find additional information online or through readily available resources, to effectively acquire these information, to apply these EMK while creating a solution, and to consider and argue for the reasonableness of their solution while abiding to specific rules.

I believe the instructions and the ambiguities in both the *Door Project* and the *Design a New School* task have effectively conveyed these goals to the students. Most students in the *Door Project* recognized they needed to consider additional tools, demonstrated their own understanding of what it means to *decorate*, and acquired additional information effectively. Similarly, all students in the *Design a New School* task applied vertical space in their designs, effectively acquired additional information for their designs, and most considered the reasonableness of their solutions.

Also, the freedom or ambiguity found in the instructions have allowed students to make their own interpretations and assumptions of the problems. These interpretations and assumptions gave students space to reduce the reality, but other times it led students to reduce the complexity of the problem. As I discuss students' interpretations and assumptions, it is important to point out that I am interpreting students' intentions to reduce the reality or complexity of the question based on the data collected, including their behaviors which I observed.

12.2. Reducing Reality

Students filter information and make decisions about the real situation during their modelling processes (Borromeo Ferri, 2006). For example, students may simplify straw bales into cylinders³² (Borromeo Ferri, 2007), or imagine the Earth as a perfect sphere (Blum & Borromeo Ferri, 2009) or make assumptions about additional fixtures such as a sandbox or a pond in a garden while arranging a system of irrigation devices³³ (Blum & Borromeo Ferri, 2009), etc. These simplification and idealization processes make the problem accessible but retain the complexity of the problem: students in the straw bale example took into account the weight of the straw bales towards the top of the pile pushing down on the bottom ones (Borromeo Ferri, 2007); and students in the irrigation devices example divided the garden into subsections while determining the optimal arrangement of the irrigation devices (Blum & Borromeo Ferri, 2009). In these examples, students simplified the situation but did not reduce the messiness of reality. They remained in the realm of reality and their assumptions idealize the situation and/or removed some negotiable aspects of the situation so they could proceed to solve the problem and arrive at a solution which fits or can be used to describe the original situation. There are a number of scenarios during both the *Door Project* and the *Design a New School* tasks where students reduced the reality of the situation.

12.2.1. A Mathematical Approach

Students took mathematical approaches in both the *Door Project* and the *Design a New School* task. These mathematical approaches allowed students to simplify the problem without removing themselves from the realm of reality. For example, group A-D in the *Door Project* initially planned to tile the face of the door with gift cards, and later on created a grid to lay gift cards on the door. The way they approached the problem suggests that while their approach is mathematical (the tiling of the door and the use of a grid), they considered the messiness of reality – they measured the door using gift cards

³² Students were asked to determine the height of a pile of straw bales where there were 5 straw bales at the bottom, 4 the next level, and then 3, 2, and finally 1 straw bale on top.

³³ Students were asked to determine the optimal arrangement of a system of irrigation devices for a garden.

rather than dividing the length or width of the door by that of a gift card. This suggests that they aimed to minimize cutting the cards.

Similarly, all students in the *Design a New School* task focused on converting the measurements of various building structures from metres to the number of squares. This is especially observed in students' treatment of the border, the soccer field, the tennis courts, and the school building. In these scenarios, students' mathematical interpretation of the problem minimizes the realistic considerations of these building structures, such as the lengths and widths of their school building in comparison to school buildings found in reality and the placement of the tennis courts. Their mathematical approach to these building structures allowed them to simplify the problem and to generate a solution that satisfy the instructions quickly. I consider this reducing reality because while students took mostly a mathematical approach, they satisfied the instructions and kept the messiness of the problem: some incorporated the parking lot into the school building to reduce the ground space these buildings occupied (group B-S and G-S), some discussed the possibility to create a 2-floor tennis courts or to incorporate the tennis courts into the school building (group A-S), and some created a multi-floor school building with different floor areas to satisfy the 11000m^2 constraint (group A-S).

12.2.2. Convenient assumptions: insufficient EMK

Another strategy which students applied to simplify the situation is by making convenient assumptions about the situation. Convenient assumptions have strong ties with students' insufficient EMK, where students overcome their insufficient EMK by making convenient assumptions rather than research on the situation. These convenient assumptions are made to address the messiness of the situation, but they do not remove the messiness of the situation.

An example of convenient assumptions due to insufficient EMK is noticed in group A-S, where Amy and Angela added an additional 50cm to the width of the vehicles to accommodate for drivers and passengers opening the doors. They did not recognize the possibility to use a parking space as their unit of measurement, but assumed that an additional 50cm to the width of their measured vehicle would suffice. They recognized

the situation is messy and made appropriate assumptions to deal with the messiness of the situation.

12.2.3. Modifying the instructions

There are a few scenarios where students deliberately modified the instructions during their modelling processes. I consider one of these scenarios a form of reducing reality.

In the *Design a New School* task, the border rule was in place to force students to consider using vertical space. Without the border rule, students would have enough space for all buildings even if they were to build everything on ground level only. As group G-S worked on the task, they modified the border rule by reducing the distance between the property line and all buildings from 12.5m to 10m.

I consider their actions a form of reducing reality because it allowed them to simplify the question, to outline the usable space using the lines on the grid rather than estimating or measuring the 1.25 squares from the edge, but their actions did not remove the messiness of the question – they were already thinking to build a multi-floor school building. Conversely, if these students were to change the rule to fit all things on ground level it would have been a case of reducing complexity.

12.3. Reducing complexity

Another type of strategy which students employed to simplify the situation is reducing complexity, where they removed the founding criteria or the non-negotiable aspects of the original real situation to avoid answering the problem, or avoid reflecting deeply about the situation and to look for a possible answer to the question as quickly as possible by removing certain variables of the problem. Modelling literature do not suggest reducing complexity as a category of strategy which students use to simplify or idealize the problem. For example, when asked to fairly split the bill fairly between two people who made different amounts of purchase, students may decide to split the bill evenly but the person who had a bigger purchase buys the other lunch to compensate for the cost. When asked to decide how much money would it be fair for students to pay in addition to the amount they fundraised for a ski trip, they may decide everyone should

pay the same additional amount of money despite of the different amount they fundraised and the difference in the cost of equipment they would require for the trip³⁴.

Students' actions in the above-mentioned examples changed the criteria or the non-negotiable aspects of the original situations to avoid answering the question or to look for a possible answer to the question without fully considering the question. In the first scenario, students avoided calculating how much each person should pay based on their purchase but compensated it by asking the person with the bigger purchase to pay for lunch. Their decision allowed them to quickly find an answer to the problem while avoiding to consider and to answer the original question, to split the bill fairly. In the second scenario, students avoided the messiness of the question, to figure out how much each person should pay based on the amount they fundraised and the equipment they required for the trip. While these are possible solutions to the problems, the students avoided the messiness of the question and they oversimplified the question. I refer to these strategies as reducing the complexity of the question.

12.3.1. A mathematical approach

In section 12.2.1 I consider a scenario where students in group A-D approached the *Door Project* mathematically but only reduced the reality of the problem. In this section, I discuss two scenarios where students reduced the complexity of the *Door Project* using a mathematical approach, including group B-D's approach and group C-D's initial approach to the problem.

Both groups B-D and C-D interpreted the *Door Project* from a mathematical perspective and aimed to tile at least one face of the door with gift cards. They divided the calculated area of the face of the door by that of a gift card. In these scenarios, both groups arrived at a solution without considering any realistic aspects of their approach. Yes, it is true that they could tile more or less every inch of the door as a way to decorate the office door. However, they did not explain how they would tile the door and their approach to divide the area of the face of the door by that of a gift card suggests that they need to cut some of the gift cards into pieces, as the cards do not align perfectly on the door. In both scenarios, students removed the messiness of the

³⁴ Both tasks can be found on <http://www.peterliljedahl.com/teachers/numeracy-tasks>. The first is the *Shoe Sale* task, the second is the *Ski Trip Fundraiser* task.

problem and oversimplified the problem. They considered the problem from a mathematical perspective and did not take into account the realistic considerations. Groups B-D and C-D took a very different path thereafter. Group C-D reflected on their solution and reconsidered their approach. They recognized the realistic aspects of the problem and eventually created a mural for the front and the back of the office door and incorporated a gift card to their design. Their work suggests that students might use multiple strategies, whether it is reducing complexity or reducing reality, throughout their modelling processes as a method to simplify and to access the problem.

Conversely, group B-D stuck with their mathematical approach, even after a classmate explicitly explained to the group the difference between decorating and tiling the office door with gift cards. As such, it could be assumed that students in group B-D understood the difference, their EMK was activated, and they recognized the affordances of their EMK. However, they chose to ignore this classmate's advice and stayed with their original mathematical approach. They chose to reduce the complexity of the question by arguing for their interpretation and approach, by ignoring the realistic approach suggested, and by grounding the question in the world of mathematics.

12.3.2. Rules are meant to be broken

There are a few scenarios where students ignored the instructions/constraints given and interpreted the instructions of the *Design a New School* task to their liking to reduce the complexity of the problem. This is especially observed when students dealt with the school building and the parking lot. These scenarios are different from that discussed in sub-section 12.2.3, as they removed the non-negotiable aspects of the situation and therefore allowed students to avoid answering the original question.

In particular, group G-S did not like their initial design (a 2-floor school building with dimensions 55m×100m) but did not bother to verify their work after they modified the dimensions of their school building to 70m×90m. They deliberately avoided the messiness of the question, to create a 11000m² school building, and produced an answer based on their interpretation that it was acceptable to modify the instructions.

Also in the *Design a New School* task, group G-S wanted to assign the entire basement of their school building as their parking lot. They understood that the basement of the

school building would be too big for a 30 car parking lot. As such, they attempted to lift the 30 car constraint so their solution could be considered an acceptable solution. This is again a form of reducing complexity, as students' actions would allow them to avoid the complexity of the problem.

12.3.3. Convenient assumptions: a way out

Unlike the convenient assumptions discussed in section 12.2.2, some convenient assumptions are made to give students a quick and easy way out and to avoid redoing parts or all of their work done. In making these convenient assumptions students did not ignore or remove the constraints provided in the question. Rather, they made assumptions about things that were not mentioned in the question to get out of a sticky situation. For example in the *Door Project*, group J-D made some convenient and somewhat unrealistic assumptions about the tools required towards the end of their modelling process, where they would provide me with every tool I needed to complete the project. While these are possible assumptions about the situation, the way they handled the situation suggests that these were convenient rather than realistic assumptions which allowed the group to produce a complete solution as quickly as possible and to avoid the consideration of additional costs.

Also in the *Door Project*, group A-D assumed that the aesthetics of the decorative design does not affect their solution. One important aspect of the door project is for students to consider what it means to decorate a door. In other words, aesthetics is something which students are required to reflect on rather than to ignore. In this scenario, while students could put gift cards in any orientation they wanted, members in the group deliberately ignored the decorative goal of the problem to avoid redoing the work leading to their "C" design, including the measurements of the door, recreation of the grid, and possibly redoing their design.

Another example, also found with group A-D, is their treatment of sales tax. When Anna pointed out the problem with sales tax towards the end of the modelling process, Amanda and Andy looked for ways to omit sales tax rather than to incorporate sales tax into their solution. Their handling of the situation and their omission of sales tax suggest that they were making convenient assumptions to find a way out and to finish solving the problem as quickly as possible.

Convenient assumptions are also found towards the end of students' modelling processes in the *Design a New School* task: group B-S assumed that a 4-floor school building is sufficient, and all students except for group A-S assumed all remaining space as green space. Students' assumptions allowed them to satisfy the instructions and to produce a complete solution quickly and to avoid reflecting on how they could distribute the green space to make the school grounds attractive or aesthetically pleasing, which was an original goal of the original problem.

12.3.4. Loaded gift cards? Free cards?

The strategy *Loaded gift cards? Free cards?* is similar to *Convenient assumptions: a way out* in the sense that students use the freedom in the question to figure a way out of a sticky situation. I separate the two strategies because *Loaded gift cards? Free cards?* involves students' ethics.

The *Door Project* provides students with the freedom to determine a reasonable amount of money to spend on decorating an office door. And one of the aims of the *Door Project* is for students to articulate the reasons behind their decisions and to argue for their spending, whether it is to put loaded gift cards on the door or to determine ways to spend the money prior to putting empty gift cards on the door; or to explain ways to justify their free gift cards.

Two groups (A-D and B-D) in the *Door Project* decided that they would use loaded gift cards for decoration purposes and two (C-D and J-D) decided they could pick up empty gift cards from the stores for the project. As I reflect on students' decisions, groups B-D, C-D, and J-D's actions are arguably a form of reducing complexity based on their interpretation of the situation and the decisions made along the way.

Group B-D required 1090 gift cards in their solution and they aimed to pay for these gift cards. They noticed something did not sit right with their solution after talking to me and their peers. They used excuses in attempts to argue their way out instead of backing up their decisions with reasons or reinterpret the question and make changes to their approach and solution. These excuses allowed them to avoid rethinking and redoing the question, which includes reconsidering their approach and recalculating the cost of the problem. Their decision to pay for the gift cards is likely related to their mathematical

approach and their interpretation of the problem as a mathematical one (see 12.3.1 for a discussion). In this sense they avoided the question and reduced the complexity of the question.

Unlike group B-D, groups C-D and J-D required 1 and 22 gift cards and they decided these gift cards were free. Group C-D stated they would “steal” a card in their submitted work and group J-D stated that they would “grab a bunch of cards” from the store and “run out of there” or ask the barista for some free cards. Both groups indicated that they understood their actions were inappropriate, and group J-D laughed it off when I confronted them. At one point group J-D reconsidered their actions and proposed to purchase a \$1 drink and pick up a card while they were at it, but reverted back to their original plans to steal gift cards later on. Students’ behaviors suggest that they refused to pay for the cards and include these cards as a part of the cost, and that they refused to recalculate the cost of the problem.

All 3 groups (B-D, C-D, and J-D) produced a solution but they avoided reflecting deeply in their solution while understanding there is something inappropriate with their solution. They experienced tensions with their solution but refused to resolve these tensions. Rather, they chose to ignore these tensions. Their behaviors throughout the modelling process suggest that they understood they needed to consider the cost of the project but one group (B-D) refused to reconsider their approach and the other two (C-D and J-D) ignored ethics to achieve their goal to keep the cost of the project low. Their actions led them to reduce the complexity of the problem by ignoring the cost of the project or their ethics. All three groups refused to recalculate the cost of their project but forced their way through the problem.

I do not include group A-D in this discussion because their actions suggest that they did not consider the cost of gift cards. In hindsight, I did not ask the group to reconsider the amount of money they spent on gift cards. There is really no telling on whether they would reconsider the amount of money spent of these gift cards or if they would ignore my comments and maintained their solution.

12.4. So what causes students to reduce complexity?

One of the first things which students do during their modelling processes is to simplify the real situation to produce a real model (Borromeo Ferri, 2006). During the simplification process, students made assumptions about the situation and determined an approach to solve the problem.

My analysis shows that students applied two major forms of strategies, reducing reality and reducing complexity, to simplify the real situation. It is no surprise that students' actions to reduce reality are in line with modelling literature, where students simplified the real situation while retaining the messiness of the problem. In these scenarios, the real situation is situated in reality and students solved the problem with various degrees of realistic perspectives depending on their understanding of the problem and their EMK (including their existing EMK and the EMK they acquired).

As I reflect deeply on students' actions to reduce complexity, I notice that students reduced the complexity of the problem for reasons more than simplifying the problem. These reasons include: students' expectations of the tasks, the avoidance of work, to situate the problem away from reality, and to keep themselves in flow.

12.4.1. It's mathematics class

This study took place in students' mathematics class, where students are expected to *do mathematics*. While students may carry different ideas and understanding to what it means for them to *do mathematics*, it is safe to assume that some of their interpretations of *doing mathematics* include various mathematical problems and exercises such as word problems. It is possible that students in this study related these two RMCTs to word problems, where they needed to extract numerical values, decide on a computation, and determine a solution using the decided computation, until they recognized that the problem is situated in reality and activated their EMK. As such, students' mathematical approaches to reduce the complexity of the task are observed prior to the activation and application of their EMK (for example, groups B-D and C-D's tiling of the door).

12.4.2. Avoidance of work

Some scenarios in this study suggest that students actively avoided the complexity or the messiness of the problem. In these scenarios, reducing complexity provided students with a way out. For example, group A-D in the *Door Project* ignored the aesthetics of their design and removed sales tax to avoid redoing their design and calculations; group B-D understood that to *decorate* does not necessarily mean to *tile* but they avoided the complexity of the problem by staying within the world of mathematics; groups C-D and J-D would steal gift cards to avoid looking into reasonable ways to obtain gift cards while keeping the cost sensible; and group J-D claimed they could provide me with all the tools I needed to avoid researching on and including the cost of these tools in their solution. Similarly in the *Design a New School* task, group B-S assumed a 4-floor school above the parking lot would suffice. These students avoided finding a solution that would satisfy the 11000m² constraint.

12.4.3. Fantasy vs. reality

I also suspect students reduced the complexity of the problem when they treated the problem as one situated in a fantasy world (Freudenthal, 1991; Van den Heuvel-Panhuizen and Drijvers, 2014) or as a fantasy task (Maaß, 2010). For example, students might find that while the context of the *Door Project* was believable, it was not real. As such, students believed that they were not responsible for their proposed actions because the *Door Project* did not represent a real situation. I strongly doubt these students would go through with their actions if they were to carry out these plans to remove up to 22 gift cards from a Starbucks store without paying for them.

12.4.4. Flow Theory

Csíkszentmihályi (1990) uses the state of flow to encapsulate the essence of optimal experience, during which “people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it” (p. 4). The state of flow is only created when there exists a balance between challenge and ability.

If students' ability exceeds the challenge offered by the activity students easily become bored. Conversely, if the challenge offered by the activity far exceeds students' ability then students are likely to feel anxious and become frustrated. The balance between challenge and skill could be represented using the following diagram:

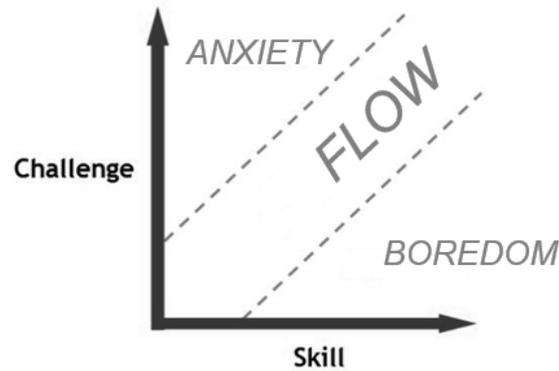


Figure 88. A graphical representation of Csíkszentmihályi's (1990) balance between challenge and skill

Liljedahl (2018) extends Csíkszentmihályi's (1990) flow theory and finds that there exists a state of tolerance between flow and boredom, where students work on repetitive tasks but do not feel bored or quit, and a state of tolerance for the mundane between flow and frustration, where students find the challenge provided exceeds their abilities but do not feel frustrated or quit. Liljedahl represents this modified state of balance between challenge and skill using the following diagram:

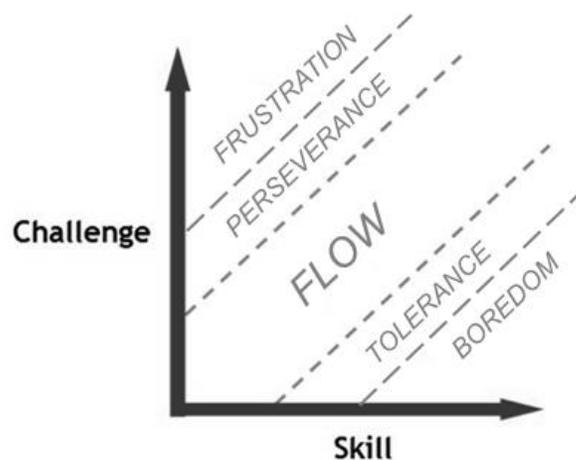


Figure 89. Liljedahl's (2018) modified graphical representation of the balance between challenge and skill

The states of tolerance and perseverance act “as buffers between flow and quitting by delaying the transition to boredom or frustration long enough for the imbalance between ability and challenge to be rebalanced. In the case of *tolerance*, this rebalancing was the result of an increase in complexity while in the case of *perseverance*, rebalancing could happen as a result of either a decrease in challenge or an increase in ability” (Liu and Liljedahl, 2019).

Within this study, the rebalance of challenge and ability is observed when students were frustrated and they were on the edge of giving up because the challenge which the problems presented were too difficult and their ability, including their intra-mathematical knowledge along with their extra-mathematical knowledge, were insufficient for them to solve the problem.

An example of the imbalance of challenge and ability includes group G-S’ work on the parking lot and the school building. Students in these scenarios reduced the complexity of the problem because they had insufficient EMK, or little understanding of the situation. As such, reducing the complexity of the problem allowed them to simplify the problem and decrease the challenge in order to produce a solution, and keep themselves in flow.

Because this imbalance have roots in students’ abilities, or more specifically their insufficient EMK, it is possible to move students from the state of frustration and perseverance back into flow by providing them with guidance and the EMK required to solve the problem so that the problem is now within their reach. This is observed in group G-S’ progress as they worked on the parking lot, where they went from designating the basement of their school building as their parking lot to a somewhat reasonable parking lot design.

However, it is not always possible to raise students’ abilities and bring students from the state of frustration or perseverance back into flow. Group B-S’s work on the parking lot demonstrates a case where students did not successfully rebalance the challenge and their abilities. During their modelling process, the group had very limited EMK to deal with the parking lot. Despite the help they received in the hopes to increase their abilities, students were riding on the edge of perseverance and frustration and they were about to give up. In this case, group B-S was not able to rebalance the challenge and their abilities. However, they did not give up on the problem either. They persevered

and finished solving the problem. Unfortunately, it was just that. After the parking lot the group further reduced the complexity of the problem and produced a solution. The imbalance of the challenge (parking lot) and their abilities had taken them out of flow.

Students' actions here were caused by the imbalance of the challenge and their abilities to determine a realistic solution. Instead of increasing their abilities, these students aimed to recreate the balance of challenge and ability and to keep themselves within the band of flow by decreasing the challenge. As such, they reduced the complexity of the problem.

12.5. Summary

Reality is messy. Students in this study purposefully simplified the situation or approached the situation in a specific way to achieve a goal. In some scenarios students aimed to simplify the situation and retain the realistic aspects of the situation. These assumptions removed some of the freedom found in the problem in order to devise a plan to solve the problem and resulted in the removal of some of the negotiable constraints of the problem. In other scenarios students made assumptions to reduce the complexity of the situation by removing the non-negotiable aspects of the situation to avoid work, to remove the messiness of reality, and to keep themselves in flow.

As I reflect on these strategies, especially reducing complexity, I do not think that reducing complexity is necessarily a strategy which we want students to completely avoid. Reducing complexity represents students' interpretation of the problem and the way which they determine a solution based on this interpretation. Reducing complexity provides students with a way out when they are stuck and a way to solve the problem. However, reducing complexity allows them to avoid the original problem, and it may remove or change the intended purpose of the original problem (such as the consideration of tools for painting in the *Door Project* and the consideration of a 30-car parking lot in the *Design a New School* task).

Does reducing complexity make the solution wrong? Not necessarily. But the results of reducing complexity often have a lot of room for improvement. In some cases students over-simplified the situation and it removed students from the realm of reality (for example, group B-D's solution to the *Door Project*). In other cases, reducing complexity

provides students with *a way* to solve the problem. It is not necessarily the way I wanted them to solve the problem or what the problem intended for students to do. But it sheds light on students' insufficient EMK and possibly their misunderstanding of the situation (for example, group G-S's assumptions about the school building and the parking lot). I do not think reducing complexity is avoidable, nor should it be avoided. Although it is not how I want students to approach and solve RMCTs, reducing complexity serves as a way for students to access the problem, and possibly a way for students to discuss their difficulties with me or with each other.

Chapter 13.

Extra-Mathematical Knowledge – a multi-faceted complex construct

I see experiences of individuals as an enrichment of their modelling route in various way (Borromeo Ferri, 2007, p. 2088)

Modelling literature use broad strokes to describe extra-mathematical knowledge, EMK, as students' non-mathematical knowledge, and imply that students naturally apply their existing EMK in their modelling processes and that their existing EMK are sufficient in the production of a real result. Modelling literature also suggest the crucial and positive role of students' existing EMK on their modelling processes, specifically when students develop a MRS and a real model, and when students interpret their mathematical results and validate their real results against the original real situation (Borromeo Ferri, 2006, 2010; Maaß, 2006).

Given that EMK is deeply intertwined with the modelling cycle, such broad-stroke descriptions of EMK seems insufficient. In this chapter, I aim to identify the facets of EMK as a complex construct by first presenting a discussion surrounding EMK related competencies. I propose that prior to the application of EMK, students need to activate and identify relevant EMK, be motivated to apply EMK, and possibility to acquire additional EMK. Students' EMK related competencies are also closely related to their engagement with RMCTs from a realistic perspective. Finally, I reflect on the EMK students applied in the two tasks and classify these EMK accordingly.

13.1. EMK Related Competencies: Activation, Acquisition, and Motivation

Borromeo Ferri's (2006, 2010) and Maaß's (2006) work suggest that students naturally apply EMK towards the beginning of their modelling cycle to produce a real result. Data in this study show that this is not necessarily the case. Students often do not recognize the relevance of EMK until later on in their modelling process, they often have insufficient EMK to solve the RMCTs, and they may choose to not apply EMK to solve the RMCTs.

13.1.1. Students' early activation of EMK

In an ideal modelling process (Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2007, 2006; Mousoulides et al., 2008), students interpret and build an understanding of the real situation and generate a MRS, and activate and apply their EMK to build a real and a mathematical model. After the generation of a mathematical result, students apply their EMK again to arrive at a real result and validate this real result against the original real situation. In other words, under ideal circumstances, students activate their EMK towards the beginning of their modelling processes and apply this EMK throughout their modelling processes.

It is true that some students recognized the realistic aspects of the *Door Project* and activated their EMK towards the beginning of their modelling processes. Group J-D in the *Door Project* is an example. They reflected on what it means to *decorate* and was the only group who did not tile at least one side of the office door with gift cards. Another example is group C-D, who considered tools that were not mentioned in the instructions at the beginning of their modelling process. However, the group got carried away as they worked on the problem and never reconsidered these tools in their modelling process. Similarly in the *Design a New School* task, group G-S activated their EMK and recognized the use of vertical space since the beginning of their modelling process and therefore did not run out of space in their design work.

Although these students activated their EMK early on during their modelling processes, this is not to say that these students did not run into EMK related problems during their modelling processes, as they considered only specific realistic aspects of the problem or they had insufficient EMK to solve the entire problem. I provide a discussion on these problems later on in this chapter. The early activation of EMK gave students the advantage of drawing from their lived-experiences and allowed them to situate the RMCT in reality rather than in the world of mathematics, which is crucial in producing a real result.

13.1.2. A later activation of EMK

Not all students activated their EMK early on in their modelling process. Many solved the real situation from a mathematical perspective and activated their EMK as they

validated their solution against the original real situation. For example, group A-D in the *Door Project* activated their EMK on what decorating the office could mean, the tools they needed for painting and adhering gift cards on the door, and the need to consider sales tax when they validated their work and recognized the realistic aspects of the problem (see Figure 16, Figure 17, and Figure 18).

Other students activated their EMK when they were stuck. In these scenarios, getting stuck forced students to reflect deeply on the real situation, which led to the activation of EMK. For example, group A-S in the *Design a New School* task did not recognize the possibility to build a multi-floor school building until they ran out of room (end of MC5, see Figure 59). Getting stuck forced the girls to reflect on the task and to stop considering it from a mathematical perspective.

These examples show that not all students activated their EMK at the beginning of their modelling processes or a modelling cycle. Students only activated their EMK when they reflected on the problem and recognized the problem is situated in reality and interpreted the problem from a realistic perspective. This could happen at any time during students' modelling processes.

The activation of EMK may lead students to identify the EMK relevant to the problem. In some cases students have the existing EMK relevant to the problem. For example, after group A-D recognized the possibility to build a school building taller than 1 floor (activation of EMK), they further drawn from their experiences that some schools are taller than others and that each floor does not need to have an equivalent floor area (identification of existing relevant EMK to be used). In other cases the activation of EMK lead students to recognize they have insufficient EMK to solve the problem. I provide a discussion on students' insufficient EMK later on in this chapter.

While an early activation is advantageous as it allows students to approach the RMCT from a realistic perspective early on in their modelling processes, a later activation of EMK allows students to reflect on their approach to solve the RMCT, to compare their mathematical approach against realistic measures, and therefore to repeat the modelling cycle in order to make improvements to their real result.

13.1.3. The specific application of EMK

During my analysis I also noticed that students applied EMK to specific aspects of the task but not the entire task. In other words, some students took into realistic considerations while solving a part of the problem but approached and solved a different part of the problem with much less realistic considerations. I will use group C-D's work as an example to illustrate the idea.

Group C-D understood they needed to consider adhesives and listed a number of possible adhesives for the *Door Project* at the beginning of their modelling process. This demonstrates that they activated their EMK, an understanding of what was needed to adhere gift cards on the door, as they started to work on the task. Immediately after, they took a mathematical approach and did not consider what it meant to *decorate* until after they tiled the face of the door with gift cards. Their work demonstrates that while they activated and applied their EMK to create a list of adhesives suitable for the task, they did not activate and apply their EMK towards decorating a door until after they reflected on their solution (see Figure34). In their modelling process, they also never applied their EMK to discuss any painting tools required to complete the project.

Group C-D's work demonstrates that the EMK activation process may not be a simple one. When students activate their EMK, they may not see how EMK could be applied to the entire problem. Rather, they might only understand how EMK is useful in a part of the problem (likely the part they were focusing on). This is possibly because students had only interpreted a part of the problem from a realistic perspective but did not realize the entire problem is situated in reality.

13.1.4. The motivation to apply EMK

The application of EMK has an underlying assumption – students are motivated to engage with the problem from a realistic perspective and to produce a realistic solution once EMK is activated. In other words, once students recognize the realistic aspects of the problem situation and the possibilities EMK offers, they are assumed to apply EMK to approach and to solve the problem from a realistic perspective. However, this is not necessarily the case – students might recognize the EMK involved but choose not to apply EMK in their modelling process. For example, group B-D in the *Door Project* took

a mathematical approach and determined the number of gift cards required to completely cover both the front and the back of the office door. During their modelling process, a classmate explicitly explained to them that *to decorate* the office door does not mean *to tile* any faces of the door with gift cards and that they could use a much smaller number of gift cards in their approach. However, group B-D chose to maintain a mathematical approach and refused to make any changes. In this scenario, it could be said that group B-D recognized the realistic aspect of the problem and their EMK was activated. However, they consciously chose not to apply their EMK and maintained their mathematical approach to solve the problem. This scenario demonstrates that in addition to recognize the realistic aspects of the real situation and to activate existing EMK, students also need to be motivated to solve the real situation from a realistic perspective prior to the application of EMK during their modelling processes.

13.1.5. The acquisition of additional EMK

The successful application of EMK during the modelling process implies a thorough understanding of the real situation and the ability to critically question the proposed real result, and to draw from one's experiences to consider various possible aspects of the real situation. As such, students with very little EMK may not be able to produce a solution as sophisticated as someone with a vast amount of EMK even though they may have similar intra-mathematical skills, unless they acquire and apply additional EMK to solve the problem.

As mentioned in the beginning of section 13.1, Borromeo Ferri's work (2006, 2010) suggests the activation of EMK at the early stages of the modelling cycle. The notion of activating EMK implies that EMK already exists, or that students have the relevant EMK to solve the problem. However, some of the EMK required in both the *Door Project* and the *Design a New School* task goes beyond students' existing EMK. As students recognized they needed to apply EMK, many also recognized their existing EMK is insufficient to solve the problem. In some scenarios, students persevered and acquired additional EMK to solve the problem. The acquisition of additional EMK is a form of rebalancing the challenge and students' abilities as discussed in Chapter 12. Unlike reducing complexity, students rebalanced the challenge and their abilities by increasing their abilities. The additional EMK allowed students to modify and improve their

solutions and to keep them in flow. Groups A-S and G-S' work on the parking lot are some examples.

However, not all students successfully acquired the EMK they needed. This could be the result of the required EMK is beyond the reach of the students' understanding (for example, group B-S' parking lot).

The *Door Project* and the *Design a New School* task demand a vast amount of EMK that are beyond students' existing EMK. As such, the successful application of EMK during students' modelling processes goes beyond the application of students' existing EMK – it also hinges on students' acquisition of additional EMK, which in turn is influenced by students' understanding of the real situation and their motivation to acquire such EMK.

13.1.6. The motivation to acquire additional EMK

In sub-section 13.1.4, I discuss students' motivation to apply their existing EMK. Similarly, as students recognized their need to acquire additional EMK, they need to be motivated to look for such EMK. Without such motivation, students might make convenient assumptions and ignore the possible contributions of the additional EMK to the quality of the generated solution. I provide two examples to illustrate the idea: Group J-D in the *Door Project* recognized they needed to include various tools but made convenient assumptions to avoid the work; group G-S in the *Design a New School* task recognized the size of the parking lot is related to the size of a vehicle but made assumptions about these measurements instead of taking an active step to acquire such measurements.

When students are motivated, they might ask for help, or independently look for what they need. For example, group A-D asked for help to validate their work and looked online for the cost of the tools required after; group A-S in the *Design a New School* task independently took measurements from the staff parking lot in order to generate a parking lot design.

Students' motivation to apply and acquire EMK and their engagement with the realistic context of the problem are closely related to Maaß's (2010) four types of modellers and to students' numeracy tools (Liljedahl and Liu, 2013). Not only do students need to be reflecting modellers who have the necessary mathematical skills and the willingness to

approach the problem task from a realistic perspective, they also need to have the appropriate and well-worn tools and the willingness to step up and use/apply these tools and to acquire additional tools to solve the problem task when needed.

Students' activation and acquisition of EMK, along with their motivation to apply EMK are important modelling competencies as they contribute to students' abilities to solve the problem task from a realistic perspective. Without these competencies students are unlikely to produce a realistic and reasonable solution.

13.1.7. Little to no EMK in the modelling process

Modelling literature and this thesis have established the importance of EMK and EMK related competencies in the quality of students' generated solutions. EMK brings in knowledge from outside of mathematics into the real situation, provides opportunities for students to consider information not mentioned in the real situation and approaches that may not be obvious, and allows for a realistic perspective of the real situation. As such, the absence or inadequacy of EMK can diminish the quality of the solutions.

Maaß (2006) suggests four types of modellers based on their attitudes towards modelling examples and mathematics, one of which being reality-distant modellers. Reality-distant modellers refer to those who hold "a positive attitude concerning context-free mathematics. They reject modelling examples and are not interested in the contexts of the real-world problem. ... [They] have problems with the construction of the real model, the validation and partially also the interpretation" (p. 138). These modellers would make unreasonable assumptions, superficial validations, and fail to connect their work with the reality context in the problem task. Maaß (2006) is unclear whether reality-distant modellers do not notice the importance of the contexts of the real-world problems and therefore focus only on the mathematics, or if they notice the importance of the contexts of the problem but choose to ignore them and focus only on the mathematics. In the following discussion I use examples found in this study to distinguish between the two types of modellers.

Students did not activate their EMK

There are a few scenarios in this study where students did not activate their EMK and therefore did not apply these EMK in their modelling processes. An example of such

scenarios includes group B-D's exclusion of painting tools and adhesives in the *Door Project*. The students in group B-D solved the *Door Project* from mostly a mathematical perspective and did not take into consideration painting tools and adhesives because they did not recognize the affordances of these tools.

Students consciously chose to avoid EMK

In scenarios where students recognized but chose not to pay attention to the EMK involved, they may end up remaining in the world of mathematics throughout their modelling cycle, and reducing the complexity of the problem (see Chapter 12). Readers can refer to 12.3.3 for a discussion where students made convenient assumptions and chose not to acquire additional EMK, and to 12.3.1 for a discussion where group B-D ignored the affordances of their EMK and maintained a mathematical approach to tile the office door with loaded gift cards.

RMCTs require students to apply both EMK and their intra-mathematical knowledge to produce a realistic solution. Students who applied little to no EMK in their modelling processes produced real results that are heavily based on mathematics and lack realistic considerations. This further illustrates the crucial role of EMK in students' modelling processes while solving RMCTs.

13.2. EMK and students' engagement

Students' EMK related competencies, including their existing EMK, their activation of EMK, and their motivation to apply and acquire EMK, greatly affect students' engagement with the problem from a realistic perspective and play a crucial role in students' successful production of a real result. Without these competencies, students would likely produce a solution that does not fit the real situation.

Students' modelling behaviors along with their submitted solutions for both RMCTs suggest that students' engagements with the problem could be categorized according to their EMK and the aforementioned EMK competencies.

Students who engage with RMCTs with realistic perspectives

Students who engage with RMCTs with realistic perspectives recognize the problem as one that is situated in reality and apply their existing EMK to solve the problem situation.

They also recognize the affordances to gather and apply additional EMK in cases where their EMK is insufficient and are motivated to do so. An example of this type of engagement include group C-D's decorative design of the door.

Students who are limited by EMK

Students who are limited by their existing EMK recognize some of the realistic aspects of the problem and EMK's possible contribution in solving the problem. They apply their existing EMK to the real situation, and they may acquire additional EMK in attempts to solve the problem. However, their work is hindered by their insufficient EMK. As such, while their solution carries some aspects of reality, they may overlook certain realistic aspects. Some examples include group A-S's work on the parking lot and students' reported volume of paint required for the *Door Project*.

Students who do not recognize EMK's contribution

Students who do not recognize EMK's contribution to RMCTs do not activate or apply their existing EMK and therefore do not engage with the problem situation from a realistic perspective. These students may hold the relevant EMK or have access to the relevant EMK to solve the problem but do not treat the RMCT as a problem situated in reality. As such, they neglect realistic aspects of the problem. Because these students have access to the relevant EMK, we can help them to recognize the realistic aspects of the RMCTs and thereby activate their EMK. Examples of students in this category include group A-D's recognition of painting tools and adhesives and group B-S's difficulties with applying vertical space to their school building.

Students who engage with RMCTs with minimal realistic perspectives

The final category includes students who have insufficient EMK and/or do not activate their EMK and/or refuse to apply their EMK. They do not engage with the problem from a realistic perspective and they do not consider realistic aspects of the question. Their produced solution is likely unrealistic. Examples of this category include group B-S's submitted parking lot and group B-D's tiling of the office door. The following is a diagrammatic representation of the four categories of modellers based on their EMK and their ability to access and apply their EMK.

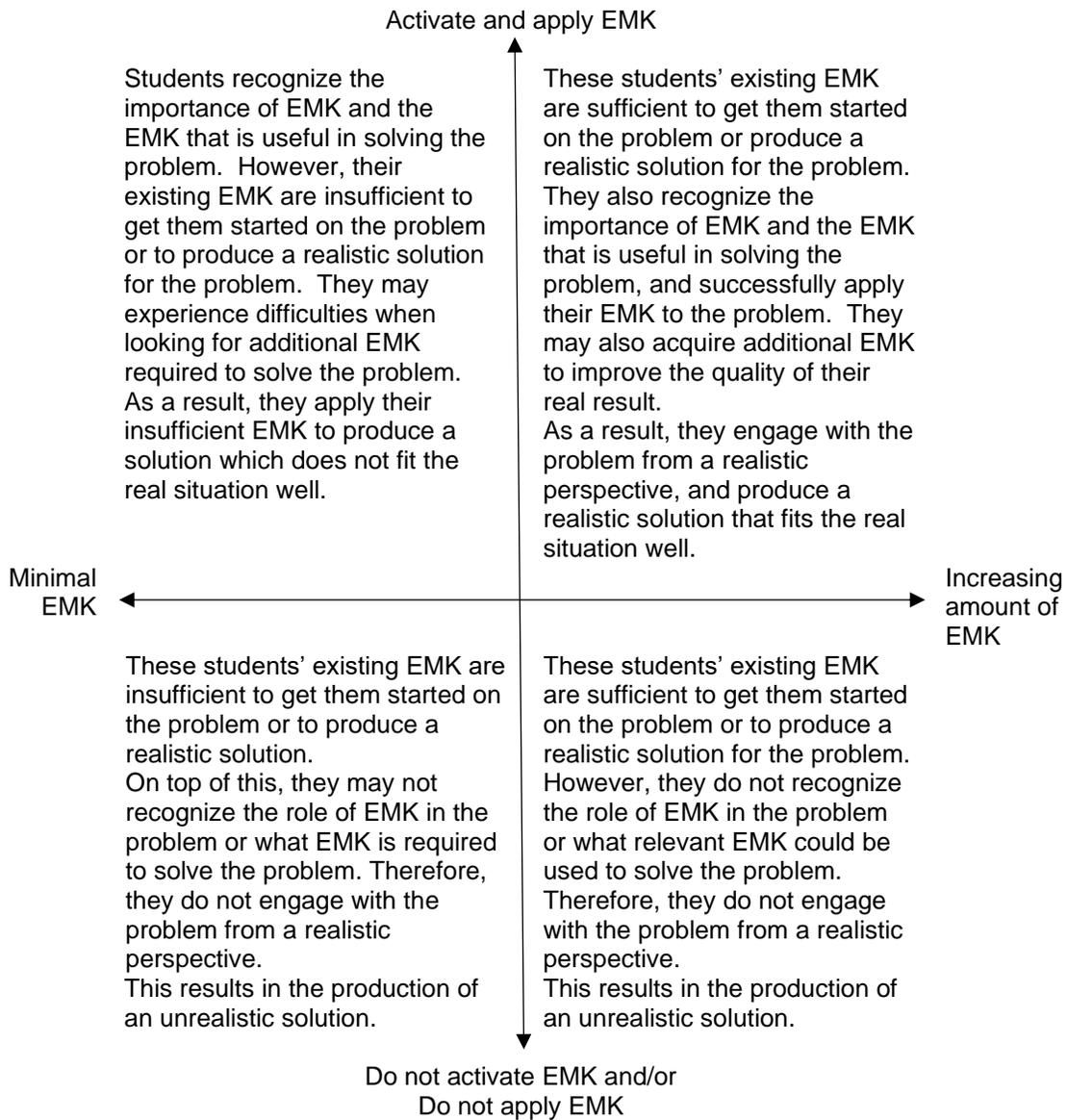


Figure 90. Students' engagement with RMCTs from an EMK perspective

Figure 90 paints a picture of students' various degrees of engagement with RMCTs based on their EMK related competencies. Under ideal circumstances, students activate their EMK towards the beginning of a modelling cycle and their existing EMK are sufficient to solve the problem (Borromeo Ferri, 2006, 2010). While this may be true for some, students often face situations where they do not have all the EMK they needed to solve the problem. As we saw from students' modelling processes and their submitted solutions from both RMCTs, the ability to acquire additional EMK plays an important role in their engagement with RMCTs and in the successful production of a realistic solution.

As I attempt to describe students' engagement with RMCTs using the categories described in Figure 90, I notice that students do not necessarily fit into a single category due to the multiple aspects found in the RMCTs used in this study. As discussed in 13.1.3, students engaged with a part of the RMCT with a more realistic aspect than another part of the RMCT. Students' engagement with the RMCT may also change when they activate or acquire additional EMK.

In other words, students approached the problem from various degrees of realistic perspectives and applied a different amount of EMK throughout their modelling processes. For example, group A-S designed parts of the school grounds (tennis courts, soccer field) from a minimal realistic perspective and produced a solution using mostly their intra-mathematical skills. However, they applied their existing EMK and acquired additional EMK to produce a parking lot design. Students' differences in realistic engagement suggest that students did not treat the overall real situation from the same realistic perspective and therefore took into different degrees of realistic considerations as they worked on the problem.

EMK plays a crucial and complex role in students' modelling process. So far in this chapter, I discuss various EMK related competencies including the activation of EMK, the acquisition of EMK, the application of EMK, the motivation to apply and acquire additional EMK, and the effects of students' EMK related competencies on their engagement with the RMCTs and the quality of their submitted solution. For the rest of this chapter, I remind readers of the EMK found in the two tasks, and then I distinguish these EMK based on their characteristics.

13.3. EMK in the two tasks

The successful production of a real result depends on students' understanding of the real situation. The *Door Project* requires students to reflect on what it means *to decorate*, to determine the appropriate amount of paint required for the project, and to consider the tools required for the project that are not mentioned in the question. Table 7 summarizes the EMK involved in the *Door Project*. I break down the EMK found in the data into 6 categories: door, decorative designs, ethics, paint, painting tools, and adhesives.

Table 7. Summary of the EMK in the Door Project

	EMK students applied while working on the <i>Door Project</i>	Groups who discussed, applied, or acquired these EMK
Door	The dimensions (physical measurements) of the office/classroom door	B-D, C-D, J-D
	Door structure (swing door vs. sliding door, solid door vs. door with a window, etc.)	J-D
	Which side(s) of the door is/are involved for decoration and painting purposes	All
	What does the door knob on the office door look like	C-D, J-D
Decorative designs and gift cards	The difference between to decorate and to tile the door with gift cards	A-D, C-D, J-D
	The possibility to incorporate gift cards into a design	C-D
	The dimensions (physical measurements) of a gift card	B-D, C-D, J-D
	The dimensions of a gift card in relation to the door	A-D
	The gift cards may not line up perfectly on the door	A-D, J-D
	The cost of a gift card – put a minimum amount in the cards	A-D, B-D
	Where to get gift cards for free – counter, friends and family, etc.	C-D, J-D
	The possible use of additional media for decoration purposes	C-D
Paint	Aesthetics – the orientation of the cards' placement	All
	Where to get paint	All
	Types of paint (interior paint vs. crayons, water colours, etc.)	All
	Cost of paint (in relations to volume)	C-D
Adhesives	Painting tools required	A-D, J-D
	How to secure the gift cards on the office door (glue, tape, wet paint, etc.)	A-D, C-D, J-D

With regards to the *Design a New School* task, a crucial extra-mathematical knowledge is vertical space. Other than that, students also need to understand the features of a parking lot, the possible features found on a soccer field, how the tennis courts normally line up, etc., to generate a realistic solution to the problem. Table 8 summarizes the EMK found in students' work on the *Design a New School* task. I break down the EMK found in the data into five categories: school building, parking lot, green space, walking paths, and overall design.

Table 8. A Summary of the EMK in the Design a New School task

	EMK found in students' work of the <i>Design a New School</i> task	Groups who discussed, applied, or acquired these EMK

School building	The school building could be taller than 1 floor	All
	Unequal floor area	A-S
Parking lot	Extra space is needed for drivers and passengers to enter/exit the vehicles when the vehicles are parked	All
	A driveway that connects all of the parking spaces is a crucial factor in the design	All
	A driveway that is wide enough for vehicles to back out of the parking spaces and to pass through is a crucial factor in the design	B-S ³⁵ , G-S
	Entrance/exit	A-S
Green space	A garden can be considered green space for the school grounds	A-S
Walking paths	The inclusion of paths that connect some/all building structures	A-S ³⁶
Overall design	The incorporation of one building structure into another	All ³⁷

Both tasks in this study require students to apply EMK that concern daily objects and to pay attention to their daily lived-experiences. While it could be assumed that participating students have encountered these daily objects, many experienced tremendous difficulties as they worked on the tasks. In the next section, I address these difficulties and classify EMK into categories according to their characteristics.

13.4. Types of EMK

The effective use of EMK is vital in students' success in solving RMCTs. In particular, the effective use of EMK provides students with a sense of direction (Maaß, 2006) of the possible realistic approaches they could take and allows students to reflect on their approaches and their generated solutions. In section 13.3, I highlight the EMK involved in the tasks. As I reflect on these EMK and the EMK that students could have applied during their modelling processes, I recognize the different characteristics of these EMK and the possibility to classify these EMK based on these characteristics. This classification scheme allows me to further reflect on the effects of students' lived-experiences and their understanding of these experiences on their modelling processes.

³⁵ Group B-S demonstrated some understanding during their conversation with me.

³⁶ Group A-S included a path that leads from the edge of the school grounds to the school building.

³⁷ Group A-S discussed the possibility of incorporating the tennis courts into the school building but eventually dismissed the idea.

Looking at the data from both the *Door Project* and the *Design a New School* task, there are at least 5 categories of EMK: everyday EMK (from everyday experiences), general EMK (common sense, some reflection required), sophisticated EMK (transformation required), special EMK of common objects, and EMK through accessing information.

13.4.1. Everyday EMK

Everyday EMK refers to the EMK students collect through their daily lives, including their experiences as they encounter with the world. I further differentiate everyday EMK into direct everyday EMK (DEMCK) and indirect everyday EMK (IEMK).

Direct everyday EMK (DEMCK) includes EMK from students' first-hand or lived-experiences; and indirect everyday EMK (IEMK) includes EMK based on students' or others' observations. Before I proceed, I will clarify and differentiate between these experiences using the following example: the hockey players on ice playing a hockey game versus the audiences watching (or listening to) the hockey game. In this example, the hockey players who are on ice playing the game have direct or lived-experiences of playing hockey (DEMCK). Conversely, the audiences who are watching (for example, in person, on TV, or through other social media) or listening to the game (for example, through radios) have indirect experiences of playing hockey. These audiences may notice how the game is played, know a lot of the terminologies and rules of the hockey game, they may be able to make accurate comments about the game, but they may have never played hockey. As they are not experiencing hockey from a first-person perspective, I'd say the audiences have indirect hockey experiences (IEMK), as their knowledge and experiences of hockey are based on their observations, including their own observations and others' observations (for example, based on what the radio host or TV commentator says).

There are plenty of scenarios where DEMCK were involved in the RMCTs. For example, in the *Door Project*, students' choice to use glue or tape to adhere gift cards on the door comes from their lived-experiences in art class or doing school projects, where they tape or glue pictures or drawings onto poster papers. Because these are students' personal experiences, they are also extremely subjective and are related to one's preference and their interpretation of the situation. For example, many opted to purchase glue and/or tape from dollar stores because that was where they usually go for school supplies.

Examples of DEMK are also found in the *Design a New School* task. One of such examples is groups B-S and G-S' underground parking lot. As mentioned, underground parking lots are commonly found at local shopping malls or apartment buildings, and it could be assumed that all students have been to underground parking lots. Students' DEMK with underground parking lots led them to incorporate their parking lot into the basement of their school building and provided them with additional space on the school grounds.

Other than DEMK, there are also scenarios where IEMK were involved in the tasks. For example, as passengers, students might have heard their parents or other drivers complained about tight parking spaces or narrow driveways. These indirect driving experiences are their IEMK. The idea of IEMK can be illustrated through group G-S' parking lot designs. Throughout the design process the group insisted on providing drivers with big parking spaces and wide driveways so drivers have an easy time maneuvering their vehicles in the parking lot. Given that these students had no driving experiences and had only experienced driving indirectly as passengers, they applied their IEMK in their design and focused on creating a parking lot which they believe suits drivers' need.

Both DEMK and IEMK play critical roles in students taking a realistic approach to solve the tasks and producing realistic solutions, as these EMK provide students with the tools to understand and approach the problem from a realistic perspective. In this study, some students had minimal DEMK and some IEMK in certain areas because they are restricted by their age, and other students had some DEMK due to the opportunities presented to them.

It is not impossible for students with limited everyday EMK (DEMK and/or IEMK) to solve the RMCTs in this study from a realistic approach and to produce realistic solutions. With regards to the parking lot in the *Design a New School* task, it is possible for students with only IEMK to understand the relationship between the size of a vehicle and that of a parking space; to notice the size of a vehicle or a parking space in comparison to other objects; to logically deduce the importance of a driveway connecting all the parking space; and to use readily available resource such as the staff parking lot to deepen their understanding of parking lots. However, without any relevant everyday

EMK, students are likely to experience difficulties to come up with these above-mentioned parking lot features or to produce a realistic parking lot layout.

13.4.2. General EMK

General EMK is related to everyday EMK in the sense that they may arise from students' everyday experiences. I differentiate general EMK from everyday EMK because while everyday EMK are based on students' everyday experiences, general EMK describes students' sense of the world around them. General EMK is similar to common sense. It requires students to reflect on their everyday experiences and make sensible decisions based on these experiences. In other words, having everyday EMK does not necessarily mean having general EMK.

I use an example in the *Door Project* to illustrate the importance of general EMK: Groups A-D and B-D decided to use loaded gift cards to decorate the office door. While this is not a wrong or unacceptable solution, it is not a sensible one, because they could spend the money on food, drinks, or merchandise prior to using the cards for decoration purposes. The choice to use loaded gift cards to decorate an office door could be viewed as a waste of money, and some may argue that this is similar to using real money to decorate the door. In these scenarios, these students did not make sensible decisions and did not demonstrate their use of general EMK during their modelling process.

During my reflection on general EMK and in my attempts to describe general EMK as one's sensibility or common sense, I find the process challenging, as general EMK is dependent on people's world view and understanding of the world around them, and it varies between people and may change over time.

To describe students' use of general EMK to determine what is sensible, a better description than common sense might be whether students are able to create logical and defensible arguments to support their decisions. Going back to the example where students used loaded gift cards to tile the office door, group B-D attempted to justify their spending's by downplaying the cost and claimed that they were *only* spending "five thousand four hundred and fifty loonies" and argued that they should spend the money because the door is "a portal to [an] amazing room". These descriptions clearly lacked

logic, as “five thousand four hundred and fifty loonies” is no less than \$5450, and “a portal to [an] amazing room” does not demonstrate any sensibility to their spending. Therefore, it could be said that these students did not demonstrate their general EMK in determining the cost of the gift cards required to decorate the office door as they were unable to provide a logical or reasonable explanation to their decisions.

General EMK is vital in deciding what is reasonable in general. It arises from one’s interpretation of the world around him/her, and requires one to be sensible to various situations in order to make logical decisions.

13.4.3. Sophisticated EMK

Sophisticated EMK arises from one’s everyday EMK but are more complex than everyday EMK. Sophisticated EMK can be considered a type of transformed everyday EMK. It requires one to reflect on his/her everyday experiences and represent these experiences in a different form. For example, all participants in the *Design a New School* task are capable of travelling from point A to point B within the school building. This represents their everyday EMK, as it is made up of their lived-experiences. Sophisticated EMK involve students using their everyday EMK to, for example, produce a floor plan or an outline of the school building. It requires students to reflect on and to transform what they know into, for example, a drawing of the school building in bird’s eyes view.

In an exercise where students were asked to make proper measurements to create a scaled floor plan of the school building, most were surprised by the ratio between the length and the width of the school building. This exercise demonstrates that while all students have everyday EMK (they know the school building like the back of their hands), they have trouble transforming this everyday EMK into a different perspective and representing this transformed knowledge (sophisticated EMK) in the form of a scaled floor plan. This exercise also explains students’ design of the school building, the tennis courts, and the soccer field from mostly a mathematical perspective. If they had transformed their everyday EMK into sophisticated EMK and had applied this EMK to their modelling processes, their designs might contain more realistic measures than the ones they submitted.

13.4.4. Special EMK of common objects

Similar to everyday and general EMK, special EMK of common objects are audience specific. It involves an understanding of a common object/place (such as an interior door or a parking lot) or a common experience (such as school). However, because these are ordinary objects found in people's everyday lives, it is not something that people would commonly notice. Such EMK are noted in both the *Door Project* and the *Design a New School* task. For example, while an interior door is a common object, it is unlikely for students to know its exact dimensions unless they have specific experiences which require such knowledge (for example, to fit a large object such as a sofa through a door). The same goes for the length and width of a gift card, a parking space, a school building, the way tennis courts line up, etc.

The application of relevant special EMK can positively affect students' modelling processes and the quality of students' real results. This is not to say that students with insufficient special EMK are unable to produce a reasonable solution, because students could acquire additional EMK from various resources.

13.4.5. EMK through accessing information

EMK through accessing information refers to the EMK students acquired from available resources. This includes knowing what questions to ask and what needs to be done to acquire additional information, and to reflect on one's approach or strategy and to act accordingly. In other words, EMK through accessing information includes metacognitive modelling competencies (Maaß, 2006) – having a sense of direction and being reflective on one's approach or strategy. On top of these metacognitive modelling competencies, EMK through accessing information is also closely related to one's research skills. For example, group A-S in the *Design a New School* task used the vehicles in the staff parking lot to help them determine the lengths and widths of a vehicle. Their ability to use resources available to them to find the information they needed to proceed led them to additional EMK. Other EMK through accessing information that could be useful in the *Door Project* and the *Design a New School* task include: knowing how to determine the minimum cost of a Starbucks gift card; knowing that one could use the floor plan of the school building to generate an outline of the school building; knowing where to find a floor plan of the school building; etc.

13.5. Summary

Modelling literature do not provide detailed descriptions of EMK but describe EMK as any knowledge or experiences that originate from outside of students' mathematical experiences and assume that students activate and apply their EMK throughout the modelling cycle.

In this chapter, I reflect on EMK and realize the complexity of EMK goes beyond what modelling literature propose. EMK is a multi-faceted complex construct. These facets include:

- 1) EMK related competencies: students' activation of EMK at different stages of their modelling processes, their specific application of EMK and their motivation to apply EMK, and their acquisition of EMK along with their motivation to acquire additional EMK;
- 2) students' engagement with the RMCTs, which makes explicit the relationship between students' existing and acquired EMK and the activation and application of these EMK (Figure 90); and
- 3) the various categories of EMK, which include but are not limited to: everyday EMK (direct and indirect EMK), general EMK, sophisticated EMK, special EMK of common objects, and EMK through accessing information. Table 9 summarizes these EMK categories discussed in this thesis.

Table 9. A summary of the characteristics of EMK found in this study

	Refers to
Everyday EMK	Everyday experiences; including direct (DEMK) and indirect (IEMK) everyday experiences
General EMK	Have roots in everyday EMK; allows one to make sensible and reasonable decisions based on one's interpretation of the world around him/her
Sophisticated EMK	Have roots in everyday EMK; reflect and transform one's everyday EMK into a different form
Special EMK of common objects	May not be common knowledge, dependent on one's experiences
EMK through accessing information	When one recognizes s/he lacks a particular type EMK and search for additional information using various resources available

As expected, RMCTs highlight the importance of EMK in students' modelling process. This is not to say that students' intra-mathematical skills are of no importance, as these skills are still required for students to produce a mathematical result. Rather, RMCTs allow students to shift their focus and energy away from mathematics and from looking

for or learning the appropriate mathematical tools to the complexity of the real situation. Both tasks in this study demand students to apply EMK and to acquire additional EMK to make sense of the real situation and to create realistic solutions. In turn, students' intra-mathematical skills support their modelling processes by enabling them to mathematically solve their generated mathematical models.

Chapter 14.

Students' validation processes

an answer to a word problem is considered correct if the student finds the applicable algorithm and carries it out successfully. But mathematical modeling demands more. [...] the real-world results must also be reasonable in that they check with the real world. (Pollak, 2011, p. 64)

Modelling literature suggest that validation happens towards the end of a modelling cycle, during which modellers interpret their real results in terms of the original situation and determine whether the generated solution fits the original situation (Borromeo Ferri, 2006; Galbraith & Stillman, 2006; Mason & Davis, 1991). The validation process is a form of procedural metacognition (Maaß, 2006). It provides modellers with a chance to reflect on their real results, and to possibly consider other realistic aspects of the problem. As such, the validation process requires an understanding of the real situation from a realistic perspective and is heavily dependent on the application of EMK. The validation process also informs modellers whether they should accept or reject their real results. If modellers choose to accept their solution, they complete their modelling process. If modellers choose to reject the solution, they may repeat parts or the entire the modelling cycle to improve their solution. Literature suggest that novice modellers seldom validate or reflect on their solution against the original situation (Blum & Leiß, 2006; Borromeo Ferri, 2006; Galbraith & Stillman, 2006) and sometimes modellers validate their solution in a superficial manner (Maaß, 2006).

14.1. Lack of validation

Many students in this study did not validate their work against the original real situation and submitted their work after they generated a complete real result. The lack of validation is seen mostly in the *Design a New School* task. Students did not consider how their building structures fit into the overall design of the school ground and they did not articulate reasons behind the locations and designs of their building structures. Rather, they focused on completing the task and submitting their work when they were done. Similarly, some students in the *Door Project* did not validate their work. For example, group C-D quickly submitted their work after they finished calculating the cost of paint. They did not look back at the notes they wrote about adhesives and therefore

did not include adhesives in their submitted solution. I suspect if they had done so, their notes would have led them to look at the problem more closely and considered painting tools in their work as well.

14.2. Superficial validation

Many students in this study did not validate their work against the real situation by considering the contextual meaning of their real result to determine whether their solution is a realistic solution that fits well with the real situation (Blum & Leiß, 2005, 2007; Borromeo Ferri, 2006). Instead, many checked to see that they completed the problem and satisfied the instructions based on their understanding and interpretation of these instructions. In other words, students had only made sure they have finished the problem, but they had not reflected on and made sense of their solution in a contextual basis. This type of validation is similar to Maaß's (2006) superficial validation, where students made sure calculations are done correctly but do not consider whether the produced solution fits the original situation. I use the following two examples to illustrate the idea.

All students in the *Design a New School* task made sure they included a 30-car parking lot in their submitted solution but did not consider their solution from a realistic perspective, such as how vehicles were to enter the parking lot from outside of the school grounds. Also in the *Design a New School* task, students included all the building structures in their submitted solution, but most did not consider how their designs could contribute to a positive school environment. In the *Door Project*, some students submitted a decorative design and included a can of paint in their real result but did not consider additional tools such as paint brushes and/or adhesives for the project. In these examples, students made sure their generated results satisfied the instructions but did not validate their results against the real situation. These superficial validation prevented students from improving their real results – to further apply their EMK and to consider additional realistic perspective. Given that participants in this study were novice modellers, these results are in line with modelling literature (Galbraith & Stillman, 2006; Maaß, 2006).

14.3. Students' validation process

Validation can happen unconsciously, where modellers validate their work based on their intuition, or feelings and reasons which they cannot clearly articulate (intuitive validation), and consciously, where modellers apply their EMK and compare their real results to the real situation (knowledge-based validation) (Borromeo Ferri, 2006).

For those who validated their work in this study, some relied on their intuition and others applied their EMK. These results are in line with literature. What is interesting about students' validation process is how they went about their intuitive or knowledge-based validation processes: some students in this study validated their solution independently within their groups, and others relied on me to help with their validation process.

14.3.1. Independent validation

Of the students who validated their work independently, some did it based on their intuition and others based on their knowledge. Students who validated their work independently based on their intuition sensed that something was wrong with their solution but had trouble articulating their feelings and/or the reasons behind these feelings. These feelings and/or reasons which they could not clearly articulate pushed them to look into their solution and the work they took to arrive at their solution. For example, after group A-D determined the number of gift cards required to tile the face of the door and discussed the orientation of the cards to be placed on the door (MC1), the girls sent Andy to re-measure the door to verify his measurements. This scenario could be classified as a form of intuition validation as the group did not provide any reasons but believed Andy might not have accurately measured the door and sent him to retake these measurements. They did not demonstrate any evidence where they used their understanding or their knowledge regarding the dimensions of the door and the gift card to question Andy's original measurements.

Other than their intuition, students also validated their work independently based on their knowledge, including their EMK and their understanding and interpretation of the problem. In these scenarios, students reflected on their solutions and identified the area(s) which they could improve on. In some cases this led to further activation of their existing EMK, and in some cases this led students to acquire additional EMK. An

example of independent knowledge-based validation is found near the end of group A-D's modelling process, where they discussed their generated real result among themselves. They applied their EMK and discussed sales tax (the need for and the avoidance of sales tax). Group A-D's work also demonstrate the fluidity of students' validation processes as they applied both validation strategies (intuition and knowledge-based) throughout their modelling process.

Borromeo Ferri's (2006) descriptions of knowledge-based validation suggest that modellers carry with them the relevant EMK they need to validate their solutions and this EMK has already been activated. This, of course, are the ideal descriptions of the modelling process. As discussed in sub-section 13.1.2, and as seen in the previous example, students sometimes only activate their EMK during the validation process. This study also finds that students often do not possess all relevant EMK to solve the RMCTs. In cases where students recognized they had insufficient EMK during knowledge-based validation, students applied their existing EMK or they acquired additional EMK (see 13.1.5 for a discussion). In other cases, students applied their existing EMK to reduce the complexity of the problem (see Chapter 12 for a discussion).

14.3.2. Dependent validation

Some students sought help in validating their work: some asked me to come to their group to validate their work with them, others initiated a conversation with me as I walked by their groups. In these scenarios, students' validation process is dependent on the help they found or provided.

Students who validated their work dependently in this study did so based on their intuition. As students experienced unsettling feelings with their solutions and sensed something was off about their solutions but were unable to clearly point out the problem, they asked for help and/or depended on others to validate their work for them. An example of dependent intuitive validation is found in group B-D's work after they calculated the cost and number of gift cards required. The group sensed that something was off but were not able to make sense of it. As such, they called me over to validate their work. Although the validation process did not lead them to solve the problem from a realistic perspective, it helped them to realize they only attended to a part of the problem.

Another possible type of dependent validation is knowledge-based validation. Dependent knowledge-based validation could be described as scenarios where students recognize they don't have sufficient EMK to validate their work but know what knowledge is required. Instead of looking for the knowledge on their own, they rely on others to provide them with the EMK or to help them validate their work. Examples of knowledge-based dependent validation is not found in this study, because the classroom culture encourages students to find out more about problems and attempt to solve problems on their own rather than to rely on me or others for answers.

Students' reliance on me to help them validate their solution might be related to their experiences in solving RMCTs. As participating students are novice modellers, they might feel uncertain whether their solution was acceptable or required improvements. Students might become more confident in validating their work on their own as they gain experiences. However, further studies are required to gain a deeper understanding on the differences in how novice and expert modellers validate their solutions.

14.4. Summary

Most students benefited from the validation process and proceeded to repeat the modelling cycle (or parts of the modelling cycle) to modify their solution to better fit the real situation. However, some students ignored these possibilities and maintained their approach: group B-D refused to reconsider how they could decorate the door and insisted to tile the front and the back of the door with gift cards; and group A-D refused to change the orientation of their gift cards and ignored the aesthetics aspect of the task. In both scenarios, students reduced the complexity of the task and refused to acknowledge the results of their validation process.

Validation is a valuable step in the modelling process. It provides students with the opportunities to reflect on their generated solution and to compare this solution against their original situation. Ideally, it is through this validation process which students decide whether to accept or reject their solutions. Some students in this study validated their solutions superficially. These students checked that they completed the problem, but did not make sense of their work against the original situation. For those who validated their work, they either did so independently based on their intuition or their knowledge, or dependently and relied on external help.

Chapter 15.

Students' modelling competencies – a response to the literature

By a “competency” we mean the ability of an individual to perform certain appropriate actions in problem situations where these actions are required or desirable. (Niss, Blum, & Galbraith, 2007, p. 12)

As mentioned in 6.2, I consciously chose not to teach participating students the modelling cycle and I purposely shy away from using any terminologies to describe the stages of the modelling cycle in class when I designed the study because I wanted to observe students' natural *mathematical Modelling* behaviors. As such, students were given the RMCTs and were asked to solve the tasks without additional information. Data show that students' modelling processes resemble the modelling cycles found in modelling literature in general (Borromeo Ferri, 2006). This is not to say they did not experience any difficulties with their modelling process, as there is no lack of examples where students had troubles with interpreting the RMCTs from a realistic perspective, identifying key variables and relevant information due to their insufficient EMK, validating their real results in terms of the original problem task, etc. But students followed the modelling cycle to produce a real result reasonably well and demonstrated their abilities to improve their solutions by repeating or focussing on specific stages of the modelling cycle. Students' *mathematical Modelling* behaviors demonstrate that they are capable of developing the modelling competencies, specifically, to carry out the stages of the modelling cycle, naturally.

On top of these competencies that are closely tied to the modelling cycle (Ludwig & Xu, 2009; Maaß, 2006, 2010; Niss et al., 2007), there also exists modelling competencies that are not directly related to the modelling cycle, including metacognitive modelling competencies, specifically declarative, procedural, and motivational metacognition (Sjuts, 2003, as cited in Maaß, 2006); a sense of direction; competencies to argue in relation to the modelling process; and modellers' attitudes towards modelling and mathematics (Maaß, 2006).

15.1. Declarative metacognition

Participating students in this study are considered novice modellers. They had minimal exposure to modelling tasks prior to the study and had no knowledge of the modelling cycle at the time of the study. As such, these students were expected to display minimal metacognitive modelling competencies, including declarative, procedural, and motivation metacognition, during their modelling processes.

Unlike previous studies where students were specifically taught the modelling cycle (for example, see Blum & Borromeo Ferri, 2009; Brand, 2014; Stender & Kaiser, 2015), students in this study were asked to solve the tasks without any prior knowledge regarding modelling or the modelling cycle. Data demonstrate that students did not impart any declarative metacognition during their modelling process, that is, they held no metaknowledge about the modelling process, and did not communicate with others about the modelling cycle at a meta level using formal terms. The absence of declarative metacognition is expected as students were not taught the modelling cycle.

15.2. Procedural metacognition

Students in this study displayed some procedural metacognition during their modelling processes, including the ability to devise a plan to solve the problem, to monitor one's actions and strategies during the modelling process, and to validate and analyze a solution in a productive manner.

15.2.1. Problem management competencies

Both the *Door Project* and the *Design a New School* task are complex, multi-part problems. The *Door Project* asks students to paint an office door and to decorate the office door using gift cards; and the *Design a New School* task asks students to design a school building, two tennis courts, a soccer field, and a parking lot while attending to a number of constraints and keeping in mind the aesthetics of their design. In what follows I discuss the way students managed their approach and devise plans to deal with the multi-part nature of both tasks.

Modelling literature describe the modelling process as a holistic approach, where modellers take on the entire problem and consider the real situation as a whole throughout the modelling cycle, and repeat the modelling process to improve the quality of their solution.

When discussing modelling competencies, Maaß (2006) describes one of the modelling competencies when working with the mathematical model as the competency to “use heuristic strategies such as division of the [mathematical model] into part problems” (Maaß, 2006, p. 116). During my analysis, I notice that many students took advantage of the multi-part nature of the tasks and applied the above-mentioned strategy to the entire tasks – they broke down the tasks into smaller, more manageable problems. I refer to this as a partitional approach.

With regards to the *Door Project*, students could have taken a holistic approach by taking into consideration both the painting and the decorating aspects of the task within the same modelling cycle, and they would possibly have to repeat the modelling cycle to improve their solution by including painting tools and adhesives – tools that are not mentioned in the question.

In the four cases of the *Door Project* presented in this thesis, groups A-D and B-D initially focused only on gift cards because they did not recognize they were also asked to repaint the door. They considered painting and decorating separately and constituted separate modelling cycles, and took a partitional approach to the *Door Project*. Conversely, groups C-D and J-D understood that they were asked to decorate and to repaint the door. Group C-D focused on gift cards in their first modelling cycle while taking paint and adhesives into consideration (Figure 34); group J-D discussed both painting and decorating in the same modelling cycle (Figure 46). It could be said that these students took a somewhat holistic approach to the *Door Project*.

Unlike the *Door Project*, participating students attended to the multi-part nature of the *Design a New School* task and took a partitional approach to solve the problem. Students’ partitional approach allowed them to focus on one feature at a time and to repeat the modelling cycle to generate a complete solution. It also removed the stress of dealing with and keeping track of a large amount of information found in the problem.

Modelling literature does not suggest students breaking down a problem task into smaller problems or working through the problem task one bit at a time. Rather, literature suggests students would work on the task holistically and repeat the modelling cycle to improve the quality of their real results. As I reviewed modelling literature, I noticed that the modelling tasks used in these research studies often involve tasks with one major focus or goal and have strong demands on both students' extra-mathematical knowledge and intra-mathematical skills. Unlike these tasks, both the *Door Project* and the *Design a New School* task have low intra-mathematical skill requirements, but they are complex tasks composed of multiple parts, and therefore would be difficult for students to piece together all the information or real results all at once. As such, the nature of these tasks possibly encourages students to take a partitional approach instead of a holistic approach as modelling literature suggests.

The partitional approach is a competency which students apply to solve complex or multifaceted tasks that are made up of multiple minor tasks. This approach allows students to break down a complex problem into smaller, more manageable pieces, and to generate the overall solution by focusing on one small problem at a time.

15.2.2. Monitoring actions and strategies

The abilities to monitor and review the actions taken and strategies applied allow modellers to arrive at solutions effectively. Literature suggest that these competencies are found more often in expert problem solvers than novice ones (Schoenfeld, 1992).

As the participants in this study are novice modellers, I did not expect students to monitor their actions and their strategies closely, and data confirmed that most students focused on completing the RMCTs without paying close attention to their actions and strategies.

Of those who paid attention to their actions and process, they also paid attention to the entire problem. For example, Group A-S recognized the orientation of their soccer field affects the placement of the tennis courts and rotated their soccer field accordingly. These students demonstrated that they understood that their work or decision on a small part of the problem could affect the rest of the problem. In another scenario, group G-S

paid attention to their progress and monitored the time spent on generating ideas and they made sure they had enough time to complete the problem.

15.2.3. Validation and analysis of solutions

In terms of validating and analyzing their work, some students compared their solution against the original situation. Most of these students acknowledged the results of their validation in a positive manner and improved their solutions. I provide a detailed discussion of validation in Chapter 14.

15.3. Motivational metacognition

Motivational metacognition refers to modellers' willingness and motivation to apply metacognition in their modelling process – to formally communicate with others about their modelling process and to monitor and regulate their modelling processes.

Motivational metacognition implies that students have a fair understanding of the modelling cycle and the affordances of planning and monitoring their strategies to solve the task and validating their generated solutions.

Students in this study did not display motivational metacognition towards declarative metacognition. This is not a surprise because students did not hold any declarative metacognition at the time of the study. Regarding procedural metacognition, students demonstrated some willingness to monitor their modelling processes but it is unclear whether they understood the affordances of procedural metacognition.

15.4. Competencies in arguing in relation to the modelling process

Students in this study used the information provided in the questions to guide them in the production of a real result and display some competencies in arguing in relation to the modelling process. Some interpreted the instructions to their liking as a way to reduce reality or complexity (see Chapter 12 for a discussion). None of them neglected the information in the instructions, purposefully bypassed the modelling process, and based their work entirely on their lived-experiences.

15.5. A Sense of Direction – Constraints and Freedom

In her discussion of modelling competencies, Maaß (2006) discusses a sense of direction as a vital modelling competency. She describes a sense of direction as “competencies for a goal-oriented proceeding” (p. 137).

Students’ sense of direction when solving RMCTs is closely related to the constraints and freedom in the tasks. Constraints and freedom are specific features of RMCTs. As the mathematics involved in RMCTs are rudimentary to the audiences, the tasks’ complexity rest on the real situation, which in turn is shaped by the freedom and constraints of the situation.

Constraints are the restrictions and the rules found in real situations which students need to satisfy. They also define the goal(s) of real situations and provide students with a possible direction to solve the problems.

There are plenty of examples of constraints in the two tasks used in this study. The two constraints found in the *Door Project* include the use of gift cards for decoration purposes and the need to paint the office door. As for the *Design a New School* task, examples of constraints include: a school building with a specific floor area, tennis courts and soccer fields that take up specific areas, distance between the border and buildings, etc. While a few students challenged these constraints (see a discussion on reducing reality and reducing complexity in Chapter 12), most students welcomed them and did not object to them. Most students used the constraints provided in the problems to organize their thinking around the tasks. In turn, these constraints provided students with a sense of direction.

For example, the 30-vehicle constraint in the *Design a New School* task shaped students organization and led them to reflect on the features found in a parking lot and their dimensions. Without the 30-vehicle constraint, students would not have reflected on the parking lot as deeply as they did, nor would they have created detailed designs of their parking lots.

Another important feature of RMCTs is freedom. Freedom refers to the opportunities students are provided to make decisions while working on the problem. Freedom in RMCTs often comes in the form of ambiguity, and ambiguities allow students to make

assumptions about the real situation and to apply their EMK to interpret the real situation. Freedom can be a double-edged sword in such tasks. Freedom in the form of ambiguity allows students to reflect on and to explore the situation and apply their creativity on both a mathematical and an interpersonal level (Galbraith & Clatworthy, 1990). It also allows students to make favorable assumptions about the problem to narrow down the situation. To use freedom in a productive manner, students need to recognize the ambiguities in the problem and make conscious decisions about their goal(s) prior to making decisions about the strategies used to solve the tasks. Conversely, freedom can act as a stumbling block as it leaves the problem wide open and may leave students feeling adrift (Liljedahl, 2010). In other words, freedom could allow students to develop their own sense of direction as they make conscious decisions about the real situation, or leave them without a sense of direction while working on the RMCT.

There is no shortage of freedom in the *Door Project* and the *Design a New School* task. Some students handled the freedom in the form of ambiguities with grace and elegance. These students had a good sense of direction in terms of where they were headed during their modelling processes. They took advantage of the freedom, made assumptions to reduce the reality of the situation, and generated realistic solutions. For example, group C-D in the *Door Project* assumed that their role to repaint and to decorate the door means for them to incorporate a gift card in a mural. Group A-S in the *Design a New School* task assumed that the green space in the instructions could mean a garden, and created a garden next to their parking lot to enhance students' school environments. These students developed a strong sense of direction of where they were heading and what they needed to do to complete the task from these assumptions.

However, many students struggled with the freedom given to them. This is not to say they do not have a sense of direction. These students had an idea what they wanted to do, but they were not certain what these freedom could offer and did not take advantage of them. For example, Group B-D aimed to completely cover the office door with gift cards. While their decision is not wrong, they have not taken advantage of the freedom (to decorate) to generate a cost reasonable solution (their solution costs over \$5000). In this scenario, students dealt with the freedom by maintaining a mathematical approach. Another example is group A-D's lack of confidence with what they could do with the freedom, which resulted in the group validating with me every time they produced a

solution to decide whether they were on the *right* path. Students' struggles with freedom are also found in the *Design a New School* task. This is especially evident in their work on the school building, where they were given a lot of freedom and a single constraint. While they could have designed school buildings with different floor areas through the use of various composite shapes, most of the designs consist of only rectangles with the same floor area throughout. In these scenarios, the freedom given in the question did not provide them with the opportunities to explore possible solutions to the problem. Rather, the freedom given in the question left students adrift.

Students' handling of the freedom found in the tasks seem to hinge on their EMK and EMK related competencies. In scenarios where students had limited EMK, they might not recognize the freedom and simply went along with the task and made assumptions to their best abilities. For example, groups A-D, B-D, and C-D simply assumed the office door in the question had the same dimensions as the classroom door. These students did not recognize the freedom in the dimensions and the style of the door, and in turn the freedom had no effect on them. In other scenarios, students' insufficient EMK prevented them to make assumptions and left them adrift – group B-S had a rough time with the 30-car parking lot. In cases where students recognized the freedom and had some EMK regarding the situation, their EMK enabled them a sense of direction and the ambiguity allowed them to explore the situation. For example, group A-S recognized buildings could have different floor areas and they applied the idea to satisfy the 11000m² constraint.

Freedom could provide students the opportunities to actively take charge in defining the problem task in their favor. As such, the productive use of freedom could be regarded as a competency that is related to a sense of direction as discussed in Maaß's (2006) work. Conversely, freedom could hinder students' sense of direction and provide them little guidance to how to solve the tasks.

The recognition and productive use of the freedom and constraints found in RMCTs can provide students with a sense of direction and directly influence students' solution, and they can provide students with opportunities when used properly. However, they could also act as a stumbling block when students fail to see what these freedom and constraints can offer.

15.6. Summary

Literature describe modelling competencies as the competencies to carry out the modelling cycle and as a complex construct, including declarative, procedural, and motivational metacognition, competencies in arguing in relation to the modelling process, and a sense of direction (Maaß, 2006). Students in this study demonstrated that they are capable of developing modelling competencies that are closely related to the modelling cycle naturally without external help. Some also used the freedom and constraints found in the RMCT effectively to develop a sense of where they were headed.

Students in this study also demonstrated some metacognitive modelling competencies during their modelling processes. They broke down the multi-faceted RMCTs into small and manageable problems and validated their work throughout their modelling processes. However, since some of the metacognitive modelling competencies are closely related to students' knowledge of the modelling cycle and students in this study lacked such knowledge, these students did not demonstrate such competencies. Overall, results in this study demonstrate that modelling competencies can arise from students' natural *mathematical Modelling* behaviors.

Chapter 16.

Conclusion

In this final chapter, I first summarize the findings of this study, the contributions I make to the mathematical modelling research community, and the limitations of this study. Afterwards, I conclude this thesis with a reflection of my journey to use mathematical modelling to highlight the relationships between reality and mathematics.

16.1. A summary of the findings

This study aims to address two big research questions, the first one surrounds students' natural *mathematical Modelling* behaviors and the second one surrounds extra-mathematical knowledge as a multi-faceted complex construct. I used a specific type of modelling tasks called rudimentary mathematics complex tasks, RMCTs, to answer these research questions.

16.1.1. Students' mathematical Modelling behaviors

In my investigation of students' *mathematical Modelling* behaviors, I find that students naturally follow the modelling cycle in general, that is, students do not need to be taught the modelling cycle explicitly in order to work on RMCTs in meaningful ways. When provided with a RMCT, students in general recognize the complexity of the situation and take into realistic considerations as they worked on the tasks.

With regard to students' modelling process, literature suggest students would simplify and idealize their understanding and interpretation of the real situation (Blum & Leiß, 2005, 2007) and make additional assumptions about the situation and apply their EMK to produce a real model (Borromeo Ferri, 2006), but do not identify these assumptions and the impact they have on students' solutions.

This study distinguishes between two categories of assumptions and students' intentions behind these assumptions: reducing reality and reducing complexity. Students who reduce the reality of the situation intend to retain as much realistic aspects of the situation as possible when they make assumptions to simplify and idealize the situation.

These assumptions lead students to produce a solution that describes the original complex situation.

However, not everyone shares this intention. Students who reduce the complexity of the situation aim to avoid the complex situation by removing the non-negotiable aspects of the situation. It is important to note that students may reduce the complexity of the situation not because they are uninterested in the RMCT or that they hold a negative attitude towards modelling or mathematics, but because of their expectations of the tasks, their interpretation of the realistic context of the situation, to avoid work, or to maintain in the state of flow (Csíkszentmihályi, 1990).

Other than reducing reality and reducing complexity as strategies to simplify the situation, students' *mathematical Modelling* behaviors also shed light on their validation processes. My analysis demonstrates that some students in this study did not validate their work and some only validated their work in a superficial manner. These findings are in line with modelling literature (Galbraith & Stillman, 2006; Maaß, 2006).

In terms of the validation process, modelling literature suggest that students independently validate their work based on their intuition or their knowledge (Borromeo Ferri, 2006), and make a logical decision based on the results of the validation process. Knowledge-based validation requires students to apply their EMK in the validation process. While this is possible for students who possess the proper EMK, this is not always the case in this study. As such, this study further differentiates students' validation processes into independent and dependent validation. In cases where students validated their work independently, students validated their work by redoing parts of their work when they sensed something was wrong with their solution (independent intuitive validation), or recognized the shortcoming of their solution and applied additional EMK to improve their solution (independent knowledge-based validation). Others validated their work dependently. Dependent knowledge-based validation includes situations where students relied on others to provide them with additional EMK to improve their work. This study does not find any cases of dependent knowledge-based validation. Dependent intuitive validation includes cases where students sensed that there was something wrong with the solution but were not able to explicitly articulate the problem. As such, students relied on the help given to make improvements to their solution. In most scenarios, students acknowledged the results of

their validation process and improved the solutions by repeating the entire or a part of the modelling cycle. In two scenarios, students rejected the results of their validation and maintained their original approaches. These students also reduced the complexity of the task.

These results from my focus on students' *mathematical Modelling* behaviors add to existing modelling literature by expanding our understanding on students' modelling process and provide additional details to the strategies students take when they face challenges in their modelling processes, especially during the generation of a real model and the validation process to determine the next course of action.

Other than these *mathematical Modelling* behaviors, this study also investigates the role of EMK in RMCTs. As RMCTs put an emphasis on the complex situation and demand students to apply EMK to solve these tasks and allow students to use their rudimentary mathematics skills to solve the tasks, it is no surprise that most of students' difficulties in this study are EMK rather than mathematics related.

16.1.2. EMK as a multi-faceted complex construct

Modelling literature provide us with a general description of EMK and its role in solving modelling tasks (Borromeo Ferri, 2006), and suggest that students activate and apply their EMK early on in the modelling cycle when they build a real model to represent their real situation, and again when they interpret and validate their solution. However, my analysis demonstrates that EMK is a multi-faceted complex construct. First, the activation and application of EMK is more complex than what modelling literature suggest: while some students activated and applied their EMK towards the beginning of their modelling process, many activated and applied EMK as they validated their work or when they were stuck and were forced to reflect deeply about the situation, and some activated but ignored the role EMK played in the problem. As such, students' successful application of EMK depends on their activation of it, where students recognize realistic aspects of the problem and the affordances of EMK in solving the problem; their acquisition of additional EMK, especially when students' existing EMK is insufficient to produce a realistic solution; and their motivation to apply their existing EMK and to acquire additional EMK.

The second facet of EMK is associated with students' engagement with the realistic context of the problem. Looking into students' engagement with the RMCT from a EMK perspective, I identify four types of modellers based on students' activation and application of EMK and students' existing and acquired EMK, including students who engage with RMCTs with realistic perspectives, students who are limited by EMK, students who do not recognize EMK's contribution, and students who engage with RMCTs with minimal realistic perspectives.

I also distinguish the characteristics of EMK and classify these EMK accordingly. I am able to categorize five types of EMK based on the data collected: everyday EMK, general EMK, sophisticated EMK, special EMK of common objects, and EMK through accessing information. Identifying the characteristics of EMK allow researchers and educators to design tasks that target audiences with specific EMK or to achieve competencies in dealing with specific EMK. It is important to note that these descriptions of EMK arose from the data in this study, and there may be additional EMK characteristics which this study does not identify.

16.2. Contributions

This study contributes to the mathematical modelling research community in three areas: the development of a rudimentary mathematics complex tasks, the differentiation of two different types of simplification strategies, and the identification of the facets of EMK as a complex construct.

16.2.1. Rudimentary mathematics complex tasks

As I looked for tasks to study students' *mathematical Modelling* behaviors, I found many modelling tasks that are realistically and mathematically well-balanced, that is, they have a strong demand for students to apply both their extra-mathematical knowledge and their intra-mathematical skills to solve the tasks. Also, many of the required intra-mathematical skills are not rudimentary to the participating students.

With my intention to highlight the relationship between reality and mathematics and to investigate students' *mathematical Modelling* behavior, I needed tasks that tip this balance. In this study, I employed a type of task called rudimentary mathematics

complex tasks (RMCTs). Since the complexity of these tasks lie in the situation rather than the mathematics, these tasks slow down the *mathematical Modelling* process and allow students to focus on approaching the task from a realistic perspective. In turn, these tasks allow researchers to investigate students' *mathematical Modelling* behaviors and the EMK involved in the tasks. These tasks also allow educators and researchers to identify and to develop interventions to help students further develop specific modelling competencies to highlight the relationships between reality and mathematics.

16.2.2. Reducing reality and reducing complexity

Literature acknowledge the process to develop a real model involves the simplification and idealization of the situation but do not provide details to the strategies involved. In this study I identify two different categories of strategies students use to simplify the tasks: reducing reality and reducing complexity.

As one of the goals of mathematical modelling is to highlight the relationship between reality and mathematics, it is important to differentiate these two strategies, as reducing reality is used to simplify the situation but remain in the realm of reality, and reducing complexity removes the complexity and therefore the realistic aspects of the problem. My analysis also provides clues to students' reasons behind their strategies. These reasons allow us to understand the difficulties they experience and therefore to develop interventions to help them overcome these difficulties in their modelling processes.

16.2.3. The facets of EMK as a complex construct

While literature acknowledge the importance of EMK, literature describe EMK with broad strokes. In this study, I identify the facets of EMK as a complex construct and my analysis adds details to our understanding of EMK. These understanding allow researchers and educators to pay attention to students' EMK related competencies and to develop specific tasks to target students' use of specific types of EMK. Since EMK is inseparable from the modelling cycle and is crucial in the successful production of a real result, students' better understanding and use of EMK in the modelling process allow them to become better modellers.

16.3. Limitations

All research studies have their limitations and room for improvement. This study employed two RMCTs to investigate students' *mathematical Modelling* behaviors and their EMK related competencies. Given that the RMCTs contain complex situations but require the use of rudimentary mathematics to solve the tasks, these RMCTs are useful for students to relate reality and mathematics but not as useful for building students' mathematical skills. As such, this study pays closer attention specific stages of the modelling cycle, including those leading up to mathematization and the interpretation and validation stages, and less on the mathematics, including the process of mathematization and students' use of intra-mathematical skills to solve their mathematical model.

Also, I have chosen to study the *mathematical Modelling* behaviors of students who came from the enriched program, where they were identified as students with strong intra-mathematical skills, are motivated in their learning, and hold positive attitudes towards mathematics. While this decision assured students would have all the intra-mathematical skills needed to tackle the tasks and allowed them to focus on the realistic complexity of the tasks, these students do not represent the general population of students in secondary schools. As RMCTs require only the use of rudimentary mathematics, students' attitudes towards and beliefs about mathematics may also play a role in their modelling processes and my data may be different if I had chosen to work with a different group of students.

Finally, acting as both the researcher and the classroom teacher, I understand the limitations of my dual role and avoided these limitations as much as I could. As I researched on my own classroom, I have established a relationship with the students and this allowed me to provide students with the appropriate help they needed to keep them in flow while working on the two tasks. To avoid interpreting and analyzing the data from a subjective perspective and to maintain an objective perspective, I based my interpretation and analysis only on students' visible modelling behaviors, including the verbal discussions and their written work, and asked students to clarify their thinking and their work during post-tasks interviews. During the entire project, I also exercised extreme self-control and refrained myself from working on any of the tasks administered to students. This allowed me to remain neutral of my opinions of the solutions to both

tasks and allowed me to analyze students' solutions objectively. Finally, as a classroom teacher, I understand that I hold certain authority when I discuss with students their modelling progress as they worked on the tasks. To avoid steering students into specific directions, I avoided answering their questions directly as much as I could, and only offered help when I sensed students were on the edge of flow, and when I sensed that students would not be able to proceed unless they received suggestions from me.

16.4. Reflection: Please mind the?

"There's something that doesn't make sense. Let's go and poke it with a stick." ~ The Eleventh Doctor, Doctor Who

In the beginning of this thesis, I reflected on the gap which I observe between the mathematics classroom and reality, where students seem to interpret reality and mathematics as two individual entities with a clear boundary between them.

And, of course, to find out more about this gap I decided to poke it with a stick. I asked questions, implemented changes to my own classroom, observed the changes that happened to the students and myself, in the hope of gaining a deeper understanding about this gap and the possibility to bring mathematics closer to reality and vice versa.

Looking back now I think it was foolish of me to believe that such a gap between the mathematics classroom and reality exists. The gap is there because I believe it is there. Rather than thinking about the gap between the mathematics classroom and reality, I should really think about the relationship between mathematics and reality that already exists. I need to highlight the relationship between mathematics and reality.

In the course of this study I was hopeful that RMCTs could prove a means to highlight the relationship between reality and mathematics. With the consistent administration of RMCTs I have observed positive changes in students' modelling behaviors and growth in their modelling competencies, that they themselves also highlight the relationship between mathematics and reality in their work.

So are RMCTs the answer to solve our problem here? I am positive that we can use tasks such as RMCTs to highlight the use of mathematics in life or to highlight the need to consider real-life perspectives in mathematics class. I think it is also possible to use RMCTs as a way to introduce to students mathematical modelling and to achieve

various learning goals. However, treating these tasks as a unit in the mathematics curriculum or as a prescription to improve students' problem solving skills defeats the purpose of these tasks. RMCTs along with other modelling tasks promote the integration of extra-mathematical knowledge and intra-mathematical skills, allow for the development of modelling competencies, and provide students a venue to discuss how mathematics could impact their daily lives. Other than bringing forth the relationship between the two, I believe RMCTs also provide students with the chance to connect with mathematics on a personal level.

With this thesis I hope I have poked at with a stick what we do not understand, that this thesis will initiate further discussions to how we can highlight the relationship between reality and mathematics better so that mathematics is not treated as an isolated subject with minimal relevance in students' lives.

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