Chapter 3

Developing Understanding in Mathematics

If the creation of the conceptual networks that constitute each individual's map of reality—including her mathematical understanding—is the product of constructive and interpretive activity, then it follows that no matter how lucidly and patiently teachers explain to their students, they cannot understand for their students.

Schifter and Fosnot (1993, p. 9)

It is a commonly accepted goal among mathematics educators that students should understand mathematics (Hiebert & Carpenter, 1992; Pirie & Kieren, 1992, 1994). The most widely accepted theory, known as *constructivism*, suggests that children must be active participants in the development of their own understanding. Constructivism provides us with insights concerning how children learn mathematics and guides us to use instructional strategies that begin with children rather than with ourselves. In the view of many educators and researchers, this theoretical perspective has become, in the past 10 years, the "watchword" for good teaching (Pirie & Kieren, 1992).

A Constructivist View of Learning

Constructivism is firmly rooted in the cognitive school of psychology and the theories of Piaget, dating back at least as far as 1960. This view of learning rejects the notion that children are blank slates who absorb ideas as teachers present them. Rather, the belief is that children are creators of their own knowledge.

The Construction of Ideas

The basic tenet of constructivism is simply this: *Children construct their own knowledge* (Pirie & Kieren, 1992). In fact, not just children, but all people, all of the time, construct or give meaning to things they perceive or think about. As you read these words, you are giving meaning to them. You are constructing ideas.

To construct or build something in the physical world requires tools, materials, and effort. How we construct ideas can be viewed in an analogous manner. The tools we use to build understanding are our existing ideas, the knowledge that we already possess. The materials we act on to build understanding may be things we see, hear, or touch—elements of our physical surroundings. Sometimes the materials are our own thoughts and ideas. The effort that must be supplied is active and reflective thought. If minds are not actively thinking, nothing happens (Janvier, 1987; Schroeder & Lester, 1989).

The diagram in Figure 3.1 is meant as a metaphor for the construction of ideas. Consider the picture to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections, or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas (dots) are used in the construction will necessarily be connected to the new idea because those were the ideas that gave meaning to it. If a potentially relevant idea that would add better meaning to the new idea is either not present in the learner's mind or is not actively engaged, then that potential connection to the new idea simply will not be made. Obviously, learners will vary in the number of connections between a new idea and existing ideas. Different learners will use different ideas to give

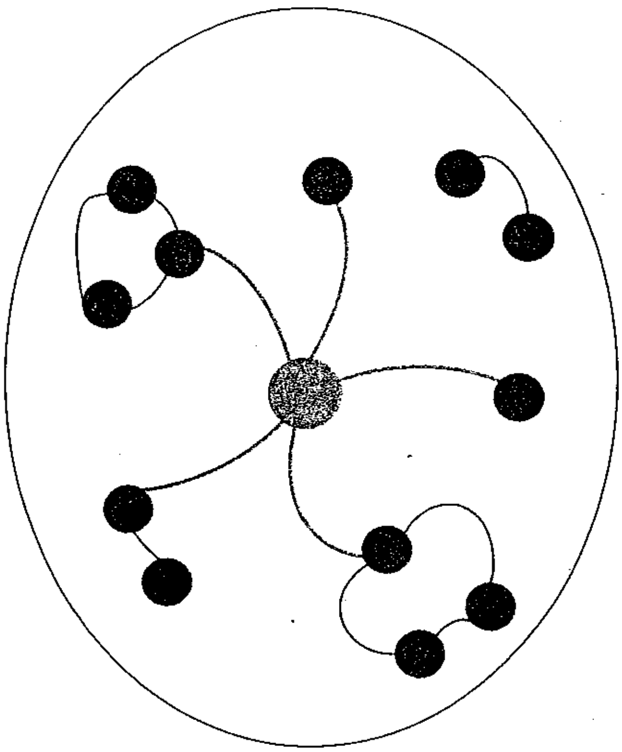


FIGURE 3.1 We use the ideas we already have (blue dots) to construct a new idea (red dot), developing in the process a network of connections between ideas. The more ideas used and the more connections made, the better we understand.

meaning to the same new idea. What is significant is that the construction of an idea is almost certainly going to be different for every learner, even within the same environment or classroom.

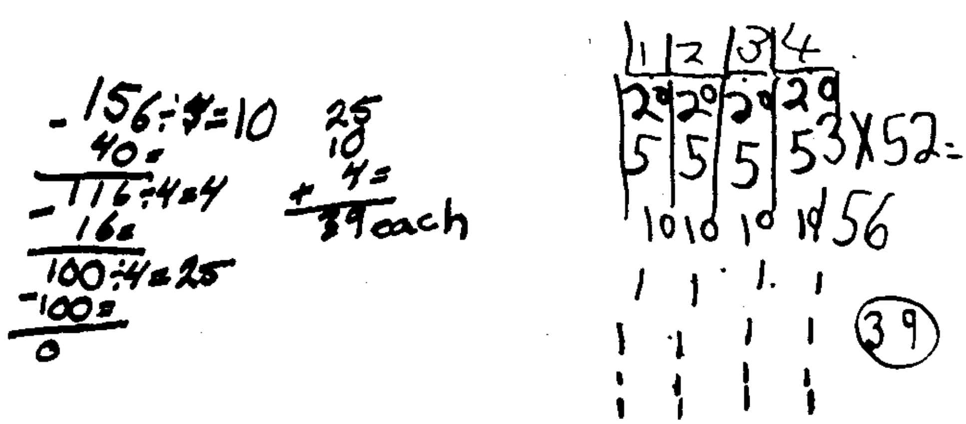
Constructing knowledge is an extremely active endeavour on the part of the learner (Pirie & Kieren, 1992; von Glasersfeld, 1990). To construct and understand a new idea requires actively thinking about it. "How does this idea fit with what I already know?" "How can I understand this idea in the face of what I currently understand about it?" Mathematical ideas cannot be "poured into" a passive learner. Children must be mentally active for learning to take place. In classrooms, children must be encouraged to wrestle with new ideas, to work at fitting them into existing networks, and to challenge their own ideas and those of others. Put simply, constructing knowledge requires *reflective thought*—actively thinking about or mentally working on an idea. Reflective thought means sifting through existing ideas in order to find those that seem to be the most useful in giving meaning to the new idea.

Integrated networks, or *cognitive schemas*, are both the product of constructing knowledge and the tools with which additional new knowledge can be constructed. As learning occurs, the networks are rearranged, added to, or otherwise modified. When there is active, reflective thought, schemas are constantly being modified or changed so that ideas fit better with what is already known.

Examples of Constructed Learning

Consider the solution methods of two grade 4 children from schools where a highly constructivist approach to mathematics had been in place for several years. The "dots" these children had at their disposal included the meanings of the basic operations and a good understanding of place-value concepts. They were asked to solve the following problem: "Four children had 3 bags of M&Ms. They decided to open all 3 bags of candy and share the M&Ms fairly. There were 52 M&M candies in each bag. How many M&M candies did each child get?" (Campbell & Johnson, 1995, pp. 35–36). Their solutions are shown in Figure 3.2.

Both children were able to determine the product 3×52 mentally. The two children used different cognitive tools to



Myka's Solution

Harjit's Solution

FIGURE 3.2 Two grade 4 children construct unique solutions to a computation.

Source: Campbell & Johnson (1995). Used with permission.

solve the problem of 156 ÷ 4. Myka interpreted the task as "How many sets of 4 can be made from 156?" She first used facts that were either easy or available to her: 10 × 4 and 4 × 4. These totals she subtracted from 156 until she arrived at 100. This seemed to cue her to use 25 fours. Myka did not hesitate to add the number of sets of 4 that she found in 156 and knew the answer was 39 candies for each child.

Harjit's approach was more directly related to the sharing context of the problem. He formed four columns and distributed amounts to each, accumulating the amounts mentally and orally as he wrote the numbers. Like Myka, Harjit used numbers that were either easy or available to him; first 20 to each, then 5, then 10, and then a series of ones. He added one of the columns without hesitation (Rowan, 1995).

If computational speed and proficiency were your goal, you might be tempted to argue that the children need further instruction. However, both children clearly constructed ideas about the computation that had meaning for them. They demonstrated confidence, understanding, and a belief that they could solve the problem.

In contrast to these two children, consider a grade 3 child in a traditional classroom. She has made a quite common error in subtraction, as shown in Figure 3.3. The class had been doing subtraction with "borrowing," more appropriately known as trading or regrouping, and the problem appeared on a mathematics worksheet. The context narrowed the choices of ways to give meaning to the situation (the "dots" she would likely use). But this problem was a little different from the child's existing ideas about "borrowing." The next column contained a 0. How could she take 1 from the 0? That part was different, creating a situation that for her was problematic. The child decided that "the next column" must mean the next one that has something in it. She therefore believed that she had to "borrow" from the 6 and ignore the 0. The child used her existing ideas to give her own meaning to the rule "borrow from the next column."

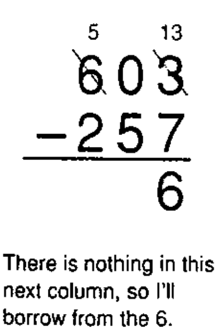


FIGURE 3.3 Children sometimes invent incorrect meanings by extending poorly understood rules.

Children rarely give random responses (Ginsburg, 1977; Labinowicz, 1985). Their answers tend to make sense in terms of their personal perspective or in terms of the knowledge they are using to give meaning to the situation. In many instances, children's existing knowledge is incomplete or inaccurate, or perhaps the knowledge we assume to be there simply is not. In such situations, as in the present example, new knowledge may be constructed inaccurately.

Construction in Rote Learning

Constructivism is a theory about how we learn. If it is correct, then it describes how *all* learning takes place, regardless of how we teach. We cannot choose to have children learn constructively on some days and not others. Even rote learning is a construction. But what tools or ideas are used for construction in rote learning? To what is knowledge learned by rote connected?

Children searching for a way to remember 7 × 8 = 56 might note that the numbers 5, 6 and 7, 8 go in order. Or they may connect the number 56 to that "hard fact" since 56 is unique in the multiplication table. (But then so is 54.) Repetition of a routine procedure may be connected to some mantra-type recitation of the rule, as in "Divide, multiply, subtract, and bring down." This sequence has even been related to the mnemonic "Dirty monkeys smell bad." New ideas learned like this are not connected to anything that can be called mathematical. Nor are they part of networks of ideas. Each newly learned bit is essentially isolated. Rote knowledge will almost never contribute to a useful network of ideas. Rote learning can be thought of as a "weak construction" (Noddings, 1993).

When mathematical ideas are used to create new mathematical ideas, useful cognitive networks are formed. Returning to 7 × 8, imagine a class where children discuss and share clever ways to figure out the product. One child might think of 5 eights and then 2 more eights. Another may have learned 7 × 7 and noted that this is just one more seven. Still another might look at a list of 8 sevens and take half of them (4 × 7) and double that. This may lead to the notion that double 7 is 14, and double that is 28, and double that is 56. Not every child will construct 7 × 8 using all of these approaches. However, the class discussion brings to the fore a wide range of useful mathematical "dots" so that the potential is there for profitable constructions.

Understanding

It is possible to say that we know something, but we might not necessarily understand it. Knowledge is something that we either have or don't have. Understanding is another matter. For example, how did you learn 7 × 8? If you learned it by rote, as most adults did, you may never have thought about the other ideas just discussed. Is your understanding of 7 × 8 just the same as that of a person who has connected some of these other ideas to that fact?

Understanding is never an all-or-nothing proposition. It depends on the existence of appropriate ideas and on the creation of new connections (Backhouse, Haggarty, Pirie & Stratton, 1992; Hiebert & Carpenter, 1992). It can be defined as a measure of the quality and quantity of connections that a new idea has with existing ideas.

One way we can think about an individual's understanding is to imagine it along a continuum (see Figure 3.4). At one extreme is a very rich set of connections. The understood idea is associated with many other existing ideas in a meaningful network of concepts and procedures. Hiebert and Carpenter (1992) refer to "webs" of interrelated ideas. Understanding at this rich interconnected end of the continuum will be referred to as *relational understanding*, borrowing a term made popular by Skemp (1978). At the other end of the continuum, ideas are largely or completely isolated. At this end, we have what we will call *instrumental understanding*, again borrowing from Skemp. Knowledge that is learned by rote is almost always understood instrumentally.

Examples of Understanding

If we accept the notion that understanding has both qualitative and quantitative differences, the question "Does she know it?" must be replaced with "How does she understand it? What ideas does she connect with it?" In the following examples, you will see how different children may well develop varied ideas about the same concept and thus have dissimilar understandings.

Computation in Two Classrooms

Schifter and Fosnot (1993) describe a grade 3 class where several children are discussing the problem of sharing 90 jelly beans among four children. They decide to use base-ten models (ones and tens). They distribute 2 tens to each group and trade a ten for 10 ones.

Next, they distribute 2 ones to each group. Then, there is a discussion of what to do with the 2 leftover ones and how to write down what they have done. One child suggests "$22\frac{1}{2}$" and another "22 R 2." They decide that the best answer for any division depends on the situation and what you want to do with the leftovers.

In a more traditional class, another grade 3 student was quite confident in her ability to do long divisions such as 24 682 ÷ 5. When asked what the "R 2" meant when she computed 32 ÷ 5, she could only identify 2 as the remainder. Asked to demonstrate 32 ÷ 5 with the base ten blocks, she began but then decided it couldn't be done. The child was at a loss to explain "R 2" in terms of the leftover counters (Schifter & Fosnot, 1993). These children all have different understandings of division. Some are very rich understandings; some are very limited.

Connections with Early Number Concepts

Consider the concept of "seven" as constructed by a child in grade 1. Seven for a first grader is most likely connected to the counting procedure and the construct of "more than" and is probably understood as less than 10 and more than 2. What else will this child eventually connect to the concept of seven as it now exists? Seven is 1 more than 6; it includes those numbers less than itself; it is 2 less than 9; it is the combination of 3 and 4 or 2 and 5; it is odd; it is small compared to 73; it is the number of days in a week; and so on. The web of ideas connected to a number can grow large and involved, depending on the level of the child's understanding.

A Web of Ideas Involving Ratio

A clear example of the potential for rich relational understanding is found in the many ideas that can be associated with the concept of "ratio" (see Figure 3.5). Unfortunately, many children learn only meaningless rules connected with ratio, such as, "Given one ratio, how do you find an equivalent ratio?"

Benefits of Relational Understanding

To teach for a rich or relational understanding requires a lot of work and effort. Concepts and connections develop over time, not in a day. Tasks must be selected. Instructional materials must be made. The classroom needs to be organized for group work and maximum interaction with and among the children. The important benefits to be derived from relational understanding make the effort not just worthwhile, but essential.

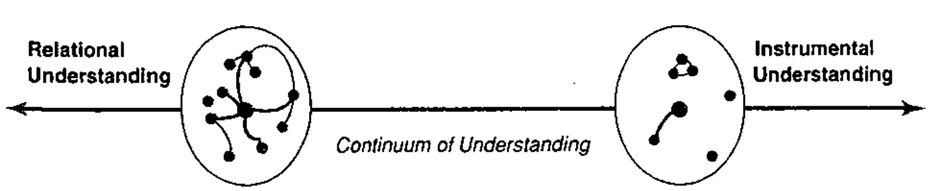
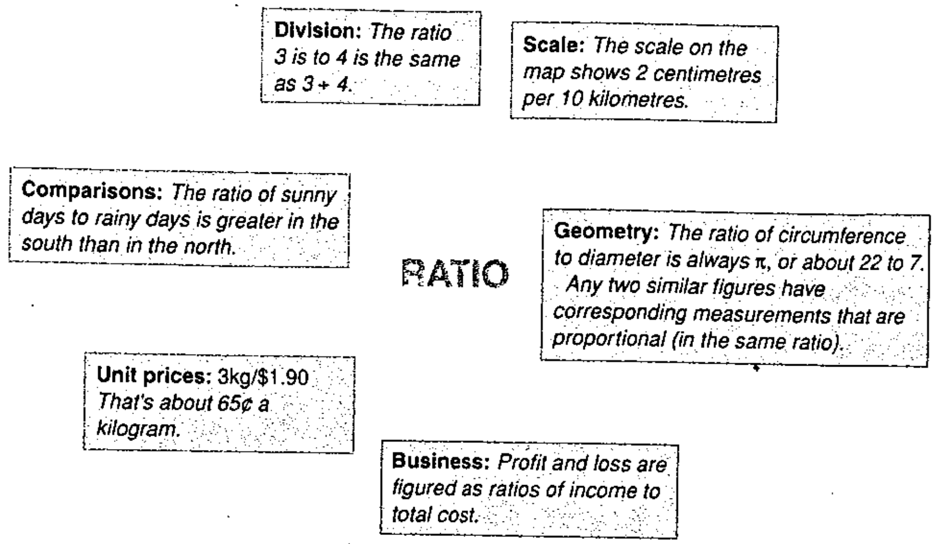


FIGURE 3.4 Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

FIGURE 3.5 Potential web of associations that could contribute to the understanding of "ratio."



It Is Intrinsically Rewarding

Nearly all people, and certainly children, enjoy learning. This is especially true when new information connects with ideas already possessed. The new knowledge makes sense; it fits; it feels good. Children who learn by rote must be motivated by external means: for the sake of a test, to please a parent, from fear of failure, or to receive some reward. Such learning is distasteful. Rewards of an extra recess or a star on a chart may be effective in the short run but do nothing to encourage a love of the subject when the rewards are removed.

It Enhances Memory

Memory is a process of retrieving information. When learning in mathematics establishes a rich set of connections, there is much less chance that the information will deteriorate; connected information is more likely than disconnected information to be retained over time. It is also easier to retrieve. Connected information provides an entire web of ideas for which a learner can reach. If what you need to recall seems distant, reflecting on related ideas will, in most instances, eventually lead to the desired idea. Attempting to retrieve disconnected information is more like looking for a needle in a haystack.

A large portion of instructional time in schools is devoted to re-teaching and review. If teaching focused more on developing relational rather than instrumental understanding, much less review time would be needed.

There Is Less to Remember

Traditional approaches have tended to fragment mathematics into seemingly endless lists of isolated skills, concepts, rules, and symbols. The lists are so lengthy that teachers and students become overwhelmed. Constructivists, for their part, talk about teaching "big ideas" (Brooks & Brooks, 1993; Hiebert et al., 1996; Schifter & Fosnot, 1993). Big ideas are really just large networks of interrelated concepts. Ideas are learned relationally when they are integrated into a larger web of information, a big idea. Frequently, the network is so well constructed that whole chunks of information are stored and retrieved as single entities rather than isolated bits. For example, knowledge of place value subsumes rules about lining up decimal points, ordering decimal numbers, moving decimal points to the right or left in decimal-percent conversions, rounding and estimating, and a host of other ideas. Similarly, knowledge of equivalent fractions ties together rules concerning common denominators, reducing fractions, and changing between mixed numbers and whole numbers.

It Helps with Learning New Concepts and Procedures

An idea fully understood in mathematics is easily extended when a new idea is learned. Understanding of number concepts and relationships helps with mastery of basic facts. Fraction knowledge and place-value knowledge come together to make decimal learning easier, and decimal concepts directly enhance an understanding of percentage concepts and procedures. Many of the ideas of elementary arithmetic become the model for understanding ideas in algebra. Reducing fractions by finding common prime factors is the same thing as dividing out common factors.

Without these connections, children will need to learn each new piece of information they encounter as a separate, unrelated idea.

It Improves Problem-Solving Abilities

The solution of novel problems requires transferring ideas learned in one context to new situations. When concepts are embedded in a rich network, transferability is significantly enhanced and, thus, so is problem solving (Schoenfeld, 1992). The most recent results of the School Achievement Indicators Program (SAIP) for mathematics, the third assessment of its type, demonstrated that significantly more 13-year-old Canadian students improved (compared to results of the previous assessment) in problem solving, as well as in mathematics content. It is suggested that exposure to reform-oriented curricula, with their emphasis on understanding, may have played a role in students' improvement (Council of Ministers of Education, 2001).

It Is Self-Generative

"Inventions that operate on understandings can generate new understandings, suggesting a kind of snowball effect. As networks grow and become more structured, they increase the potential for invention" (Hiebert & Carpenter, 1992, p. 74). Skemp (1978) noted that when gaining knowledge is found to be pleasurable, people who have had that experience of pleasure are likely to seek or invent new ideas on their own, especially when confronting problematic situations.

It Improves Attitudes and Beliefs

Relational understanding has an affective as well as a cognitive effect on the learner. When relational learning occurs, the learner tends to be more positive about his or her ability to learn and understand mathematics. There is a definite sense of "I can do this! I understand!" There is no reason to fear or be in awe of knowledge learned relationally. Mathematics then makes sense. It is not some mysterious world that only "smart people" dare to enter. At the other end of the continuum, instrumental understanding may produce mathematics anxiety, a real phenomenon that involves fear and avoidance behaviour.

Relational understanding also promotes a positive view of mathematics itself. Sensing the connectedness and logic of mathematics, students are more likely to gravitate toward it or to describe the discipline in positive terms.

Types of Mathematical Knowledge

All knowledge, mathematical or otherwise, consists of internal or mental representations of ideas that the mind has constructed. For some time now, mathematics educators have found it useful to distinguish between two types of mathematical knowledge: conceptual knowledge and procedural knowledge (Hiebert & Lindquist, 1990).

Conceptual Knowledge of Mathematics

Conceptual knowledge of mathematics consists of logical relationships constructed internally and existing in the mind as a part of a network of ideas. It is the type of knowledge Piaget referred to as logico-mathematical knowledge (Kamii, 1985, 1989; Labinowicz, 1985). By its very nature, conceptual knowledge is knowledge that is understood (Hiebert & Carpenter, 1992).

Ideas such as seven, rectangle, ones/tens/hundreds (as in place value), sum, product, equivalent, ratio, and negative are all examples of mathematical relationships or concepts.

Figure 3.6 shows three different types of Dienes' Base-Ten Multibase Arithmetic Blocks, commonly used to represent ones, tens, and hundreds. (The blocks were designed by Zoltan Dienes, a mathematics educator whose experience extends to a number of Canadian provinces.) By the middle of grade 2, most children have seen pictures of these or have used the actual blocks. It is quite common for these children to be able to identify the rod as the "ten" piece and the large square block, the flat, as the "hundred" piece. Does this mean that they have constructed the concepts of ten and a hundred? All that is known for sure is that they have learned the names for these objects, the conventional names of the base ten blocks. The mathematical concept of a ten is that *a ten is the same as ten ones*. Ten is not a rod. The concept is the relationship between the rod and the small cube. It is not the rod or a bundle of ten sticks or any other model of a ten. It is this relationship called "ten" that children must create in their own minds.

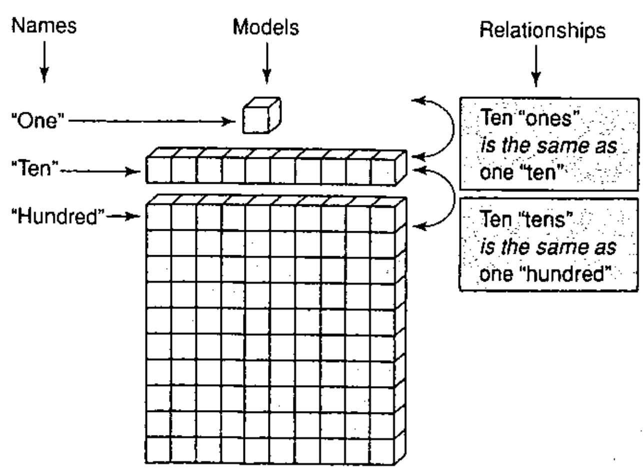


FIGURE 3.6 Objects and names of objects are not the same as relationships between objects.

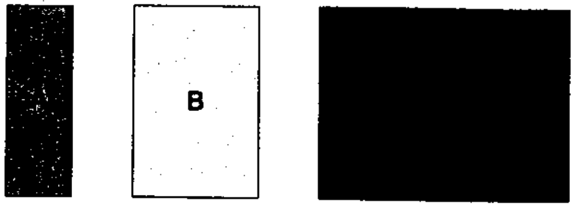


FIGURE 3.7 Three shapes, different relationships.

In Figure 3.7, the shape labelled A is a rectangle. But if we call shape B "one" or a "whole," then we might refer to shape A as "one-half." The idea of "half" is the *relationship* between shapes A and B, a relationship that must be constructed in our mind. It is not in either rectangle. In fact, if we decide to call shape C the whole, shape A becomes "one-fourth." The actual rectangle did not change in any way. The concepts of "half" and "fourth" are not in rectangle A; we construct them in our mind. The rectangles help us "see" the relationships, but what we see are rectangles, not concepts.

Procedural Knowledge of Mathematics

Procedural knowledge of mathematics is knowledge of the rules and the procedures that one uses in carrying out routine mathematical tasks, and of the symbolism that is used to represent mathematics. Knowledge of mathematics consists of more than concepts. Step-by-step procedures exist for performing tasks such as multiplying 47 × 68. Concepts are represented by special words and mathematical symbols. These procedures and symbols can be connected to or supported by concepts, but very few cognitive relationships are needed to have knowledge of a procedure.

Procedures are the step-by-step routines learned to accomplish a task. "To add two three-digit numbers, first add the numbers in the right-hand column. If the answer is 10 or more, put the 1 above the second column, and write the other digit under the first column. Proceed in a similar manner for the next two columns in order." We can say that someone who can accomplish a task such as this has knowledge of that procedure. Again, the conceptual understanding that may or may not support this procedural knowledge can vary considerably from one student to the next.

Some procedures are very simple and may even be confused with conceptual knowledge. For example, grade 7 children may be shown how to add the integers $^-7$ and $^+4$ by combining 7 red "negative" counters with 4 yellow "positive" counters. Pairs consisting of 1 red and 1 yellow counter are removed, and the result is noted. In this example, there would be 3 red negative counters remaining, and the students would record $^-3$ as the sum. This might be called a manipulative or physical procedure. Notice that it is conceivable that a student could master a procedure such as this with very little understanding, or it could be integrated with a conceptual web related to integers and thus be well understood.

Symbolism includes expressions such as $(9 - 5) \times 2 = 8$, π, $\leq$, $\geq$, and $\neq$. The meaning attached to this symbolic knowledge depends on how it is understood—what concepts and other ideas the individual connects to the symbols. Symbolism is part of procedural knowledge, whether it is understood or not.

Procedural Knowledge and Doing Mathematics

Procedural knowledge of mathematics plays a very important role both in learning and in doing mathematics. Algorithmic procedures help us do routine tasks easily and thus free our minds to concentrate on more important tasks. Symbolism is a powerful mechanism for conveying mathematical ideas to others and for "doodling around" with an idea as we do mathematics. But even the most skillful use of a procedure will not help develop conceptual knowledge that is related to that procedure (Hiebert, 1990). Doing endless long-division exercises will not help a child understand the meaning of division. In fact, students who become skillful with a particular procedure are very reluctant, after the fact, to attach meaning to it.

From the perspective of learning mathematics, the question of how procedures and conceptual ideas can be linked is much more important than the usefulness of the procedure itself (Hiebert & Carpenter, 1992). Recall the two children who used their own invented procedure to solve 156 ÷ 4 (see Figure 3.2, p. 29). Clearly, there was an active and useful interaction between the procedures the children invented and the ideas they were constructing about division.

It is generally accepted that procedural rules should never be learned in the absence of a concept. Unfortunately, that happens far too often.

The Role of Models In Developing Understanding

It has become a cliché that good teachers use a "hands-on" approach to teach mathematics. Manipulatives, or physical materials used to model mathematical concepts, are certainly important tools available for helping children learn mathematics. But they are not the panacea that some educators seem to believe them to be. It is important that you have the appropriate perspective on how manipulatives can help or fail to help children construct ideas.

Models for Mathematical Concepts

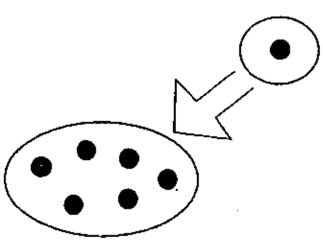
Return for a moment to the idea of a mathematical concept as a relationship, a logical idea. There are no physical embodiments of mathematical concepts in the physical world. The concept of "hundred," for example, is a quantity relationship that exists between a group of 100 items and a single item of the same type. We can talk of 100 people, 100 dollars, or 100 acts of kindness. None of those sets is a hundred. Hundred is only a relationship that the group has with one thing like those in the group. It is impossible to imagine "hundred" without first understanding "one."

A *model for a mathematical concept* refers to any object, picture, or drawing that represents the concept, or onto which the relationship for that concept can be imposed. In this sense, any group of 100 objects can be a model of the concept "hundred" because we can impose the 100-to-1 relationship on the group, and on a single element of the group.

It is incorrect to say that a model "illustrates" a concept. To illustrate implies showing. That would mean that when you look at the model, you would see an example of the concept. Technically, all that you actually see with your eyes is the object; only your mind can impose the mathematical relationship on the object (Thompson, 1994). If a person does not yet possess the relationship, the model does not illustrate the concept *for that person.*

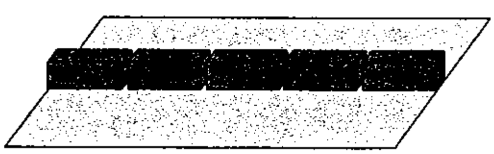
Examples of Models

As noted, physical materials have become enormously popular as tools for teaching mathematics. They can run the gamut from common objects, such as lima beans for counters, to commercially produced materials, such as wooden rods or plastic geometric shapes. Figure 3.8 shows six common examples of models for six different concepts. Consider each of the concepts and the corresponding model. Try to separate the physical model from the relationship that you must impose on it in order to "see" the concept.



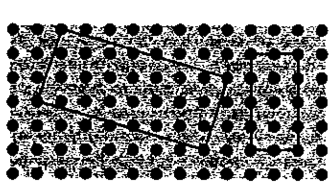
Countable objects can be used to model "number" and related ideas such as "one more than."

(a)



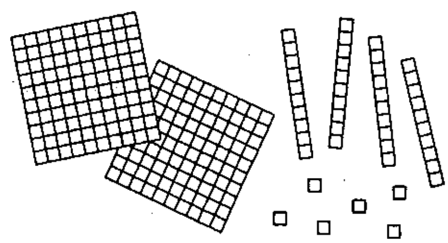
"Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.

(b)



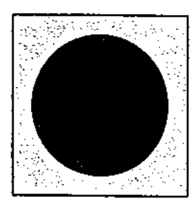
"Rectangles" can be modelled on a dot grid. They involve length and spatial relationships.

(c)



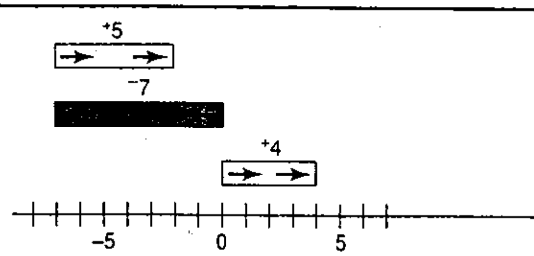
Base-ten concepts (ones, tens, hundreds) are frequently modelled with strips and squares. Sticks and bundles of sticks are also commonly used.

(d)



"Chance" can be modelled by comparing outcomes of a spinner.

(e)



"Positive" and "negative" integers can be modelled with arrows with different lengths and directions.

(f)

FIGURE 3.8 Examples of models to illustrate mathematics concepts.

For the examples in Figure 3.8:

(a) The concept of "six" is a relationship between sets that can be matched to the words *one, two, three, four, five, six*. Changing a set of counters by adding one alters the relationship. The difference between the set of 6 and the set of 7 is the relationship of "one more than."

(b) The concept of "length" could not be developed without making comparisons of the length attribute of different objects. The length measure of an object is a comparison relationship between the length of the object and the length of the unit.

(c) The concept of "rectangle" is a combination of spatial and length relationships. By drawing on dot paper, the relationships of opposite sides that are equal in length and parallel, and the adjacent sides' meeting at right angles, can be illustrated.

(d) The concept of "hundred" is not in the larger square but in the relationship of that square to the strip ("ten") and to the little square ("one").

(e) "Chance" is a relationship between the frequency of an event's occurrence compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sectors of the spinner. Note how chance and probability are integrated with ideas of fractions and ratio.

(f) The concept of a "negative integer" is based on the relationship "is the opposite of." Negative quantities exist only in relation to positive quantities. Arrows on the number line are not themselves negative quantities but model the "opposite of" relationship in terms of direction and size, or magnitude in terms of length.

Staying with integers for a moment, this concept is often modelled with counters in two colours, perhaps red for negative quantities and yellow for positive. The "opposite" aspect of integers can be imposed on the two colours. The "magnitude" aspect is found in the quantities of red and yellow counters. Although coloured counters and arrows are physically very different, the same relationships can be imposed on each. Children must construct relationships in order to "see" positive and negative integers in either model.

It is important to include calculators in any list of common models. The calculator models a wide variety of numerical relationships by quickly and easily demonstrating the effects of these ideas. For example, if the calculator is made to count by increments of 0.01 (press [+] 0.01 [=]), the relationship of one-hundredth to one whole is illustrated. Press 3 [+] 0.01. How many presses of [=] are required to get from 3 to 4? Doing the required 100 presses and observing how the display changes along the way is quite impressive. Especially note what happens after 3.19, 3.29, and so on.

Models and Constructing Mathematics

In order to "see" in a model the concept that it represents, you must already have that concept—that relationship—in your mind. If you do not, then you would have no relationship to impose on the model. This is precisely why models are often more meaningful to the teacher than to the students. The teacher already has the concept and can see it in the model. A student without the concept sees only the physical object.

Thus a child needs to know the relationship before imposing it on the model. If the concept does not come *from* the model—and it does not—how does the model help the child get it?

Mathematical concepts that children are in the process of constructing are not the well-formed ideas conceived by adults. New ideas are formulated little by little over time. As children actively reflect on their new ideas, they test them out through as many different avenues as we might provide. For example, this is where the value of student discussions and group work comes in. Talking through an idea, arguing for a viewpoint, listening to others, and describing and explaining are all mentally active ways of testing an emerging idea against external reality. As this testing process goes on, the developing idea gets modified and elaborated and further integrated with existing ideas. When there is a good fit with external reality, the likelihood of a concept being formed correctly is good.

Models can also play the role of a testing ground for emerging ideas. They can be thought of as "thinker toys," "tester toys," and "talker toys."* It is difficult for students (of all ages) to talk about and test out abstract relationships using words alone. Hence, models give learners something to think about, explore with, talk about, and reason with.

Expanding the Idea of a Model

Lesh, Post, and Behr (1987) talk about five "representations" for concepts, two of which are manipulative models and pictures (see Figure 3.9). In their research, they also consider written symbolism, oral language, and real-world situations as representations or models of concepts. Their research has demonstrated that children who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations. Strengthening children's ability to move between and among these representations improves their conceptual growth.

The five representations illustrated in Figure 3.9 are simply an expansion of the model concept. The more ways that children are given to think about and test out an

*The term *thinker* toy is taken from Seymour Papert's book *Mindstorms* (1980), in which the inventor of the Logo computer language describes the computer as a powerful and flexible device that encourages learners to play with ideas and work through problems. "Tester toys" and "talker toys" were suggested in the current context by Laura Domalik, a grade 1 teacher.

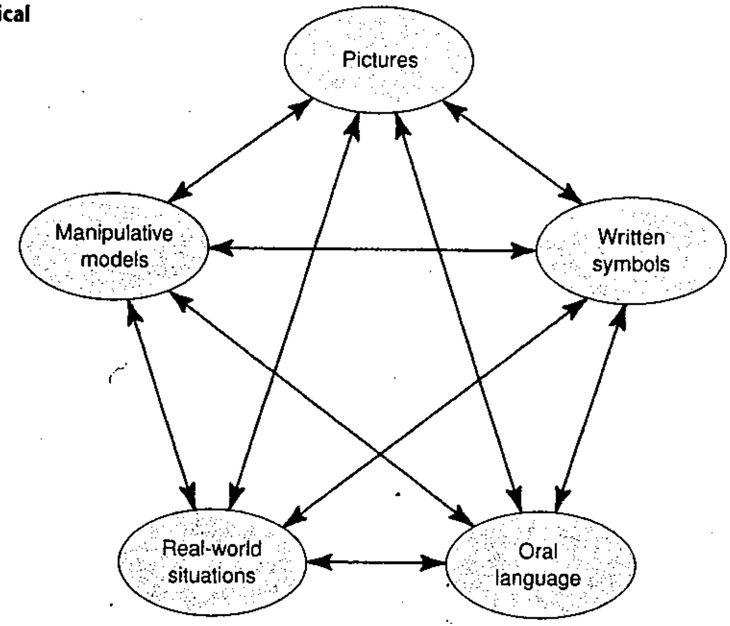


FIGURE 3.9 Five different representations of mathematical ideas. Translations between and within each can help develop new concepts.

emerging idea, the better chance the idea has of being formed correctly and integrated into a rich web of ideas and relational understanding.

Using Models in the Classroom

If we think of models as thinker toys or talker toys, we can identify three related uses for them in a developmental approach to teaching:

1. To help children develop new concepts or relationships
2. To help children make connections between concepts and symbols
3. To help educators assess children's understanding

Developing New Concepts

Models help children as they think and reflect on new ideas. Students should be encouraged to select and use materials to help them work through a problem or explain an idea to their group. To that end, a variety of models should be available to students so that they may use them freely when thinking through an important idea. Students should be free to select those models that make sense to them and not be forced to use a particular model.

You will undoubtedly encounter situations in which you use a model that you think clearly illustrates an idea, but the child just doesn't get it. Remember that you already possess the well-formed concept, so you are able to impose it on the model. Children are often able to see connections and relationships between concepts and models that we as adults miss because of our own well-formed concepts. Always encourage children to share their ideas with one another.

A child in the process of creating a concept may use a model to test an emerging idea. Your job is to get children to think with models, to work actively at the test-revise-test-revise process until the new concept fits with the physical model you have offered. It is not possible to show mathematics with models. You can only provide models on which mathematical relationships or concepts can be imposed. When the child's concept fits the model, the child sees the concept. When the concept does not seem to fit, the child cannot see it in the model. The child's concept is different from the one that you impose on the model and so must undergo further construction or revision.

Connecting Symbols and Concepts

Teachers will say, "But when they try to do it without manipulatives, they can't." Yet it is unrealistic to expect children automatically to transfer newly formed ideas to symbolic procedures without some guidance. Models can serve as a link between concepts and symbols as well as a means for developing concepts.

A general approach is to have students write down how they have used the models. "Write an equation to tell what you just did." "I see how you did that problem with the blocks. How would you go about recording what you did?" When children see written mathematics as expressions or recordings of ideas that they have already developed, the written or symbolic form is more likely to make sense.

Assessing Children's Understanding

When children in the classroom use models in ways that make sense to them (rather than following your directions), the manner in which they are used provides a wonderful window into their minds. Classroom observation of your students then becomes a student-by-student assessment.

If you want more detailed information about the understanding children have constructed, have them use manipulative materials to explain their ideas. The models give children the words they need to express themselves when abstract ideas prove difficult for them to explain. This might be done in a diagnostic setting, where you sit down, one-on-one, with a child and try to find out what he or she is thinking. Remember that drawings are also models. When students write explanations for their answers or describe their ideas in writing, always encourage them to draw pictures to help show what they are thinking. (Assessment is discussed in depth in Chapter 5.)

Incorrect Use of Models

The most widespread misuse of manipulative materials occurs when the teacher tells students, "Do as I do." There is a natural temptation to get out the materials and show children exactly how to use them. Children will blindly follow the teacher's directions, and it may even look as if they understand. It is just as possible to get students to move blocks around mindlessly as it is to teach them to "invert and multiply" mindlessly. Neither promotes thinking or aids in the development of concepts (Ball, 1992; Clements & Battista, 1990).

A natural result of overly directing the use of models is that children begin to use them as "answer-getting" devices rather than as thinker toys. When getting answers rather than solving problems becomes the focus of a lesson, children will gravitate to the easiest method available to get the answers. For example, if you have carefully shown and explained to children how to get an answer with a set of counters, then an imitation of that method is what they will most likely select. By strictly following your directions, little or no reflective thought will go into exploring the concepts involved. When an activity is not reflective, little real growth occurs, and little understanding is constructed.

Teaching Developmentally

Teaching involves decision-making. Decisions are made as you plan lessons. *What is the best task to propose tomorrow? Considering what happened today, what will move the children forward?* And decisions are made minute to minute in the classroom. *How should I respond? Should they struggle some more, or should I intervene? Is progress being made? How can I help Suzy move in the correct direction without discouraging her?*

The ideas that have been discussed in this chapter provide a theoretical foundation for making those decisions.

Foundations of a Developmental Approach

Following is a summary of the major implications of the theory that has been discussed. A teacher who keeps these ideas in mind can be said to be basing his or her instruction on a constructivist view of learning or, in the terminology of this book, a *developmental approach.*

1. *Children construct their own knowledge and understanding; we cannot transmit ideas to passive learners.* Each child comes to us with a unique but rich collection of ideas. These ideas are the tools that will be used to construct new concepts and procedures as students wrestle with ideas, discuss solutions, challenge their own and others' conjectures, explain their methods, and solve engaging problems. Ideas cannot be poured into children as if they were empty vessels.
2. *Knowledge and understanding are unique for each learner.* Each child's network of ideas is different from that of the next child. As new ideas are formed, they will be integrated into that web of ideas in a unique way as well. We should not try to make all children the same.
3. *Reflective thinking is the single most important ingredient for effective learning.* In order to create new ideas and to connect them in a rich web of interrelated ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of new ideas and the solutions to new problems. Only by being mentally engaged with the task at hand can relational understanding of new ideas ever develop. "Passive learning" is an oxymoron!
4. *Effective teaching is a child-centred activity.* In a constructivist classroom, the emphasis is on learning rather than teaching. Students are given the task of learning. The role of the teacher is to engage the students by posing good problems and creating a classroom atmosphere of exploration and sense-making. The source of mathematical truth is found in the reasoning carried out by the class. The teacher is not the arbiter of what is mathematically correct.

Strategies for Effective Teaching

How can we structure lessons to promote appropriate reflective thought? Purposeful mental engagement or reflective thought about the ideas we want students to develop is the single most important key to effective teaching. Without active thinking about the important concepts of the lesson, learning will not happen. How can we make it happen? Here are seven suggestions based on the perspectives of this chapter. Perhaps you will be able to add to the list.

1. Create a mathematical environment.
2. Pose worthwhile mathematical tasks.
3. Use cooperative learning groups.
4. Use models and calculators as thinking tools.
5. Encourage discourse and writing.
6. Require justification of student responses.
7. Listen actively.

Creating a Mathematical Environment

In a mathematical environment, students feel comfortable trying out ideas, sharing insights, challenging others, seeking advice from other students and the teacher, explaining their thinking, and taking risks. No one is permitted to be a passive observer. An environment with these features is built around expectations, respect, and the belief that *all* children can learn. Learning takes effort, and children need to know that as a class, their task is to work at doing mathematics. The interactions of a mathematical environment require students and teachers alike to respect one another, to listen attentively, and to learn to disagree without offending.

We cannot simply tell children how to think or what habits to acquire. Processes and habits of thought are developed over time within a community where such processes and thinking are the norm. In a community of mathematical discourse, students evaluate their own assumptions and those of others and argue about what is mathematically true (Corwin, 1996; Lampert, 1990; Nova Scotia Department of Education and Culture, 1993). The goal is to let all students believe that they are the authors of mathematical ideas and logical arguments. In this environment, reasoning and mathematical argument—not the teacher—are the sources of an idea's legitimacy. "Doing mathematics is an act of sense-making" (Schoenfeld, 1994, p. 60). The classroom environment should be a place where figuring it out and "sense-making" are common practices, not just for individuals, but for the class as a whole.

In an urban school in Montreal, a class of grade 5 students was observed during a discussion about the meaning of area. As one child wrote on the blackboard, "area is P = l × w = 2 + 2," another said, "I disagree with Andre. I think area is something different." Another child commented, "I would like to add to what Marcel just said." All students faced the speaker and listened attentively as he spoke. In another classroom nearby, grade 2 students raised their hands with their index finger pointing up to indicate "a point of interest," a polite way to disagree. In both classrooms, it was clear that teachers had spent time and effort developing this atmosphere of respect. "Creating contexts where students can safely express their own mathematical ideas is a central teaching task and a step toward developing students' mathematical power" (Smith, 1996, p. 397).

Posing Worthwhile Mathematical Tasks

The single most important principle for reform in mathematics is to allow students to make the subject of mathematics problematic (Hiebert et al., 1996). By problematic, these authors mean "allowing students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities. It means that both curriculum and instruction should begin with problems, dilemmas, and questions for students" (p. 12). When students are actively looking for relationships, analyzing patterns, finding out which methods work and which don't, justifying results, or evaluating and challenging the thoughts of others, they are necessarily and optimally engaging in reflective thought about the ideas involved.

Tasks or problems must be designed to engage students in the concepts of the curriculum. The tasks given a class should be based on the students' knowledge of the mathematical content and an informed guess about the concepts they bring to the task (Fennema, Carpenter, Franke, & Carey, 1993; Flewelling & Higginson, 2000; Simon, 1995). Time must be given to permit students to wrestle with these tasks individually or in groups and also to discuss solutions and strategies with the class as a whole.

The selection of good tasks requires listening each day to the way students are thinking about whatever mathematics is currently being discussed. The next day's task should be chosen to help students reflect on the new ideas you want them to develop. Look for explorations that embody the big ideas of the chapter. As students wrestle with these problems, the tiny skills and ideas of the traditional curriculum will emerge. In a good task, students will "bump into" the important mathematics you have in mind for them to learn (Lappan & Briars, 1995).

Using Cooperative Learning Groups

Placing children in cooperative groups of three or four to work on a problem is an extremely useful strategy for encouraging the discourse and interaction envisioned in a mathematical community. A classroom arranged in small cooperative groups has much more interaction and discussion going on than can be accomplished in a full-class setting. It also encourages greater accountability on the part of the students. Frequently, a simple pairing of students is all that is necessary. In groups or pairs, children are much more willing and able to speak out, explore ideas, explain things to their peers, question and learn from one another, pose arguments, and have their own ideas challenged in a friendly atmosphere of learning. Children are more willing, within a small group, to take risks that they would never dream of taking in front of an entire class. Groups should usually be heterogeneous in ability so that all students are exposed to good thinking and reasoning (Bennett, Rolheiser-Bennett, & Stevahn, 1991).

While the groups are at work, the teacher has the opportunity to listen actively to six or more different discussions. Time should always be allotted for full-class discussions so that group members can share their group's ideas and the teacher can focus attention on important ideas. (See Chapter 21 for a more detailed discussion of cooperative groups.)

Using Models and Calculators as Thinking Tools

The use of models has already been thoroughly discussed, but it is worth repeating that models help children as they explore ideas and attempt to make sense of them. Many good explorations can be initiated with the use of concrete materials. For example, "Try to find different ways to make the number 437 using ones, tens, and hundreds pieces. What patterns can you find? What else do you notice about the ways you can make 437?" Here the model is the focus of the problem, rather than a means of exploring a different task.

Manipulatives and calculators should always be readily available for student use as a regular part of your classroom environment—a recommendation that is just as true for middle school classrooms as for kindergarten.

Encouraging Discourse and Writing

To explain an idea orally or in written form forces us to wrestle with that idea until it is really ours and we personally understand it. The more we try to explain something or argue reasonably about it, the more connections we will search for and use in our explanation or in our argument. Talking gets the talker involved.

When children are asked to respond to and critique others, they are similarly forced to attend to and assimilate what is being said into personal mental schemes. Frequently, when we get involved verbally with an idea, we find ourselves changing or modifying the idea in midstream. The reflective thought required to make an explanation or argue a point is a true learning experience in itself (Corwin, 1996; Whitin & Wilde, 1995; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990).

Writing can be a part of nearly every problem posed. It can include journals, formal essays, and reports. It is also a useful tool for assessment of students' progress (Marks-Krpan, 2001). Not only does writing help children structure their thoughts, it obliges them to commit to an idea and to rehearse an explanation or defence in preparation for a class discussion. Countryman (1992) states, "The writer reflects on, returns to, and builds upon what has gone before" (p. 59).

Requiring Justification of Student Responses

Requiring children to explain their ideas in detail or to defend their responses has a positive effect on how they view mathematics and their own mathematical abilities. It communicates that mathematics is not mysterious or unfathomable, and that the teacher is not necessarily the source of all mathematical truth. It also promotes confidence and self-worth.

Having to justify responses forces students to think reflectively. It also eliminates guessing or responses based on rote learning. Thus, having children explain their answers is another excellent mechanism for achieving the same benefits as from discourse and writing.

Listening Actively

To promote reflective thinking requires teaching to be child-centred, not teacher-centred. By focusing on children's thoughts instead of our own, we encourage children to do more thinking and hence to search for and strengthen more internal connections—in short, to develop understanding. When children respond to questions or make an observation in class, an interested but nonevaluative response from the teacher is a way to have them elaborate their ideas or provide additional information: "Tell me more about that, Karen" or "I see. Why do you think that?" Even a simple "Um-hmm" followed by silence is very effective, as it permits the child and others to continue their thinking.

Active listening requires that we believe in children's ideas. Waiting 45 seconds, a minute, or even longer for a child to find a response or formulate even a simple idea is much easier when we believe that whatever the child says reflects a unique and valuable understanding. When you believe in children, they sense it and respond accordingly.

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